Kernelizations for Parameterized Counting Problems

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Overview

1. Counting Kernelizations - Definitions and Concepts

2. Counting Kernelization Examples
   - The Vertex Cover Problem
   - The Hitting Set Problem
   - The Unique Hitting Set Problem

3. Conclusions and Future Work

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Motivation

Recall the concepts of kernelization and fixed parameter tractability for regular decision problems.

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Definition

For a decision problem $Q \subseteq \Sigma^*$ and parameterization $k \in \mathbb{N}$, a kernelization is a mapping $\Psi : \Sigma^* \to \Sigma^*$ satisfying the following three conditions.

- Ψ *is a function computable in polynomial time.*
- *There exists a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $|\Psi(x)| \leq g(k)$ for all $x \in \Sigma^*$.*
- $x \in Q \iff \Psi(x) \in Q$ *for all $x \in \Sigma^*$.*

We say that Q is kernelizable if there exists such a mapping Ψ for Q.

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Let FPT denote the class of all fixed parameter tractable decision problems and let (Q, k) be a parameterized decision problem. Then $(Q, k) \in FPT$ if and only if (Q, k) is kernelizable.

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With the success of kernelization in the design of fixed parameter tractable algorithms for decision problems, we would like to generalize and expand this notion to counting problems as well.

Counting Kernelizations

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Definition

Let $F : \Sigma^* \to \mathbb{N}$ be a counting problem and let $k \in \mathbb{N}$ be a parameterization of F.

1. *A mapping $\Psi : \Sigma^* \to \Sigma^*$ is k-bounded if there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $|\Psi(x)| \leq g(k)$ for all $x \in \Sigma^*$.*
2. *A relation $Y \subseteq \Sigma^* \times \{0,1\}^*$ is Ψ-aware for a k-bounded mapping Ψ if there are computable functions $f : \mathbb{N} \to \mathbb{N}$ and $h : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \Sigma^*$ and every $z \in \{0,1\}^*$:*
   - *Given $\Psi(x)$, it can be decided in time $\mathcal{O}(f(k))$ whether or not $(\Psi(x), z) \in Y$ holds.*
   - *If $(\Psi(x), z) \in Y$, then $|z| \leq h(k)$.*

Counting Kernelizations

As it stands, we are still missing a characterization of the solutions of the reduced kernel instance $\Psi(x)$. The notion of Ψ-awareness, in conjunction with the notion of witnesses, will be used to create this desired characterization. Informally speaking, we can think of a witness as an encoding of a solution for a particular problem instance.

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Definition

Let $Y \subseteq \Sigma^* \times \{0,1\}^*$ be a relation. For any $x \in \Sigma^*$, we define $w_Y(x) := \{z : (x,z) \in Y\}$ to be the set of witnesses of x in Y.

Counting Kernelizations

Furthermore, it is possible for the reduced kernel instance $\Psi(x)$ to have fewer solutions than x itself! In other words, there may not necessarily be a bijective correspondence between the solutions of x and $\Psi(x)$.

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Hence, we need a function μ such that, for each solution z of $\Psi(x)$, μ computes the number of solutions of x that are represented by z.
Putting everything together gives the following definition of counting kernelization for a given parameterized counting problem.

Counting Kernelizations

Definition

Let $F : \Sigma^* \to \mathbb{N}$ be a counting problem and let $k \in \mathbb{N}$ be a parameterization of F. A counting kernelization of (F, k) is a pair (μ, Ψ) of computable functions $\mu : \Sigma^* \times \Sigma^* \times \{0,1\}^* \to \mathbb{N}$ and $\Psi : \Sigma^* \to \Sigma^*$ such that the following three conditions are satisfied.

1. *μ and Ψ are computable in polynomial time.*
2. *Ψ is a k-bounded mapping.*
3. *There is a Ψ-aware relation $Y \subseteq \Sigma^* \times \{0,1\}^*$ such that:*

$$F(x) = \sum_{z \in w_Y(\Psi(x))} \mu(x, \Psi(x), z)$$

We call $|\Psi(x)|$ the size of the kernel.

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Though this theorem may not provide any practical purpose, it does however provide an interesting theoretical result relating fixed parameter tractability and counting kernelizations.

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Vertex Cover - Problem Definition

In this section, we will see how to adapt the Crown Rule Reduction to counting problems. In particular, we will see how to apply this kernelization technique to count vertex covers.

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Input: A graph $\mathcal{G} := (V, E)$ *and* $k \in \mathbb{N}$
Parameter: k
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Question: Compute the number of vertex covers of cardinality k *in* $\mathcal{G}$

We will use the notation $vc(\mathcal{G})$ to denote the size of a minimum vertex cover in $\mathcal{G}$, and we will also use $\#vc(\mathcal{G}, k)$ to denote the number of vertex covers of size k in $\mathcal{G}$.

The Crown Reduction Rule

We begin by recalling the definition of a crown in $\mathcal{G}$.

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Definition

Let $\mathcal{G} := (V, E)$ be a graph. A crown in $\mathcal{G}$ is a bipartite subgraph $\mathcal{C} := (I, N(I), F)$ of $\mathcal{G}$ satisfying the following three conditions.

1. *I is an independent set in $\mathcal{G}$ and $N(I)$ is the set of all neighbours of vertices from I in $\mathcal{G}$.*
2. *F contains all of the edges of E that connect vertices from I to $N(I)$.*
3. *$\mathcal{C}$ has a matching of cardinality $|N(I)|$.*

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For simplicity, we will asume that $\mathcal{G}$ has no isolated vertices - The paper shows how to handle this case.

The Crown Reduction Rule - Recap

Recall from the Crown Reduction Rule seminar that we have a crown consutruction algorithm for the vertex cover problem that can produce one of three different results.

- Determine if $vc(\mathcal{G}) > k$ (a no-instance).
- Check whether $|V| \leq 3k$.
- Construct a crown $\mathcal{C} = (I, N(I), F)$ on at least $|V| - 3k$ vertices.

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Recall also that we had shown that $|N(I)| < |I|$. We will use this fact again later on.

Counting Vertex Covers Using The Crown Reduction Rule

Let $X_{\mathcal{C}}$ be a vertex cover of $\mathcal{C}$ and let $\mathcal{G}' := (V', E')$ be the graph obtained from $\mathcal{G}$ by removing the vertices and edges that are covered by $X_{\mathcal{C}}$. In particular, we have that $\#vc(\mathcal{G}', k - |X_{\mathcal{C}}|)$ is equal to the number of vertex covers X of size k in $\mathcal{G}$ with $X \supseteq X_{\mathcal{C}}$.

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Since $|V'| \leq |\mathcal{G} \backslash \mathcal{C}| + |N(I)| \leq 4k$, we have that $\#vc(\mathcal{G}', k - |X_{\mathcal{C}}|)$ can be computed in time depending on k only. Hence it suffices to show how enumerate the vertex covers of $\mathcal{C}$.

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Lemma

Let $X_{\mathcal{C}}$ be a vertex cover of $\mathcal{C}$ such that there exists a vertex $y \in N(I) \backslash X_{\mathcal{C}}$. If $v \in I$ is a vertex with $y \in N(v)$, then $[v] \subseteq X_{\mathcal{C}}$.

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Proof.

Assume that there is a vertex $w \in [v]$ such that $w \notin X_{\mathcal{C}}$. Then there exists an uncovered edge $\{y, w\}$ in $X_{\mathcal{C}}$, contradicting the fact that $X_{\mathcal{C}}$ is a vertex cover. □

Counting Vertex Covers Using The Crown Reduction Rule

Consider a vertex cover $X_{\mathcal{C}}$ of $\mathcal{C}$ and let $Y = N(I) \cap X_{\mathcal{C}}$ and $Z = I \cap X_{\mathcal{C}}$.

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$$\mathcal{L}(Y) := \bigcup_{[v] \in \mathcal{B}(Y)} [v].$$

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Note that the previous lemma implies that $\mathcal{L}(Y) \subseteq X_{\mathcal{C}}$, which also implies that $\mathcal{L}(Y) \subseteq Z$.

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Note that the previous lemma implies that $\mathcal{L}(Y) \subseteq X_{\mathcal{C}}$, which also implies that $\mathcal{L}(Y) \subseteq Z$.
Furthermore, we can observe that $Y \cup \mathcal{L}(Y)$ forms a minimal vertex cover, since a vertex is not in $\mathcal{L}(Y)$ if all of its neighbours are already in the vertex cover (hence all of the adjacent edges are already covered).

Counting Vertex Covers Using The Crown Reduction Rule

We now define our kernelization procedure $\Psi(\mathcal{G}, \mathcal{C}, k) := (\mathcal{G}', \mathcal{C}', k)$, where $\mathcal{C}' := (I', N(I'), F')$ is obtained from $\mathcal{C}$ by keeping one vertex from I per equivalency class and then deleting the rest - $\mathcal{G}' := (V', E')$ is obtained from $\mathcal{G}$ in the same manner.

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Hence we have that $\|\mathcal{C}'\| \leq 2^k + k$ and $\|\mathcal{G}'\| \leq 2^k + 4k$, and so Ψ is a k-bounded mapping computable in polynomial time.
We can define μ as follows. Let X' be a vertex cover of $\mathcal{G}'$ such that $|X'| \leq k$ and $X' \cap (I' \cup N(I'))$ is minimal with respect to the vertices in $\mathcal{C}'$. Furthermore, we let $Y' := N(I') \cap X'$. Defining μ as follows gives us the desired result.

$$\mu((\mathcal{G}, \mathcal{C}, k), (\mathcal{G}', \mathcal{C}', k), X') := \binom{|I \backslash \mathcal{L}(Y')|}{k - |Y'| - |\mathcal{L}(Y')|}$$

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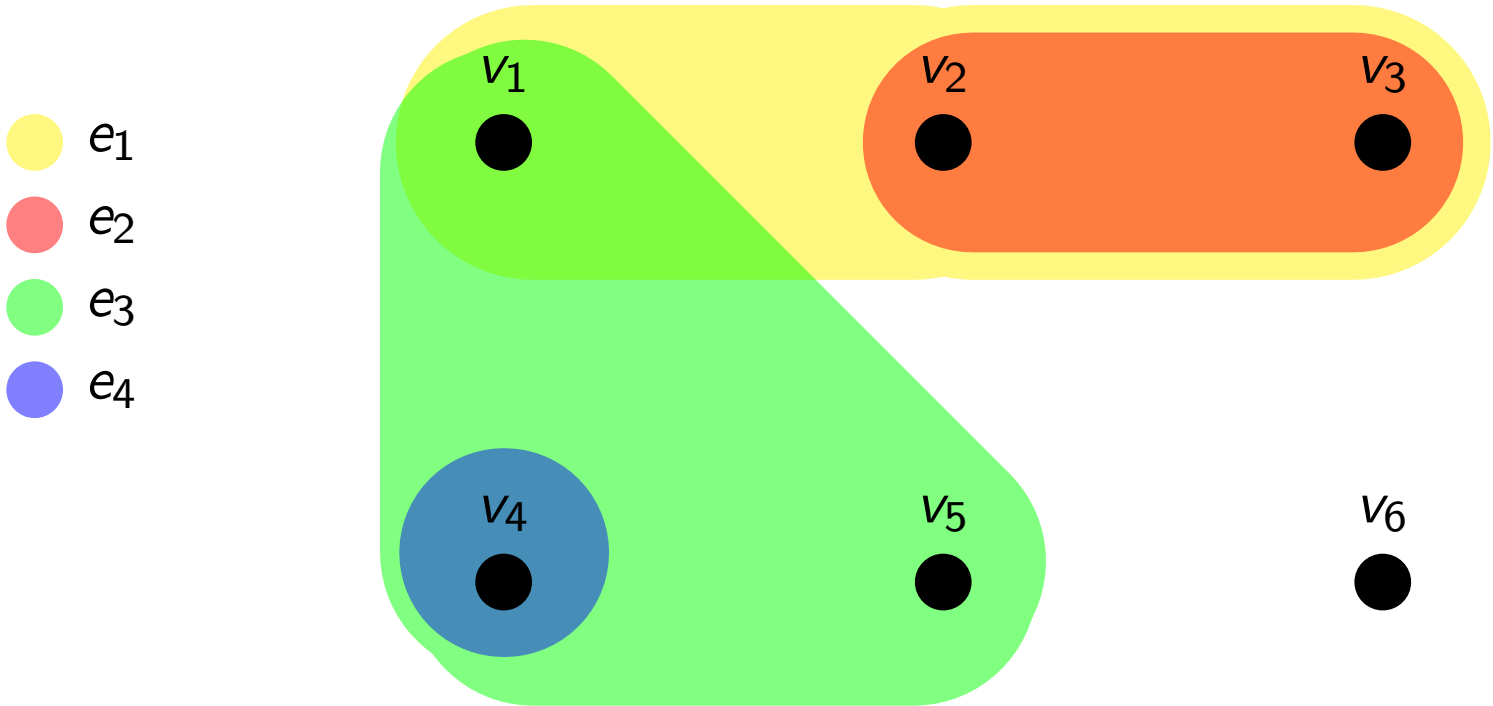

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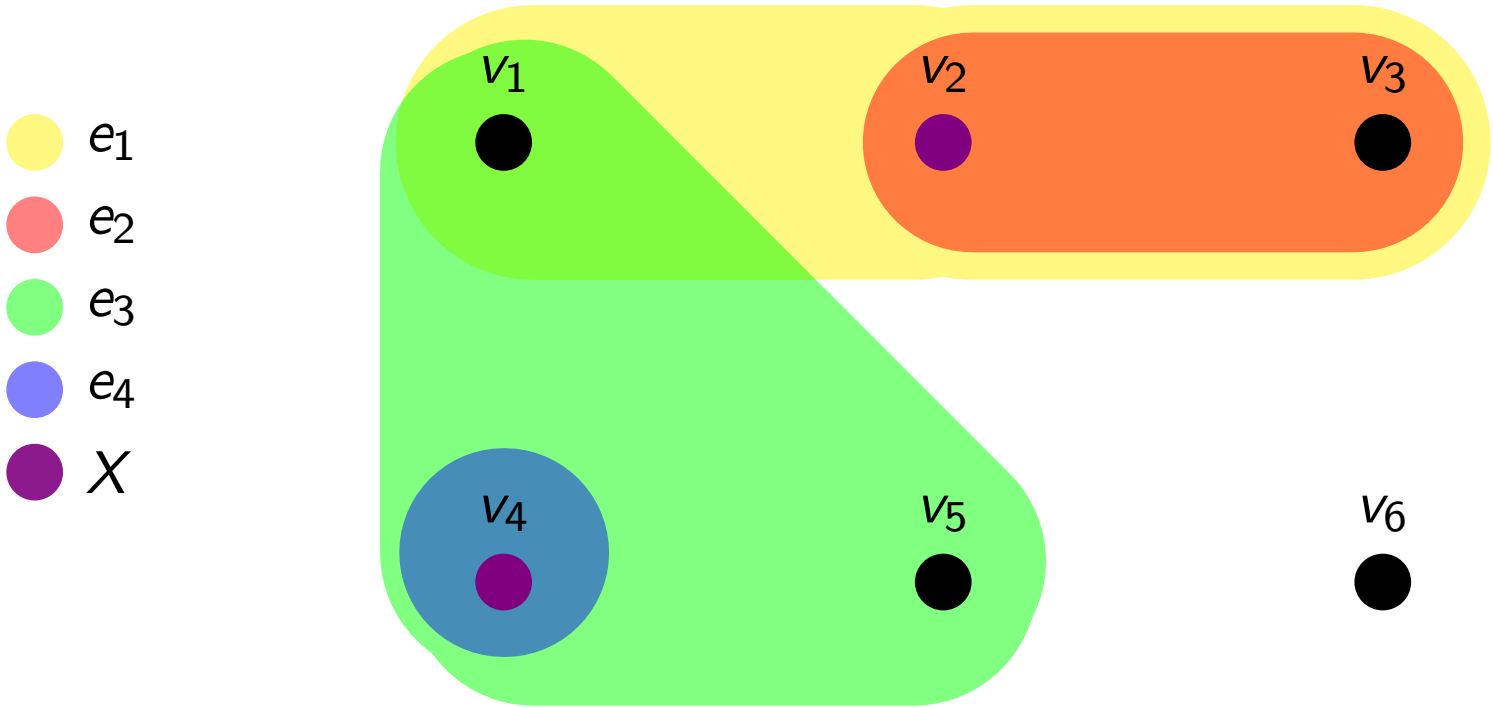

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We will also use the notation $\#hs(\mathcal{H}, k)$ to denote the number of hitting sets of size k in the hypergraph $\mathcal{H}$.

Hitting Set - Data Reduction Rule

Definition

Given a hypergraph $\mathcal{H} := (V, E)$, a sunflower in $\mathcal{H}$ is a family of hyperedges $\mathcal{S} = \{S_1, \ldots, S_k\} \subseteq E$ such that there exists a set $\mathcal{C} \subseteq V$ satisfying

$$S_i \cap S_j = \mathcal{C} \qquad \text{for all } 1 \leq i < j \leq k$$

We call $\mathcal{C}$ the core of the sunflower $\mathcal{S}$, while $S_i \backslash \mathcal{C}$ is called a petal of $\mathcal{S}$ if it is nonempty.

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The following lemma is known as the Sunflower Lemma. It will be the primary tool used in the kernelization process of our problem.

Lemma

Let $\mathcal{H} := (V, E)$ be a d-uniform hypergraph with more than $(k-1)^d \cdot d!$ hyperedges, for some $k, d \in \mathbb{N}$. Then there exists a sunflower $\mathcal{S}$ of cardinality k in $\mathcal{H}$.

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Lemma

Let $\mathcal{S}$ *be a sunflower in* $\mathcal{H}$ *with* $k+1$ *petals and core* $\mathcal{C}$. *Also, let* $\mathcal{H}' := (V, E')$ *be a hypergraph with* $E' := E \backslash \mathcal{S} \cup \{\mathcal{C}\}$. *Then we have that* $\#hs(\mathcal{H}, k) = \#hs(\mathcal{H}', k)$.

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- Every hitting set of size k in $\mathcal{H}'$ has a nonempty intersection with the core of $\mathcal{S}$, since the core of $\mathcal{S}$ is a hyperedge in $\mathcal{H}'$. Therefore, every hitting set of size k in $\mathcal{H}'$ is also a hitting set of size k in $\mathcal{H}$.

Hitting Set - Kernelization Procedure

Input: $\mathcal{H} := (V, E)$ and $k \in \mathbb{N}$
Output: Kernelized instance of $\mathcal{H}$
$d \leftarrow \max_{e \in E} |e|$;
for $i \leftarrow 1$ **to** d **do**
  $E_i \leftarrow \{e \in E : |e| = i\}$;
  $V_i \leftarrow \bigcup_{e \in E_i} e$;
  while $\mathrm{FindSunflower}((V_i, E_i), k+1) \neq (\emptyset, \emptyset)$ **do**
    Let $(\mathcal{S}, \mathcal{C})$ be the sunflower returned by $\mathrm{FindSunflower}$;
    $E_i \leftarrow (E_i \backslash \mathcal{S}) \cup \{\mathcal{C}\}$;
    $V_i \leftarrow (V_i \backslash \bigcup_{X \in \mathcal{S}} X) \cup \mathcal{C}$;
  end
end
$V' \leftarrow V_1 \cup \ldots \cup V_d$;
$E' \leftarrow E_1 \cup \ldots \cup E_d$;
return (V', E')

Hitting Set - Putting It All Together

Lemma

Let $\mathcal{H}$ be a hypergraph, let $k \in \mathbb{N}$, and let $\mathcal{H}' := (V', E')$ be the kernelized instance of $\mathcal{H}$. If $I = V \backslash V'$, then

$$\#hs(\mathcal{H}, k) = \sum_{j=0}^{k} \#hs(\mathcal{H}', j) \cdot \binom{|I|}{k-j}.$$

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Hitting Set - Putting It All Together

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- For any hitting set X' of size j in $\mathcal{H}'$ and for every set $I' \subseteq I$ of size $k - j$, the union $X' \cup I'$ is a hitting set of size k in $\mathcal{H}$.

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It can be shown that the kernelization procedure takes $\mathcal{O}(d^2 \cdot |E| \cdot \|\mathcal{H}\|)$ time. Moreover, for the hypergraph $\mathcal{H}' := (V', E')$, the Sunflower Lemma implies that $|E'| \leq k^d \cdot d! \cdot d$ and that $|V'| \leq k^d \cdot d! \cdot d^2$.

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- In particular, this implies a kernel of size $\|\mathcal{H}'\| \leq 2 \cdot k^d \cdot d! \cdot d^2$.

Following our framework for counting kernelizations, we can define functions $\Psi(\mathcal{H}) := \mathcal{H}'$ and $\mu(\mathcal{H}, \mathcal{H}', X) := \binom{|I|}{k-|X|}$ that are computable in polynomial time.

Unique Hitting Set - Problem Definition

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We will develop the kernelization technique and reduction rule in the upcoming slides.

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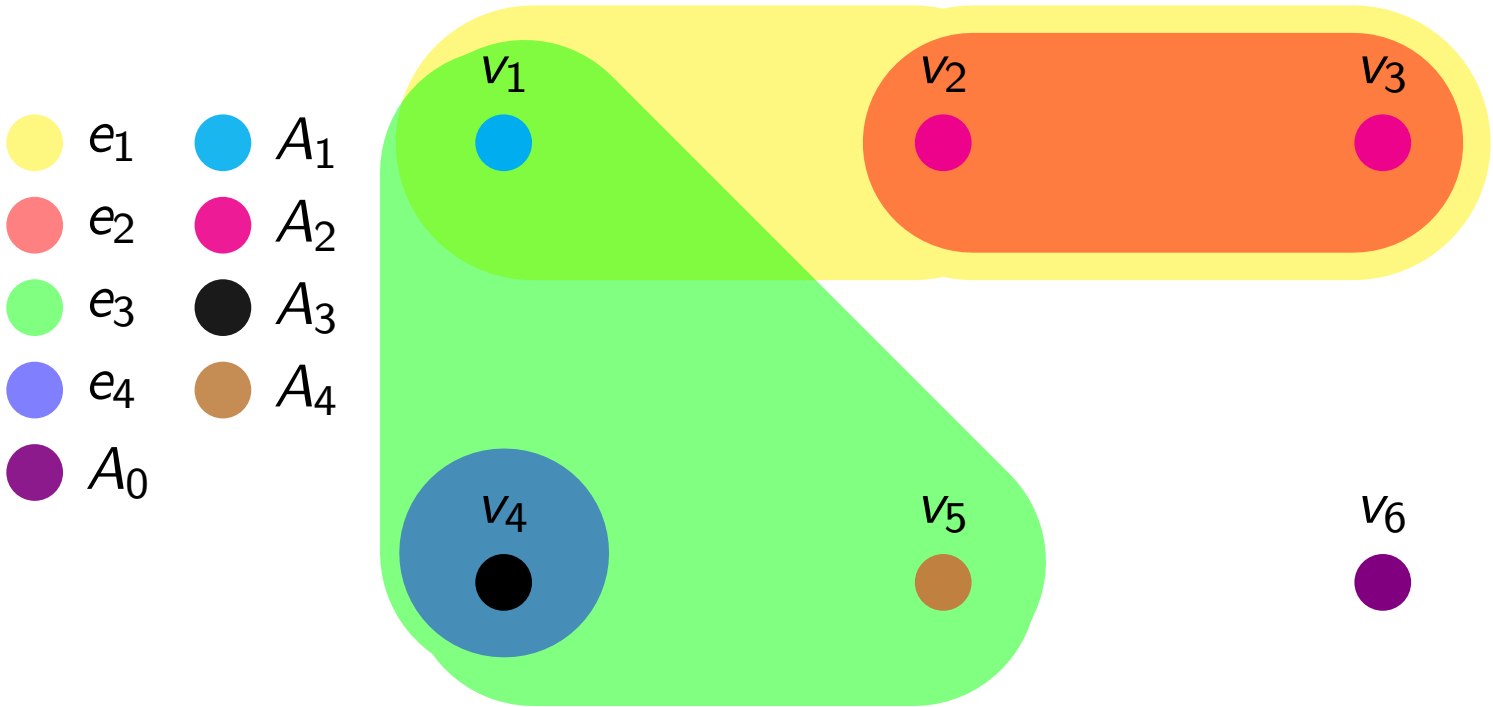

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2. Let $f : 2^V \to 2^{\mathcal{A}'}$ be an analogous function defined for sets of vertices, where $f(\{u_1, \ldots, u_b\}) := \{g(u_1), \ldots, g(u_b)\}$.

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We now claim that the hypergraph $\mathcal{H}' := (\mathcal{A}', \mathcal{E}')$ will be the kernel in our algorithm, where $\mathcal{E}' := \{f(e) : e \in E\}$.

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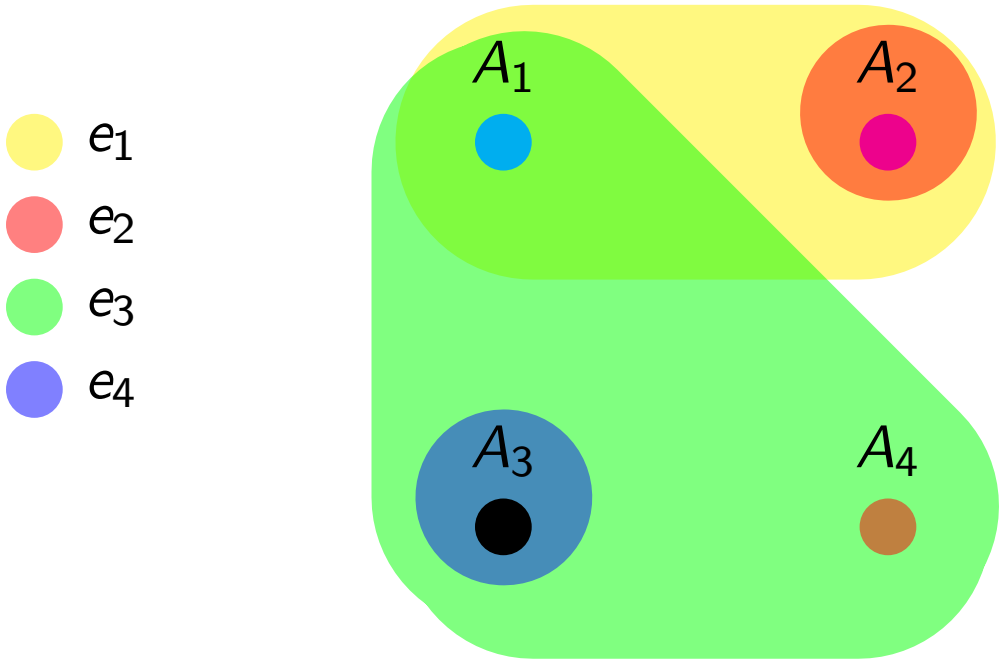

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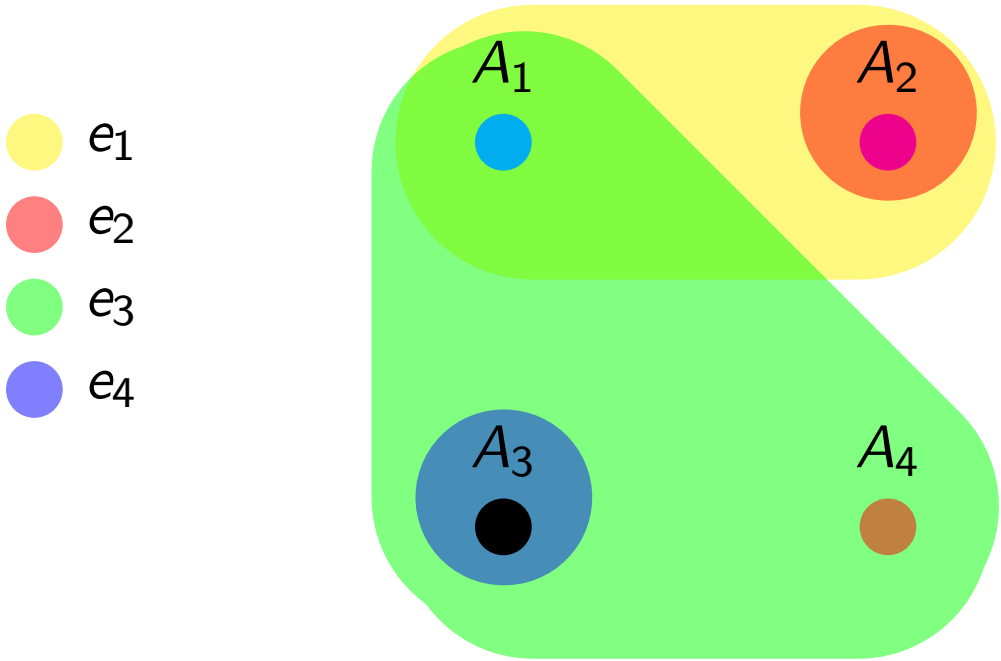

Lemma

Let $\mathcal{H}$, $\mathcal{H}'$ and $\mathcal{A}'$ be defined as above, and let $\mathcal{U}'$ be the set of all unique hitting sets in $\mathcal{H}'$. Then the number $\#uhs(\mathcal{H})$ of unique hitting sets in $\mathcal{H}$ satisfies:

$$\#uhs(\mathcal{H}) = \sum_{S' \in \mathcal{U}'} 2^{|A_0|} \prod_{A \in S'} |A|.$$

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- Following our framework of counting kernelizations, we can define $\Psi(\mathcal{H}) := \mathcal{H}'$ and $\mu(\mathcal{H}, \mathcal{H}', S') := 2^{|A_0|} \prod_{A \in S'} |A|$ for any unique hitting set S' of $\mathcal{H}'$.

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- Following our framework of counting kernelizations, we can define $\Psi(\mathcal{H}) := \mathcal{H}'$ and $\mu(\mathcal{H}, \mathcal{H}', S') := 2^{|A_0|} \prod_{A \in S'} |A|$ for any unique hitting set S' of $\mathcal{H}'$.
- We have that $|\mathcal{E}'| = |E| = k$ and $|\mathcal{A}'| \leq 2^k$, and so the size of $\mathcal{H}'$ is a function of k. Furthermore, we have that $\mathcal{H}'$ can be constructed in polynomial time, while μ can be computed in polynomial time.

1. Counting Kernelizations - Definitions and Concepts

2. Counting Kernelization Examples
   - The Vertex Cover Problem
   - The Hitting Set Problem
   - The Unique Hitting Set Problem

3. Conclusions and Future Work

Concluding Remarks

We have developed a formal framework for the kernelization of counting problems, and have applied this notion to develop parameterized algorithms for three different counting problems on graphs.

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We have also related the concept of counting kernelization to the class of fixed parameter tractable counting problems ($FFPT$).

Concluding Remarks

We have developed a formal framework for the kernelization of counting problems, and have applied this notion to develop parameterized algorithms for three different counting problems on graphs.
We have also related the concept of counting kernelization to the class of fixed parameter tractable counting problems ($FFPT$).
Lastly, we have shown that several well known kernelization techniques for decision problems can be adapted to solve their counting counterpart, such as the Crown Rule Decomposition and the Sunflower Kernel, all of which provide a solid structural framework for the development of other parameterized algorithms.

Future Work

In terms of future work, it could be worth investigating if other well known kernelization techniques can be adapted to counting problems as well, such as the Buss Kernelization for vertex cover.

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How can kernelization be combined with other techniques seen in class to solve counting problems? For example, does it make sense to use color-coding or branching with kernelization to help obtain even smaller kernels?
There are still many problems whose counting counterparts are not known to be in $FFPT$ or in $\#W[1]$-hard.

- p-DisjointTriangles
- p-(n-k)-Coloring

References

Marc Thurley (2007)

Kernelizations for Parameterized Counting Problems

TAMC, volume 4484 of Lecture Notes in Computer Science, 703 – 714.

Thank You!