Combinatorial Changes of Euclidean Minimum Spanning Tree of Moving Points in the Plane

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Abstract

In this paper, we enumerate the number of combinatorial changes of the the Euclidean minimum spanning tree (EMST) of a set of $n$ moving points in 2-dimensional space. We assume that the motion of the points in the plane, is defined by algebraic functions of maximum degree $s$ of time. We prove an upper bound of $O(n^3\beta_2s(n^2))$ for the number of the combinatorial changes of the EMST, where $\beta_s(n) = \frac{\lambda_s(n)}{n}$ and $\lambda_s(n)$ is the maximum length of Davenport-Schinzel sequences of order $s$ on $n$ symbols which is nearly linear in $n$. This result is an $O(n)$ improvement over the previously trivial bound of $O(n^4)$.

1 Introduction

The minimum spanning tree for a weighted graph $G(V, E)$ is a connected sub-graph $G'(V, E')$ of $G$ where sum of the weights of its edges is the minimum possible. For a set $P = \{p_1, p_2, \ldots, p_n\}$ of $n$ points, we can construct a complete weighted graph with these points as its nodes and the weight of an edge is the Euclidean distance between its end points. Finding the minimum spanning tree for this graph is known as the Euclidean minimum spanning tree problem (EMST for short). This problem has many applications in geometry and graph theory and has been studied extensively before [2, 3].

We consider the kinetic version of the EMST problem in which the points are moving in the plane and we want to enumerate the total number of the combinatorial changes of the EMST during the motion. In this setting, the points are moving independently and we assume that the position of a point $p_i$ at time $t$, denoted by $p_i(t)$, is defined by an algebraic function of maximum degree $s$. We are supposed to find an upper bound for the number of these changes from time $t = 0$ to time $t = \infty$.

For a restricted version of this problem in which the trajectory of each point is defined by a linear function of time, Katoh et al. [4] proved that the maximum number of the minimum spanning tree changes for $L_1$ and $L_\infty$ metrics is $O(n^{5/2}\alpha(n))$ where $\alpha(n)$ is the inverse Ackerman’s function. They obtained an upper bound of $O(n^{3/2}\alpha(n))$ for the number of the combinatorial changes of the minimum spanning tree in the $L_2$ metric for the special case of linearly moving points (in terms of time). Basch et al. [1] presented an approximate method for $(1 + \epsilon)$-EMST which considers $O(\epsilon^{−(d−1)n^3})$ combinatorial changes in $d$-dimensional space when the points follow algebraic trajectories of fixed degree. However, for the case of our problem in which the points are moving according to algebraic functions in the plane and the distances are measured by the $L_2$ metric, the known upper and lower bounds of the number of combinatorial changes of the EMST is the trivial $O(n^4)$ and $\Omega(n^2)$ bounds.

In this paper, we show that the number of the combinatorial changes of the EMST for our setting is $O(n^3\beta_2s(n^2))$. Here, $\beta_s(n)$ is an extremely slowly growing function of $n$. Precisely, $\beta_s(n) = \frac{\lambda_s(n)}{n^2}$ where $\lambda_s(n)$ is the maximum length of Davenport-Schinzel sequences of order $s$ on $n$ symbols which is nearly linear in $n$. To be exact, $\lambda_s(n) = n\alpha(n)^{O(\alpha(n)^{s−1})}$, for $s > 3$ where $\alpha(n)$ is the inverse Ackerman’s function[5].

2 Preliminaries

We first consider the problem of enumerating the number of changes on the lower envelope of a set of functions in the plane. Let $\mathcal{F} = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a collection of continuous polynomial functions. The lower envelope of $\mathcal{F}$ is defined as: $\mathcal{L}\mathcal{E}\mathcal{F}(x) = \min_{1 \leq i \leq n} f_i(x)$. A breakpoint arises on the lower envelope whenever two functions $f_i$ and $f_j$ intersect on $\mathcal{L}\mathcal{E}\mathcal{F}$ (See Figure 1). In other words, the lower envelope is a sequence of partially defined functions connected at the breakpoints. The length of a lower envelope is the length of its sequence. We have the following theorem about the length of the lower envelope:

Theorem 1 (Corollary 2.2 [5]) For any collection
Proof. Two consecutive functions $f_i$ and $f_{i+j}$ are not equal and the path $P_{LE}$ does not contain any alternating subsequence that uses two different curves $C_j$ and $C_{j'}$ and has length $s + 2$. In particular, there are no $s + 2$ indices $r_1 < r_2 < \ldots < r_{s+2}$ where $f_{r_1} = f_{r_3} = \ldots = f_{r_j}$ and $f_{r_2} = f_{r_4} = \ldots = f_{r_{j'}}$. Therefore, the number of all curves minus one is equal to the number of the breakpoints in the lower envelope of $n$ polynomial functions of at most degree $s$ [5].

From Lemma 2, we can rewrite Theorem 1 as follows:

**Theorem 3** For any collection $\mathcal{F}' = \{f'_1, f'_2, \ldots, f'_n\}$ of $n$ $\mathcal{LE}_{\mathcal{F}'}$-continuous, $\mathcal{LE}_{\mathcal{F}'}$-total, univariable functions, each pair of whose graphs intersect in at most $s$ points, the length of the lower envelope sequence is at most $\lambda_s(n)$.

### 3 Combinatorial Changes of the EMST

Let $\mathcal{E}(CG)$ and $\mathcal{E}(EMST)$ be respectively the set of edges of the complete graph and the edges of the EMST of a set of moving points $P$ in the plane and $path(p_i, p_j)$ be the simple path between $p_i$ and $p_j$ in the EMST.

After the initial computation of $\mathcal{E}(EMST)$ and during the motion, we must replace some edge $e_i \in \mathcal{E}(EMST)$ by another edge $e_j \in \mathcal{E}(CG) - \mathcal{E}(EMST)$. To enumerate the number of these changes we have to know such replacement candidate pairs. For each edge $p_i p_j \in \mathcal{E}(CG) - \mathcal{E}(EMST)$ there is a simple path $path(p_i, p_j)$ in the EMST that connects $p_i$ and $p_j$. Assume that $p_s p_t$ has the maximum Euclidean length among edges of $\mathcal{E}(EMST)$.

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From Lemma 2, we can rewrite Theorem 1 as follows: assume
that $g_i(t) = |e(PK_i)|^2$ and $f_{ij} = |e_{ij}|^2$ for all edges $e_{ij} \in PK_i$. Then, the lower envelope of $F_i = \{f_{i1}, ..., f_{ik}\} \cup \{g_i\}$ defines the edges with the minimum length during the motion. It must be noted that $\mathcal{L}E_{F_i}$ always lies of $g_i$ and whenever another function $f_{ij}$ intersects $g_i$ its corresponding edge will be added to the EMST and will be the edge $e(PK_i)$ and $PK_i$ will be subject to some changes (insertion and deletion).

Therefore, an important notice about the $PK_i$ and $F_i$ sets is that these sets are not fixed and are subject to changes during the motion from time $t = 0$ to time $t = \infty$.

Using the above discussion we have the following Theorem about the number of the combinatorial changes of the EMST:

**Theorem 4** The total number of the combinatorial changes of the EMST of a set of $n$ moving points in the plane is equal to the number of the breakpoints of the lower envelope of the functions $F_i$.

It is simple to show that the sets $PK_i$’s are disjoint and therefore the $F_i$’s sets of functions are disjoint. Moreover, $\bigcup_i (PK_i \cup e(PK_i)) = \mathcal{E}(CG)$ and therefore $|\bigcup_i F_i| = O(n^2)$ which means that $|F_i| = O(n^2)$.

**Lemma 5** Whenever an edge $e$ is added or removed from a set $PK_i \cup e(PK_i)$ at time $t$, its corresponds function $f(t) = |e|^2$ lies above the lower envelop of $F_i$ at this time.

**Proof.** Assume that at time $t$ an edge $e$ is removed from $PK_i \cup e(PK_i)$ and it is inserted into $PK_j \cup e(PK_j)$. We know that for each $PK_k$ the lower envelope $\mathcal{L}E_{F_k}$ lies on the function $g_k = |e(PK_k)|^2$. Therefore to prove the Lemma it is enough to show that at time $t - \epsilon$ we have $e \neq e(PK_i)$ and $e \neq e(PK_j)$ at time $t + \epsilon$. The correctness of these are directly derived from the definition of $PK$ sets. An edge $p_k p_s$ at time $t$ belongs to $PK_i$ if and only if $e(PK_i) \in \text{path}(p_k p_s)$ at time $t$ and has the maximum length. Therefore, if at time $t$ we must remove $e$ from $PK_i \cup e(PK_i)$ and add it to $PK_j \cup e(PK_j)$ we must have $e(PK_i) \in \text{path}(e)$, $e(PK_j) \in \text{path}(e)$ and $|e(PK_i)| < |e(PK_j)|$ at time $t + \epsilon$. While $e(PK_i)$ and $e(PK_j)$ belong to the EMST, $e$ can not be equal to any one of these edges at time $t$.

Using the results of the above Theorems, Lemma and discussions we have the following result:

**Theorem 6** For a set of $n$ points moving in the plane according to some algebraic functions of maximum degree $s$, the number of the combinatorial changes of the EMST is $O(n^3 \beta_{2s}(n^2))$.

**Proof.** As the points are moving according to function of maximum degree $s$, the functions $g_i$ and $f_{ij}$ are defined by algebraic functions of maximum degree $2s$. Combining the results of Theorem 3 and Lemma 5 and the fact that $|F_i| = O(n^2)$, the number of the breakpoints of the lower envelope of each $F_i$ is $\lambda_{2s}(n^2)$. We have $n - 1$ sets of $F_i$ corresponding to the $n - 1$ labels and therefore the total number of the breakpoints of the lower envelopes is $O(n \lambda_{2s}(n^2))$ which according to Theorem 4 is equal to the number of the combinatorial changes of the EMST.

4 Conclusion

In this paper, we investigated the combinatorial changes of the minimum spanning tree of a set of moving points in $L_2$ metric. We proved an upper bound of $O(n^3 \beta_{2s}(n^2))$ which is an improvement of $O(n^3)$ over the previously known bound.

Proving the tight bounds for the number of the changes of the EMST and extending it to higher dimensions are further directions of this research.

**References**


