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A trust region and affine scaling interior point method for nonconvex minimization with linear inequality constraints**

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Abstract. A trust region and affine scaling interior point method (TRAM) is proposed for a general nonlinear minimization with linear inequality constraints [8]. In the proposed approach, a Newton step is derived from the complementarity conditions. Based on this Newton step, a trust region subproblem is formed, and the original objective function is monotonically decreased. Explicit sufficient decrease conditions are proposed for satisfying the first order and second order necessary conditions.

The objective of this paper is to establish global and local convergence properties of the proposed trust region and affine scaling interior point method. It is shown that the proposed explicit decrease conditions are sufficient for satisfy complementarity, dual feasibility and second order necessary conditions respectively. It is also established that a trust region solution is asymptotically in the interior of the proposed trust region subproblem and a properly damped trust region step can achieve quadratic convergence.

Key words. trust region – interior point method – Dikin-affine scaling – Newton step

1. Introduction

For many nonlinear programming problems, the number of function and derivative evaluations is often regarded as the main computational cost indicator. This cost can greatly surpass that of the linear algebra work required by the optimization procedure. Hence it can be desirable to have an algorithm for a nonlinear programming problem which requires as few evaluations as possible. For convex programming problems, interior point methods have proven to be an efficient approach; see [18] for a comprehensive bibliography on these methods. Using these methods, a small number of iterations is typically required to solve a large problem [1,4,26]. The quest for similarly successful interior point algorithms for nonconvex programming problems has become increasingly important [3,7,9,10,13,15,20,27].

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Negative curvature in a nonconvex programming problem implies that there can be many local minimizers: typically, a computational method is able to compute one of them. Assume that an initial feasible point x_0 is available. An algorithm for which the original objective function is monotonically decreased can be desirable in the nonconvex minimization context. For example, assuming an initial feasible point is given, one would expect an algorithm to yield a local minimizer with smaller objective function value than that of the initial point. It is not clear how a non-monotone algorithm can achieve this. The majority of interior point methods, e.g., a path following (see, e.g., [16]) or a potential function reduction method (see, e.g., [24]), do not have this monotonicity property; this does not pose a problem for convex programming problems. Despite lack of polynomial convergence properties, an affine scaling approach, e.g., [1,4,12, 26], is the only interior point strategy which approaches a solution by monotonically decreasing the original objective function.

For various structured problems, affine scaling Newton methods have been proposed [2,5–7,10,19,20]. Using these methods, a sequence of interior points $\{x_k\}$, with the objective function values monotonically decreasing, are generated to converge quadratically to a solution. To extend this approach to a general nonlinear programming problem, we consider the problem of minimizing a nonlinear (nonconvex) function subject to linear inequality constraints,

$$\begin{aligned} &\min_{x\in\Re^n} f(x) \\ \text{subject to} \quad &Ax \geq b, \end{aligned} \tag{1}$$

where $A^T = [a_1, \cdots, a_m] \in \Re^{n\times m}$. In this paper, $\mathcal{F} \overset{\text{def}}{=} \{x : Ax \geq b\}$ denotes the feasible region. Moreover, it is assumed that the interior of the feasible region $int(\mathcal{F}) \overset{\text{def}}{=} \{x : Ax > b\}$ is not empty, $f(x)$ is at least continuously differentiable in $\mathcal{F}$ and twice continuously differentiable if second order convergence is considered. Moreover, a strictly feasible initial point $x_0 \in int(\mathcal{F})$ is assumed to be given.

In [8] a trust region and affine scaling interior point method (TRAM) is proposed for solving (1). Given the current interior point x_k, an improved strictly feasible iterate $x_{k+1} \in int(\mathcal{F})$ with $f(x_{k+1}) < f(x_k)$ is generated by solving a trust region subproblem with a 2-norm trust region measure. Asymptotically, solutions of the trust region subproblems generate approximate affine scaling Newton steps for the complementarity conditions. Preliminary computational results are given in [8].

The main objective of this article is to analyze global and local convergence properties of the proposed method [8]. In §2, the TRAM method, and its explicit decrease conditions are described. The global convergence properties are established in §3 and local convergence properties in §4. Concluding remarks are given in §5.

2. The proposed TRAM

The idea behind a trust region method for unconstrained minimization is intuitive and simple. In a neighborhood of the current point x_k, a quadratic function is minimized over a region of trust to yield a sufficient decrease of the original objective function. The size of the trust region is updated according to the agreement of the objective function with its approximation.

Constraints make it difficult to formulate a similar subproblem for which a global solution can be computed by existing software. In the proposed TRAM [8], constraints in the original minimization do not appear in the subproblem explicitly; difficulties imposed by constraints are overcome using affine scaling to formulate an appropriate quadratic function and trust region. A trust region mechanism with a 2-norm trust region measure is used. We briefly motivate and describe TRAM next.

Let x_k be the current strictly feasible iterate and λ_k be an approximation to the Lagrangian multipliers. Let

$$D(x) \stackrel{\text{def}}{=} \text{diag}(Ax - b), \quad \text{and} \quad D_k \stackrel{\text{def}}{=} D(x_k). \tag{2}$$

Assume $\psi_k(d)$ denotes the quadratic approximation to $f(x)$ at x_k, i.e.,

$$\psi_k(d) \stackrel{\text{def}}{=} \nabla f_k^T d + \frac{1}{2} d^T B_k d, \tag{3}$$

where B_k is symmetric and approximates the Hessian $\nabla^2 f_k$ of $f(x)$ at x_k.

The trust region subproblem proposed in [8] can be derived from the Newton step for the first order necessary conditions of (1). Ignoring primal and dual feasibility constraints, the first order necessary conditions of (1) can be expressed as an $(m+n)$-by-$(m+n)$ system of nonlinear equations (see, e.g., [14]),

$$\text{diag}(Ax - b)\lambda = 0 \quad \text{and} \quad \nabla f - A^T\lambda = 0. \tag{4}$$

For any $x \in \Re^n$, $y \in \Re^m$, $(x; y)$ denotes the vector in $\Re^{n+m}$ with the first n components equal to x and the last m components equal to y. The Newton step $(\Delta x_k; \Delta\lambda_k)$ for (4) satisfies

$$\begin{bmatrix} \nabla^2 f_k & -A^T \\ \text{diag}(\lambda_k)A & D_k \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta\lambda_k \end{bmatrix} = -\begin{bmatrix} \nabla f_k - A^T\lambda_k \\ D_k\lambda_k \end{bmatrix}.$$

The Newton step Δx_k may not be a descent direction for $f(x)$ when far away from a solution. To globalize, we generate a modified Newton step by replacing $\text{diag}(\lambda_k)$ by $C_k \stackrel{\text{def}}{=} \text{diag}(|\lambda_k|)$; this modified Newton step is subsequently denoted by $(p_k^N; \Delta\lambda_k^N)$ with $\lambda_{k+1}^N \stackrel{\text{def}}{=} \lambda_k + \Delta\lambda_k^N$, i.e.,

$$\begin{bmatrix} \nabla^2 f_k & -A^T \\ C_k A & D_k \end{bmatrix} \begin{bmatrix} p_k^N \\ \Delta\lambda_k^N \end{bmatrix} = -\begin{bmatrix} \nabla f_k - A^T\lambda_k \\ D_k\lambda_k \end{bmatrix}. \tag{5}$$

The modified Newton step can be shown to sufficiently approximate the exact Newton step, asymptotically, to achieve fast convergence. Moreover, the modified Newton step p_k^N is a minimizer of the augmented quadratic $\nabla f_k^T d + \frac{1}{2} d^T \nabla^2 f_k d + \frac{1}{2} d^T A^T D_k^{-1} C_k A d$, which can be considered as a quadratic convex regularization of the constrained problem (1) at x_k. Using the augmented quadratic as the objective function, a trust region subproblem consistent with the modified Newton step p_k^N is

$$\begin{array}{ll} & \min_{d \in \Re^n} \nabla f_k^T d + \frac{1}{2} d^T \nabla^2 f_k d + \frac{1}{2} d^T A^T D_k^{-1} C_k A d \\ \text{subject to} & \left\| \left(d; D_k^{-\frac{1}{2}} A d \right) \right\|_2 \leq \Delta_k. \end{array} \quad (6)$$

The decrease of the quadratic approximation of the objective function $f(x)$ at x_k, $\nabla f_k^T d + \frac{1}{2} d^T \nabla^2 f_k d$, is always less than the decrease of the objective function of (7) since C_k is positive semidefinite. Note also that D_k depends on the primal variable x_k and C_k depends on the dual variable λ_k. Define the transformation $\hat{d} \stackrel{\text{def}}{=} D_k^{-\frac{1}{2}} A d$. When $\|(p_k^N; \hat{p}_k^N)\|_2 < \Delta_k$, it can be easily verified that $(p_k^N; \hat{p}_k^N)$ is the Newton step of the following trust region subproblem

$$\begin{array}{ll} & \min_{d \in \Re^n, \hat{d} \in \Re^m} \nabla f_k^T d + \frac{1}{2} d^T \nabla^2 f_k d + \frac{1}{2} \hat{d}^T C_k \hat{d} \\ \text{subject to} & A d - D_k^{\frac{1}{2}} \hat{d} = 0 \\ & \|(d; \hat{d})\|_2 \leq \Delta_k. \end{array} \quad (7)$$

If (1) is actually a simple bound constrained problem, $\min_{l \leq x \leq u} f(x)$, the modified Newton step is the same as that defined in [7] with the affine scaling matrix equaling $\text{diag}(\min(x_k - l, u - x_k))$. Let $\hat{H}_k$ denote the Hessian of the objective function in (7), i.e.,

$$\hat{H}_k \stackrel{\text{def}}{=} \begin{bmatrix} \nabla^2 f_k & 0 \\ 0 & C_k \end{bmatrix}.$$

Let $(p_k; \hat{p}_k)$ denote a solution to (7) and the columns of Z_k denote an orthonormal basis for the null space of $[A, -D_k^{\frac{1}{2}}]$. The reduced Hessian of the subproblem (7) is $Z_k^T \hat{H}_k Z_K$. The first order necessary conditions of (7) imply that there exists a parameter $\nu_k \geq 0$ such that

$$(\hat{H}_k + \nu_k I)(p_k; \hat{p}_k) = -\begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix} + \begin{bmatrix} A^T \\ -D_k^{\frac{1}{2}} \end{bmatrix} \lambda_{k+1}^p, \quad (8)$$

with $\nu_k(\Delta_k - \|(p_k; \hat{p}_k)\|_2) = 0$. Clearly, $\lambda_{k+1}^p = \lambda_{k+1}^N = \lambda_k + \Delta\lambda_k^N$ when $\nu_k = 0$. Using the second order necessary conditions of (7), the projected Hessian $Z_k^T(\hat{H}_k + \nu_k I)Z_k$ is positive semi-definite, i.e.,

$$Z_k^T(\hat{H}_k + \nu_k I)Z_k = R_k^T R_k, \quad (9)$$

where R_k is an upper triangular matrix.

Stepsize choice is important for computational and theoretical convergence behaviors of an affine scaling method for linear programming [21,25]. One would like to take as large a step as possible for optimal decrease but this may bring the iterates close to the boundary prematurely. Nonlinearity of the problem (1) can both alleviate and exacerbate this problem. On the one hand, the iterates may approach a boundary more slowly due to nonlinearity of $f(x)$. On the other hand, once close to the boundary, nonlinearity of $f(x)$ may make it harder to move away. Two strategies are proposed in [8] to overcome this difficulty. Firstly, a simple reflection technique can be used which facilitates departure from the boundary. Secondly, when the Lagrangian multiplier approximation is sufficiently accurate and suggests that iterates are approaching a hyperplane $a_{j0}^T x - b_{j0} = 0$, which should not be binding at a solution, a perturbed diagonal scaling $\tilde{D}_k \stackrel{\text{def}}{=} \tilde{D}(x_k)$ is used. For example, in this situation we use

$$(\tilde{D}(x))_{ii} \stackrel{\text{def}}{=} \begin{cases} (D(x))_{ii} & \text{if } i \neq j_0, \\ 1 & \text{otherwise,} \end{cases} \tag{10}$$

to facilitate iterates to move away from the constraint j_0. Geometrically, $\tilde{D}_k$ changes the shape of the trust region so that it is elongated along the normal a_{j0} of the j_0th constraint. This elongation encourages a step away from the nearly binding constraint j_0. Let $\lambda(x)$ be a least squares approximation to the Lagrangian multipliers of (1) at x. Various techniques for identifying j_0 are possible; theoretical convergence requires only that, whenever iterates are converging to a point satisfying complementarity with some $(\lambda(x))_{j0} < 0$, the iterates leave the hyperplane $a_{j0}^T x - b_{j0} = 0$ eventually. For example, we identify j_0 as follows,

$$(\lambda(x))_{j_0} = \min\{(\lambda(x))_i \ : \ |a_i^T x - b_i| < -(\lambda(x))_i\}. \tag{11}$$

Note that $\tilde{D}$ depends on both x and λ. If $(\lambda(x))_i < 0$, then when the iterate x_k is sufficiently close to the hyperplane $a_i^T x - b_i = 0$, perturbation $\mathcal{D} \neq D$. When there is no i with $|a_i^T x - b_i| < -(\lambda(x))_i$, it is assumed that $j_0 = 0$ and $D(x) = \tilde{D}(x)$. In particular, $D(x) = \tilde{D}(x)$ if $\lambda(x) \geq 0$.

In general a perturbed scaling $\mathcal{D}$ only needs to be considered when the Lagrangian multiplier approximation is sufficiently accurate, e.g., near points satisfying complementarity. For simple bound constrained minimization, $\nabla f(x)$ directly provides local information about which bound the iterates should not approach; $\nabla f(x)$ converges to the Lagrangian multipliers asymptotically. A scaled steepest descent direction with scaling perturbation for all components whose corresponding Lagrangian multiplier approximation have the wrong sign can be used to move away from all incorrect bounds simultaneously. For the inequality constrained problem (1), perturbation $\tilde{D}$ differs from D in at most one component. This is one of the main differences between the proposed method here for the inequality constrained problem and that of [7] for simple bound constrained minimization.

We denote the trust region subproblem using either affine scaling D_k or $\tilde{D}_k$ as,

$$\begin{aligned} &\min_{d \in \Re^n} \psi_k(d) + \frac{1}{2} d^T A^T S_k^{-1} C_k A d \\ &\text{subject to} \quad \left\| \left(d; S_k^{-\frac{1}{2}} A d \right) \right\|_2 \leq \Delta_k, \end{aligned} \tag{12}$$

where S_k is either D_k or $\mathcal{D}_k$; typically S_k equals D_k. The diagonal scaling $D_k^{-\frac{1}{2}}$ in the 2-norm trust region bound constraint serves a purpose similar to the Dikin affine scaling [12] for a linear programming problem. The classical Dikin affine scaling uses D_k^{-1} rather than $D_k^{-\frac{1}{2}}$. The scaling matrix S_k in the trust region subproblem (12) is typically D_k while $\tilde{D}_k$ is occasionally used when it is necessary to change the shape of the trust region to encourage departure from a nearly binding constraint.

An approximate trust region solution needs to be damped in order to maintain strict feasibility. Assume $0 < \theta_0 < 1$. Let $\theta_k \in [\theta_0, 1)$ be a damping parameter. Let d_k be any descent direction for the objective function of the trust region subproblem (12). The damped step s_k along d_k is defined as:

$$s_k \stackrel{\text{def}}{=} \alpha_k d_k, \quad \alpha_k \stackrel{\text{def}}{=} \theta_k \alpha_k^*, \tag{13}$$

where α_k^* is the minimizer of the augmented quadratic objective along d_k within the *feasible* trust region, i.e.,

$$\begin{aligned} &\min_{\alpha \geq 0} \ \psi_k(\alpha d_k) + \frac{\alpha^2}{2} d_k^T A^T S_k^{-1} C_k A d_k \\ \text{subject to} \quad &\left\| \alpha \left(d_k; S_k^{-\frac{1}{2}} A d_k \right) \right\|_2 \leq \Delta_k \\ &x_k + \alpha d_k \in \mathcal{F}. \end{aligned} \tag{14}$$

If d_k equals a solution p_k to the trust region subproblem (12), then $\alpha_k^* = \min(1, \beta_k)$ where β_k is the stepsize to the boundary of $\mathcal{F}$ along p_k.

The new iterate $x_{k+1} = x_k + s_k$, where s_k is defined in (13), can be computed. The trust region size Δ_k is subsequently adjusted based on the agreement between the original objective function and its approximation. This process is then repeated yielding the proposed method TRAM which is described in Fig. 1.

The quadratic objective in the subproblem (12) is an approximation to the original objective function augmented by a quadratic term adding positive curvature in the space spanned by the constraint normals. This is similar to an augmented Lagrangian function for a constrained minimization. In §3, it is shown rigorously that the presence of the affine scaling in both the objective function and the trust region bound constraint ensures the damped step of the trust region solution to yield a sufficient decrease for convergence to a solution.

Explicit sufficient decrease conditions for achieving optimality are useful in computing approximate trust region solutions. In unconstrained minimization, sufficient decrease is measured against reductions of the quadratic objective along the gradient direction and the solution to the trust region subproblem. For the constrained nonlinear minimization problem (1), similar explicit decrease conditions can be formulated based on the trust region subproblem (12).

A sufficient decrease condition for satisfying complementarity (4) can be derived from the projected gradient direction $g_k \stackrel{\text{def}}{=} g(x_k)$ of the trust region subproblem (12)

TRAM Let $0 < \mu < \eta < 1$ and $x_0 \in int(\mathcal{F})$.

Step 1 Evaluate f_k, ∇f_k and $B_k \approx \nabla^2 f_k$; compute a least squares Lagrangian multiplier approximation λ_k and let $C_k = \mathrm{diag}(|\lambda_k|)$.
Step 2 Compute a step s_k, $x_k + s_k \in int(\mathcal{F})$, based on the trust region subproblem

$$\min_{d \in \Re^n} \psi_k(d) + \frac{1}{2} d^T A^T S_k^{-1} C_k A d$$

$$\text{subject to} \quad \left\| \left[d; S_k^{-\frac{1}{2}} A d \right] \right\|_2 \leq \Delta_k.$$

Step 3 Compute

$$\rho_k = \frac{f(x_k + s_k) - f(x_k)}{\psi(s_k)}.$$

Step 4 If $\rho_k > \mu$ then set $x_{k+1} = x_k + s_k$. Otherwise set $x_{k+1} = x_k$. Update Δ_k as specified.

Updating trust region size Δ_k
Let $0 < \gamma_1 < 1 < \gamma_2$ be given.

1. If $\rho_k \leq \mu$ then set $\Delta_{k+1} \in (0, \gamma_1 \Delta_k]$.
2. If $\rho_k \in (\mu, \eta)$ then set $\Delta_{k+1} \in [\gamma_1 \Delta_k, \Delta_k]$.
3. If $\rho_k \geq \eta$ then set $\Delta_{k+1} \in [\Delta_k, \gamma_2 \Delta_k]$.

Fig. 1. A trust region and affine scaling interior point method

with $S_k = D_k$, i.e.,

$$g_k \stackrel{\text{def}}{=} -(\nabla f_k - A^T \lambda_k), \quad \begin{bmatrix} A^T \\ -D_k^{\frac{1}{2}} \end{bmatrix} \lambda_k \stackrel{\text{LS}}{=} \begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix}. \tag{15}$$

Denote the damped step of g_k as $g_k^* = \alpha_k g_k$, where $\alpha_k = \theta_k \alpha_k^*$ and α_k^* is the stepsize defined by (14) with $d_k = g_k$. Let P_k denote the orthogonal projection onto the null space of $[A, -D_k^{\frac{1}{2}}]$. Then

$$\nabla f_k^T g_k = -\left\| P_k \begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix} \right\|_2^2 = -\left(\|\nabla f_k - A^T \lambda_k\|_2^2 + \left\| D_k^{\frac{1}{2}} \lambda_k \right\|_2^2 \right). \tag{16}$$

Due to the affine scaling $D_k^{-\frac{1}{2}}$, the direction g_k becomes increasingly tangential to any constraint i with $a_i^T x_k - b_i$ close to zero. In §3, we will prove that the stepsize $\{\alpha_k\}$ is sufficiently large to drive $\{\nabla f_k^T g_k\}$ to zero. Therefore, from (16), a sufficient decrease of $\psi_k(d)$ measured against the decrease from the damped minimizer g_k^* leads to satisfaction of complementarity:

$$\lim_{k \to \infty} \|\nabla f_k - A^T \lambda_k\|_2 = 0 \quad \text{and} \quad \lim_{k \to \infty} \left\| D_k^{\frac{1}{2}} \lambda_k \right\|_2 = 0.$$

Let $0 < \beta_{cs}, \beta_s < 1$ and $\|s_k\|_2 \leq \beta_s \Delta_k$. The proposed sufficient decrease condition for achieving complementarity is:

(AS.1) $\quad \psi_k(s_k) < \beta_{cs}\big(\psi_k(g_k^*) + \frac{1}{2} g_k^{*T} A^T D_k^{-1} C_k A g_k^*\big).$

Note that (AS.1) is satisfied if $s_k = g_k^*$. The result stating that condition (AS.1) is sufficient for achieving complementarity is rigorously established in §3.

In addition, in order to achieve dual feasibility, we consider the projected gradient $\tilde{g}_k$ corresponding to the trust region subproblem (12) with $S_k = \tilde{D}_k$: $\tilde{g}_k \stackrel{\text{def}}{=} \tilde{g}(x_k)$ is defined below,

$$\tilde{g}(x) \stackrel{\text{def}}{=} -(\nabla f(x) - A^T \tilde{\lambda}(x)), \quad \begin{bmatrix} A^T \\ -\tilde{D}(x)^{\frac{1}{2}} \end{bmatrix} \tilde{\lambda}(x) \stackrel{\text{LS}}{=} \begin{bmatrix} \nabla f(x) \\ 0 \end{bmatrix}. \tag{17}$$

To see how dual feasibility can be achieved using $\tilde{g}$, we demonstrate that the first order necessary conditions can be expressed as $\tilde{g}(x) = 0$, assuming $[A, -D(x)]$ has full row rank. If there exists λ such that $\nabla f(x) = A^T \lambda$, $D(x)\lambda = 0$ and $\lambda \geq 0$, clearly $\tilde{D}(x) = D(x)$ and the result holds. Assume now that there exists $\tilde{\lambda}$ satisfying the conditions

$$\tilde{D}(x)\tilde{\lambda} = 0 \quad \text{and} \quad \nabla f(x) - A^T \tilde{\lambda} = 0, \tag{18}$$

where $\tilde{D}(x)$ is defined by (10, 11). Then $\tilde{\lambda}_{j_0} = 0$ and

$$D(x)\tilde{\lambda} = 0 \quad \text{and} \quad \nabla f(x) - A^T \tilde{\lambda} = 0.$$

Since $[A, -D(x)]$ is assumed to have full rank, the least squares solution λ to (15) equals $\tilde{\lambda}$. From definition (11) of the index j_0 and $\lambda_{j_0} = 0$, $\lambda \geq 0$.

Let $\tilde{P}_k$ denote the orthogonal projection to the null space of $[A, -\tilde{D}_k^{\frac{1}{2}}]$. Similar to (16),

$$\nabla f_k^T \tilde{g}_k = -\left\| \tilde{P}_k \begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix} \right\|_2^2 = -\left(\| \nabla f_k - A^T \tilde{\lambda}_k \|_2^2 + \left\| \tilde{D}_k^{\frac{1}{2}} \tilde{\lambda}_k \right\|_2^2 \right). \tag{19}$$

Equation (19) suggests that a "good" decrease of the quadratic objective function in (12) along the projected gradient $\tilde{g}_k$ can lead to dual feasibility: $\lim_{k\to\infty} \|\nabla f_k - A^T \tilde{\lambda}_k\|_2 = 0$ and $\lim_{k\to\infty} \|\tilde{D}_k^{\frac{1}{2}} \tilde{\lambda}_k\|_2 = 0$. In §3, we establish that, assuming the complementarity conditions are satisfied asymptotically, the decrease from the damped step retains a sufficient portion to achieve dual feasibility. Denote the damped step of $\tilde{g}_k$ as $\tilde{g}_k^* = \alpha_k \tilde{g}_k$, where $\alpha_k = \theta_k \alpha_k^*$ and α_k^* is defined in (14) with $d_k = \tilde{g}_k$. Assume that $0 < \beta_{df} < 1$. This leads to the proposed condition (AS.2) below which can be used to ensure dual feasibility:

(AS.2) $\quad \psi_k(s_k) < \beta_{df}\big(\psi_k(\tilde{g}_k^*) + \frac{1}{2} \tilde{g}_k^{*T} A^T \tilde{D}_k^{-1} C_k A \tilde{g}_k^*\big).$

Note that (AS.2) is satisfied if $s_k = \tilde{g}_k^*$.

When $S_k = D_k$ and $B_k = \nabla^2 f_k$ in the trust region subproblem (12), the solutions to this subproblem become increasingly accurate approximations to the Newton steps for the complementarity condition (4) when $\{x_k\}$ approaches to a point satisfying the first order optimality. Intuitively, sufficient decrease of the objective function for the second order convergence can be measured against the decrease generated by the solution p_k to the trust region subproblem (7). Assume that $0 < \beta_q < 1$, $S_k = D_k$, and $B_k = \nabla^2 f_k$ in the trust region subproblem (12). Let p_k^* denote the damped step $p_k^* = \alpha_k p_k$ where $\alpha_k = \theta_k \alpha_k^*$ and α_k^* is defined in (14) with $d_k = p_k$. The proposed condition (AS.3) below is a reasonable sufficient decrease requirement for second order necessary conditions and fast local convergence:

(AS.3) $\psi_k(s_k) + \frac{1}{2} s_k^T A^T D_k^{-1} C_k A s_k < \beta_q \big(\psi_k(p_k^*) + \frac{1}{2} p_k^{*T} A^T D_k^{-1} C_k A p_k^*\big)$, $B_k = \nabla^2 f_k$.

Clearly (AS.3) is satisfied with $s_k = p_k^*$.

Conditions (AS.1), (AS.2) and (AS.3) are proposed as sufficient decrease criteria for satisfying complementarity, dual feasibility and second order necessary conditions respectively. We establish next that, under these conditions, desirable convergence properties can indeed be achieved.

3. Global convergence

We now establish that conditions (AS.1), (AS.2) and (AS.3) are sufficient for the proposed TRAM to achieve complementarity, dual feasibility, and second order necessary conditions respectively. Equation (16) suggests that complementarity can be satisfied if decrease of the quadratic approximation $\psi_k(s_k)$ is comparable to the minimum of the objective of the trust region subproblem (12) along g_k. Therefore, condition (AS.1) is sufficient if the decrease of the damped step g_k^* retains a significant portion of the decrease from the exact minimizer along g_k. Similar remarks are applicable to $\tilde{g}_k^*$ for dual feasibility. An important component of the global convergence analysis is therefore to establish that, due to affine scaling, the *posterior* damping for feasibility defined by (13) does not prohibit sufficient decrease of the quadratic objective function of the trust region subproblem.

We subsequently make the typical assumption on the compactness of the level set and full rank of the constraint matrix, i.e.,

(AS.0) Given an initial point $x_0 \in int(\mathcal{F})$, it is assumed that the level set $\mathcal{L}$ is compact, where $\mathcal{L} \stackrel{\text{def}}{=} \{x : x \in \mathcal{F} \text{ and } f(x) \le f(x_0)\}$. Moreover, $[A, -D(x)^{\frac{1}{2}}]$ is assumed to have full row rank for all $x \in \mathcal{L}$.

If each diagonal component of $D(x)$ is positive, then the matrix $[A, -D(x)^{\frac{1}{2}}]$ clearly has full row rank. It can be easily verified that the matrix $[A, -D(x)^{\frac{1}{2}}]$ has full row rank if and only if the vectors $\{a_i : a_i^T x - b_i = 0\}$ are linearly independent. In addition $[A, -\tilde{D}(x)^{\frac{1}{2}}]$ has full row rank if $[A, -D(x)^{\frac{1}{2}}]$ has full row rank.

Under the assumption (AS.0), λ_k and $\tilde{\lambda}_k$ can be computed via the normal equations of (15) and (17), i.e.,

$$AA^T \lambda_k + D_k \lambda_k = A \nabla f_k, \tag{20}$$

and

$$AA^T\tilde{\lambda}_k + \tilde{D}_k\tilde{\lambda}_k = A\nabla f_k. \tag{21}$$

Equation (20) and (21) give the relations below which are used in the subsequent analysis:

$$Ag_k = -D_k\lambda_k \quad \text{and} \quad A\tilde{g}_k = -\tilde{D}_k\tilde{\lambda}_k. \tag{22}$$

Lemma 1. *Assume that (AS.0) holds and $f(x) : \mathcal{F} \to \Re$ is continuously differentiable. Let $D(x)$, $\tilde{D}(x)$, $g(x)$, $\tilde{g}(x)$, $\lambda(x)$ and $\tilde{\lambda}(x)$ be defined as in (2), (10), (15), and (17). Then*

(a) $g(x)$ and $\lambda(x)$ are continuous in $\mathcal{L}$;
(b) $\tilde{g}(x)$ and $\tilde{\lambda}(x)$ are bounded in $\mathcal{L}$;
(c) $D(x)^{-\frac{1}{2}}Ag(x)$ and $\tilde{D}(x)^{-\frac{1}{2}}A\tilde{g}(x)$ are bounded in $\mathcal{L}$.

Proof. From (AS.0), $[A, -D(x)^{\frac{1}{2}}]$ has full row rank for any x in $\mathcal{L}$. Then $[A, -\tilde{D}(x)^{\frac{1}{2}}]$ has full row rank for any x in $\mathcal{L}$. Therefore, $(AA^T + D(x))$ and $(AA^T + \tilde{D}(x))$ are nonsingular for any x in $\mathcal{L}$. Hence $g(x)$, $\lambda(x)$, $\tilde{g}(x)$ and $\tilde{\lambda}(x)$ are defined for all x in $\mathcal{L}$.

From definition (15) and (17), $g(x) = -(\nabla f(x) - A^T\lambda(x))$ and $\tilde{g}(x) = -(\nabla f(x) - A^T\tilde{\lambda}(x))$, where $\lambda(x) = (AA^T + D(x))^{-1}A\nabla f(x)$ and $\tilde{\lambda}(x) = (AA^T + \tilde{D}(x))^{-1}A\nabla f(x)$. Since $\nabla f(x)$ and $D(x)$ are continuous in $\mathcal{F}$ and $AA^T + D(x)$ is nonsingular in $\mathcal{L}$, $g(x)$ and $\lambda(x)$ are continuous in $\mathcal{L}$. From compactness of $\mathcal{L}$, $g(x)$ and $\lambda(x)$ are bounded in $\mathcal{L}$. Similarly, applying continuity of $\nabla f(x)$ and nonsingularity of $AA^T + \tilde{D}(x)$ in the compact set $\mathcal{L}$, $\tilde{g}(x)$ and $\tilde{\lambda}(x)$ are bounded in $\mathcal{L}$. Using (22), $D(x)^{-\frac{1}{2}}Ag(x)$ and $\tilde{D}(x)^{-\frac{1}{2}}A\tilde{g}(x)$ are bounded in $\mathcal{L}$. The proof is completed.

□

Lemma 2 below further describes the relationship between complementarity and the projected gradient g_k, and dual feasibility and $\tilde{g}_k$.

Lemma 2. *Assume that (AS.0) holds, $\{B_k\}$ is bounded, and $f(x)$: $\mathcal{F} \to \Re$ is continuously differentiable. Moreover assume that there exists λ_* such that $\nabla f_* = A^T\lambda_*$ and $D_*\lambda_* = 0$ at $x_* \in \mathcal{L}$. Then for any $\{x_k\}$ in $\mathcal{L}$ which converges to x_*, $\{\nabla f_k^T g_k\}$ converges to zero. Moreover, if there exists a sequence $\{x_k\}$ in $\mathcal{L}$ converging to x_* with $\{\nabla f_k^T \tilde{g}_k\}$ converging to zero, then $\nabla f_* = A^T\lambda_*$, $D_*\lambda_* = 0$, and $\lambda_* \geq 0$.*

Proof. From Lemma 1, $g(x)$ is continuous in $\mathcal{L}$. This implies that $\{\nabla f_k^T g_k\}$ converges to zero for any $\{x_k\}$ in $\mathcal{L}$ which converges to x_*.

Assume now that there exists a sequence $\{x_k\}$ in $\mathcal{L}$ converging to x_* with $\{\nabla f_k^T \tilde{g}_k\}$ converging to zero. From $\lim_{k\to\infty} \nabla f_k^T \tilde{g}_k = 0$, there exists $\tilde{\lambda}_*$ such that $\nabla f_* = A^T\tilde{\lambda}_*$ and $\tilde{D}_*\tilde{\lambda}_* = 0$. Then $\lim_{k\to\infty}(\tilde{\lambda}_k)_{j_0} = 0$ and

$$\lim_{k\to\infty} D_k\tilde{\lambda}_k = 0 \quad \text{and} \quad \lim_{k\to\infty} (\nabla f_k - A^T\tilde{\lambda}_k) = 0.$$

Since $[A, -D_*]$ is assumed to have full rank, $\lim_{k\to\infty} \tilde{\lambda}_k = \lambda_*$. From definition (11) of the index j_0 and $(\lambda_*)_{j_0} = 0$, $\lambda_* \geq 0$. The proof is completed.

□

Lemma 3 relates decrease required by (AS.1) and (AS.2) with complementarity and dual feasibility. Its proof is similar to that of Lemma (4.8) in [22].

Lemma 3. *Assume that $\{x_k\}$ is generated from TRAM, (AS.0) is satisfied, $f(x) : \mathcal{F} \to \Re$ is continuously differentiable, and $\{B_k\}$ is bounded. If (AS.1) is satisfied at the kth iteration, then there exist constants $\chi_\beta, \chi_\zeta, \chi_g > 0$ such that*

$$-\psi_k(s_k) \geq \frac{\beta_{cs}\theta_0^2}{2\chi_g}\left(-\nabla f_k^T g_k\right)\min\left\{\Delta_k, \frac{-(\nabla f_k^T g_k)}{\chi_\zeta \chi_g}, \chi_\beta\sqrt{\|g_k\|_2^2 + \left\|D_k^{\frac{1}{2}}\lambda_k\right\|_2^2}\right\}.$$

Let the index j_0 be as defined in (11). If (AS.2) is satisfied and $(\tilde{\lambda}_k)_{j_0} < 0$, then there exist $\chi_{\tilde{\beta}}, \chi_\zeta, \chi_{\tilde{g}} > 0$ such that

$$-\psi_k(s_k) \geq \frac{\beta_{df}\theta_0^2}{2\chi_{\tilde{g}}}\left(-\nabla f_k^T \tilde{g}_k\right)\min\left\{\Delta_k, \frac{-(\nabla f_k^T \tilde{g}_k)}{\chi_\zeta \chi_{\tilde{g}}}, \chi_{\tilde{\beta}}\sqrt{\|\tilde{g}_k\|_2^2 + \left\|\tilde{D}_k^{\frac{1}{2}}\tilde{\lambda}_k\right\|_2^2}\right\}.$$

Proof. Define $\phi(\tau) : \Re \to \Re$ by

$$\phi(\tau) = \psi_k(\tau d_k) + \frac{\tau^2}{2} d_k^T A^T D_k^{-1} C_k A d_k \quad \text{if} \quad d_k = \frac{g_k}{\sqrt{\|g_k\|_2^2 + \left\|D_k^{-\frac{1}{2}} A g_k\right\|_2^2}},$$

and

$$\phi(\tau) = \psi_k(\tau d_k) + \frac{\tau^2}{2} d_k^T A^T \tilde{D}_k^{-1} C_k A d_k \quad \text{if} \quad d_k = \frac{\tilde{g}_k}{\sqrt{\|\tilde{g}_k\|_2^2 + \left\|\tilde{D}_k^{-\frac{1}{2}} A \tilde{g}_k\right\|_2^2}}.$$

Hence $\phi(\tau) = \tau(\nabla f_k^T d_k) + \frac{1}{2}\tau^2 \zeta_k$, where

$$\zeta_k = d_k^T B_k d_k + d_k^T A^T D_k^{-1} C_k A d_k, \quad \text{if} \quad d_k = \frac{g_k}{\sqrt{\|g_k\|_2^2 + \left\|D_k^{-\frac{1}{2}} A g_k\right\|_2^2}},$$

and

$$\zeta_k = d_k^T B_k d_k + d_k^T A^T \tilde{D}_k^{-1} C_k A d_k, \quad \text{if} \quad d_k = \frac{\tilde{g}_k}{\sqrt{\|\tilde{g}_k\|_2^2 + \left\|\tilde{D}_k^{-\frac{1}{2}} A \tilde{g}_k\right\|_2^2}}.$$

From Lemma 1, $\{\lambda_k\}$ and $\{\tilde{\lambda}_k\}$ are bounded. By definition of d_k,

$$\left\|D_k^{-\frac{1}{2}} A d_k\right\|_2 \leq 1 \quad \text{if} \quad d_k = \frac{g_k}{\sqrt{\|g_k\|_2^2 + \left\|D_k^{-\frac{1}{2}} A g_k\right\|_2^2}},$$

and

$$\left\|\tilde{D}_k^{-\frac{1}{2}} A d_k\right\|_2 \leq 1 \quad \text{if} \quad d_k = \frac{\tilde{g}_k}{\sqrt{\|\tilde{g}_k\|_2^2 + \left\|\tilde{D}_k^{-\frac{1}{2}} A \tilde{g}_k\right\|_2^2}}.$$

In addition, $\{B_k\}$ is also bounded. Hence there exists $\chi_\zeta > 0$ such that $|\zeta_k| \le \chi_\zeta$. Let τ_k^* be the minimizer of $\phi(\tau)$ on $[0, \min\{\Delta_k, \beta_k\}]$ where β_k is the stepsize along d_k to the boundary of $\mathcal{F}$, i.e.,

$$\beta_k = \min\left\{-\frac{a_i^T x_k - b_i}{a_i^T d_k} : -\frac{a_i^T x_k - b_i}{a_i^T d_k} > 0\right\},$$

with $\beta_k = +\infty$ if $-\frac{a_i^T x_k - b_i}{a_i^T d_k} \le 0$ for all i. If $\tau_k^* \in [0, \min\{\Delta_k, \beta_k\})$, then $0 < \zeta_k \le \chi_\zeta$ and $\tau_k^* = -(\nabla f_k^T d_k)/\zeta_k$. From $0 < \theta_0 \le \theta_k < 1$ and $0 < \zeta_k \le \chi_\zeta$,

$$\phi(\theta_k \tau_k^*) = -\theta_k \frac{(\nabla f_k^T d_k)^2}{\zeta_k} + \frac{1}{2}\theta_k^2 \frac{(\nabla f_k^T d_k)^2}{\zeta_k} \le -\frac{\theta_0^2}{2}\frac{(\nabla f_k^T d_k)^2}{\chi_\zeta}. \tag{23}$$

Assume $\tau_k^* = \Delta_k$. Since $\zeta_k \Delta_k \le -(\nabla f_k^T d_k)$ when $\zeta_k > 0$, and $\phi(\theta_k \tau_k^*) \le \theta_0^2 \Delta_k (\nabla f_k^T d_k)$ otherwise,

$$\phi(\theta_k \tau_k^*) = \phi(\theta_k \Delta_k) \le \frac{\theta_0^2}{2}\Delta_k(\nabla f_k^T d_k). \tag{24}$$

Assume $\tau_k^* = \beta_k$. Since $\zeta_k \beta_k \le -(\nabla f_k^T d_k)$ when $\zeta_k > 0$, and $\phi(\theta_k \tau_k^*) \le \theta_0^2 \beta_k (\nabla f_k^T d_k)$ otherwise,

$$\phi(\theta_k \tau_k^*) = \phi(\theta_k \beta_k) \le \frac{\theta_0^2}{2}\beta_k(\nabla f_k^T d_k). \tag{25}$$

Considering all the three cases,

$$\phi(\theta_k \tau_k^*) \le \frac{\theta_0^2}{2}(\nabla f_k^T d_k) \min\left\{\Delta_k, \frac{-(\nabla f_k^T d_k)}{\chi_\zeta}, \beta_k\right\}. \tag{26}$$

Next we establish lower bounds on β_k for the two definitions of d_k respectively.

First consider $d_k = g_k/\sqrt{\|g_k\|_2^2 + \left\|D_k^{-\frac{1}{2}} A g_k\right\|_2^2}$. By definition, there exists some $1 \le j \le m$ such that

$$\beta_k \ge \left|\frac{a_j^T x_k - b_j}{a_j^T d_k}\right|.$$

From (22), boundedness of $\{\lambda_k\}$ and $\{x_k\}$, there exists $\chi_\beta > 0$ such that,

$$\beta_k \ge \left|\frac{a_j^T x_k - b_j}{a_j^T d_k}\right| = \left|\frac{\sqrt{\|g_k\|_2^2 + \left\|D_k^{-\frac{1}{2}} A g_k\right\|_2^2}}{(\lambda_k)_j}\right| \ge \chi_\beta \sqrt{\|g_k\|_2^2 + \left\|D_k^{\frac{1}{2}} \lambda_k\right\|_2^2}. \tag{27}$$

Consider the second case when $d_k = \tilde{g}_k/\sqrt{\|\tilde{g}_k\|_2^2 + \left\|\tilde{D}_k^{-\frac{1}{2}} A\tilde{g}_k\right\|_2^2}$. By assumption $(\tilde{\lambda}_k)_{j_0} < 0, a_{j_0}^T d_k > 0$ from (22). Hence there exists some $j \neq j_0$ such that

$$\beta_k \geq \left| \frac{a_j^T x_k - b_j}{a_j^T d_k} \right|.$$

From (22), $(\tilde{D}_k)_{ii} = (D_k)_{ii}$ when $i \neq j_0$, and the boundedness of $\{\tilde{\lambda}_k\}$, there exists $\chi_{\tilde{\beta}} > 0$ such that

$$\beta_k \geq \chi_{\tilde{\beta}} \sqrt{\|\tilde{g}_k\|_2^2 + \left\|\tilde{D}_k^{\frac{1}{2}} \tilde{\lambda}_k\right\|_2^2}. \tag{28}$$

Using Lemma 1 and (22), there exist $\chi_g, \chi_{\tilde{g}} > 0$ such that

$$\sqrt{\|g_k\|_2^2 + \left\|D_k^{-\frac{1}{2}} A g_k\right\|_2^2} \leq \chi_g \quad \text{and} \quad \sqrt{\|\tilde{g}_k\|_2^2 + \left\|\tilde{D}_k^{-\frac{1}{2}} A\tilde{g}_k\right\|_2^2} \leq \chi_{\tilde{g}}. \tag{29}$$

Since $\psi_k(s_k) \leq \beta_{cs}\phi(\theta_k \alpha_k^*)$ with $d_k = g_k/\sqrt{\|g_k\|_2^2 + \left\|D_k^{-\frac{1}{2}} A g_k\right\|_2^2}$ if (AS.1) is satisfied and $\psi_k(s_k) \leq \beta_{df}\phi(\theta_k \alpha_k^*)$ with $d_k = \tilde{g}_k/\sqrt{\|\tilde{g}_k\|_2^2 + \left\|\tilde{D}_k^{-\frac{1}{2}} A\tilde{g}_k\right\|_2^2}$, if (AS.2) is satisfied, the results follow immediately from (26), (27), (28), and (29).

□

Lemma 4 and 5 are auxillary results to prove the first order optimality.

Lemma 4. *Assume that (AS.0) holds, $f(x) : \mathcal{F} \to \Re$ is continuously differentiable, $\{B_k\}$ is bounded and $\{x_k\}$ is generated by TRAM. Then there cannot exist constants $\chi_0, \chi_1 > 0$ such that, for sufficiently large k,*

$$-\psi_k(s_k) \geq \frac{1}{2}\chi_0 \min\{\Delta_k, \chi_1\}.$$

Proof. The lemma is proved by contradiction. Assume that there exist constants $\chi_0, \chi_1 > 0$ such that, for sufficiently large k,

$$-\psi_k(s_k) \geq \frac{1}{2}\chi_0 \min\{\Delta_k, \chi_1\}. \tag{30}$$

We first prove that

$$\sum_{k=1}^{\infty} \Delta_k < +\infty. \tag{31}$$

If there are a finite number of successful iterations then $\Delta_{k+1} \leq \gamma_1 \Delta_k$ for all k sufficiently large. Hence

$$\sum_{k=1}^{\infty} \Delta_k < +\infty.$$

If there is an infinite sequence $\{k_i\}$ of successful iterations, using (30), for a successful step and sufficiently large k,

$$f(x_{k_i}) - f(x_{k_i+1}) \geq -\mu\psi_k(s_{k_i}) \geq \frac{1}{2}\mu\chi_0 \min\{\Delta_{k_i}, \chi_1\}. \tag{32}$$

Since $f(x_{k+1}) \leq f(x_k)$ for all k and $\{f(x_k)\}$ is bounded below,

$$\sum_{i=1}^{\infty} \Delta_{k_i} < +\infty.$$

The trust region updating rules of TRAM specify that $\Delta_{k+1} \leq \gamma_1\Delta_k$ for an unsuccessful iteration and $\Delta_{k+1} \leq \gamma_2\Delta_k$ for a successful iteration. Hence

$$\sum_{k=1}^{\infty} \Delta_k \leq \left(1 + \frac{\gamma_2}{1-\gamma_1}\right)\sum_{i=1}^{\infty} \Delta_{k_i}.$$

Again (31) holds.

From (31) and $\|x_{k+1} - x_k\|_2 = \|s_k\|_2 \leq \beta_s\Delta_k$, $\{x_k\}$ converges, and $\{s_k\}$ converges to zero. Since $\{B_k\}$ is bounded, there exists $\chi_B > 0$ such that $\|B_k\|_2 \leq \chi_B$. From continuity of $\nabla f(x)$, $f(x_k + s_k) - f(x_k) = \nabla f(x_k + \xi_k s_k)^T s_k$ with $0 \leq \xi_k \leq 1$. Hence

$$\begin{aligned} |f(x_k + s_k) - f(x_k) - \psi_k(s_k)| &\leq \left|\frac{1}{2}s_k^T B_k s_k\right| + \left|(\nabla f(x_k + \xi_k s_k) - \nabla f_k)^T s_k\right| \\ &\leq \frac{1}{2}\chi_B\beta_s^2\Delta_k^2 + \beta_s\Delta_k\|\nabla f(x_k + \xi_k s_k) - \nabla f_k\|_2. \end{aligned}$$

Since $\nabla f(x)$ is continuous in the compact level set $\mathcal{L}$ and $\{x_k\}$ converges, there exists a sequence $\{\epsilon_k\}$ converging to zero such that

$$|f(x_k + s_k) - f(x_k) - \psi_k(s_k)| \leq \epsilon_k\Delta_k.$$

By assumption, for sufficiently large k, $-\psi_k(s_k) \geq \frac{1}{2}\chi_0\Delta_k$. It is readily obtained that $\{|\rho_k - 1|\}$ converges to zero. Hence $\{\Delta_k\}$ cannot converge to zero, which is a contradiction to (31). The proof is completed.

□

Lemma 5. *Assume that (AS.0) holds and $f(x) : \mathcal{F} \to \Re$ is continuously differentiable. Assume further that* $\lim_{k\to\infty} \nabla f_k^T g_k = 0$ *and* $\liminf_{i\to\infty}(-\nabla f_{m_i}^T \tilde{g}_{m_i}) > 0$ *for a subsequence $\{m_i\}$. Then $(\tilde{\lambda}_{m_i})_{j_0} < 0$ for sufficiently large i.*

Proof. Using the first system of equations in (22),

$$\begin{aligned} \nabla f_k^T \tilde{g}_k &= (A^T\lambda_k)^T \tilde{g}_k + (\nabla f_k - A^T\lambda_k)^T \tilde{g}_k \\ &= -\lambda_k^T \tilde{D}_k \tilde{\lambda}_k + (\nabla f_k - A^T\lambda_k)^T \tilde{g}_k. \end{aligned}$$

From $\lim_{k\to\infty} \nabla f_k^T g_k = 0$ and (16), $\lim_{k\to\infty}(\nabla f_k - A^T\lambda_k) = 0$ and $\lim_{k\to\infty}(\lambda_k)_j(\tilde{D}_k)_{jj} = 0$ for all $j \neq j_0$. Since $\liminf_{i\to\infty}(-\nabla f_{m_i}^T \tilde{g}_{m_i}) > 0$, $(\lambda_{m_i})_{j_0}(\tilde{\lambda}_{m_i})_{j_0} > 0$ for sufficiently large i. From $\lim_{k\to\infty} \nabla f_k^T g_k = 0$ and $\liminf_{i\to\infty}(-\nabla f_{m_i}^T \tilde{g}_{m_i}) > 0$, $(\lambda_{m_i})_{j_0} < 0$ for sufficiently large i. This implies that $(\tilde{\lambda}_{m_i})_{j_0} < 0$ for sufficiently large i.

□

Noting that $a_{j0}^T \tilde{g}_k = -(\tilde{\lambda}_k)_{j_0}$, Lemma 5 indicates that, assuming that the complementarity conditions are satisfied asymptotically, the iterates will leave the hyperplane $a_{j0}^T x - b_{j_0} = 0$ and move into the feasible region if $\liminf_{k\to\infty}(\lambda_k)_{j_0} < 0$.

Lemma 6. *Assume that (AS.0) holds, $f(x) : \mathcal{F} \to \Re$ is continuously differentiable, and $\{B_k\}$ is bounded. Let $\{x_k\}$ be generated by TRAM.*

(a) If (AS.1) holds at every iteration then $\liminf_{k\to\infty} \nabla f_k^T g_k = 0$.
(b) If (AS.2) holds for sufficiently large k and $\lim_{k\to\infty} \nabla f_k^T g_k = 0$, *then* $\liminf_{k\to\infty} \nabla f_k^T \tilde{g}_k = 0$.

Proof. We prove each result by contradiction.

(a) Assume that there is $\epsilon > 0$ such that $-\nabla f_k^T g_k \geq \epsilon$ for all sufficiently large k. From (16), there exists $\bar{\epsilon} > 0$ such that, for sufficiently large k,

$$\|\nabla f_k - A^T \lambda_k\|_2^2 + \left\|D_k^{\frac{1}{2}} \lambda_k\right\|_2^2 \geq \bar{\epsilon}.$$

Since (AS.1) is satisfied, using Lemma 3 there exist $\chi_0, \chi_1 > 0$ such that, for sufficiently large k,

$$-\psi_k(s_k) \geq \frac{1}{2}\chi_0 \min\{\Delta_k, \chi_1\}.$$

But Lemma 4 indicates that this is not possible. Hence

$$\liminf_{k\to\infty} \nabla f_k^T g_k = 0.$$

(b) Assume now that (AS.2) holds for sufficiently large k and $\lim_{k\to\infty} \nabla f_k^T g_k = 0$ but $-\nabla f_k^T \tilde{g}_k \geq \epsilon$ for sufficiently large k. From (19), there exists $\bar{\epsilon} > 0$ such that, for sufficiently large k,

$$\|\nabla f_k - A^T \tilde{\lambda}_k\|_2^2 + \left\|\tilde{D}_k^{\frac{1}{2}} \tilde{\lambda}_k\right\|_2^2 \geq \bar{\epsilon}.$$

Lemma 3 and 5 apply and there exist $\chi_0, \chi_1 > 0$ such that, for sufficiently large k,

$$-\psi_k(s_k) \geq \frac{1}{2}\chi_0 \min\{\Delta_k, \chi_1\}.$$

Using Lemma 4, this is impossible. Hence

$$\liminf_{k\to\infty} \nabla f_k^T \tilde{g}_k = 0.$$

The proof is completed. □

The next theorem establishes that complementarity and dual feasibility can be satisfied at every limit point.

Theorem 1. *Assume that (AS.0) holds and $f(x)$: $\mathcal{F} \to \Re$ is continuously differentiable and $\{B_k\}$ is bounded. Let $\{x_k\}$ be generated by TRAM. Assume that (AS.1) holds at every iteration. Then*

(a) $\lim_{k\to\infty} \nabla f_k^T g_k = 0$, *i.e., the complementarity conditions are satisfied at every limit point of* $\{x_k\}$*;*
(b) If, in addition, (AS.2) is satisfied for sufficiently large k*, then* $\lim_{k\to\infty} \nabla f_k^T \tilde{g}_k = 0$. *Thus the first order necessary condition is satisfied at every limit point.*

Proof. The proof is again by contradiction. We consider each result in turn.

(a) Let ϵ_1 in $(0, 1)$ be given and assume that there is a sequence $\{m_i\}$ such that $\|\nabla f_{m_i}^T g_{m_i}\|_2 \geq \epsilon_1$. Lemma 6 guarantees that for any ϵ_2 in $(0, \epsilon_1)$ there is a subsequence of $\{m_i\}$ (without loss of generality we assume that it is the full sequence) and a subsequence $\{l_i\}$ such that

$$\left|\nabla f_k^T g_k\right| \geq \epsilon_1, \quad m_i \leq k < l_i, \quad \left|\nabla f_{l_i}^T g_{l_i}\right| < \epsilon_2. \tag{33}$$

Using (16),

$$\left\|\nabla f_k - A^T \lambda_k\right\|_2^2 + \left\|D_k^{\frac{1}{2}} \lambda_k\right\|_2^2 \geq \epsilon_1, \quad m_i \leq k < l_i,$$

where λ_k is defined in (15).
If the k-th iteration is successful, using Lemma 3 and noting that $g_k = \nabla f_k - A^T \lambda_k$,

$$f(x_k) - f(x_{k+1}) \geq \frac{\mu \epsilon_1 \beta_{cs} \theta_0^2}{2\chi_g} \min\left\{\Delta_k, \frac{\epsilon_1}{\chi_\varsigma \chi_g}, \chi_\beta \sqrt{\epsilon_1}\right\}, \quad m_i \leq k < l_i. \tag{34}$$

Since $f(x)$ is bounded below on $\mathcal{L}$ and $f(x_{k+1}) \leq f(x_k)$, $\{f(x_k)\}$ converges and $\{f(x_k) - f(x_{k+1})\}$ converges to zero. From $\|x_{k+1} - x_k\|_2 \leq \beta_s \Delta_k$, it follows that, for sufficiently large i,

$$f(x_k) - f(x_{k+1}) \geq \epsilon_3 \|x_{k+1} - x_k\|_2, \quad m_i \leq k < l_i, \tag{35}$$

where $\epsilon_3 = \frac{\mu \epsilon_1 \beta_{cs} \theta_0^2}{2\chi_g \beta_s}$. Consider a subsequence of $\{l_i\}$ such that $\{x_{l_i}\}$ converges to x_* and $\{\nabla f_{l_i}^T g_{l_i}\}$ converges to zero. Without loss of generality, the subsequence is still denoted by $\{l_i\}$. Using (35) and the triangle inequality,

$$f(x_{m_i}) - f(x_{k_i}) \geq \epsilon_3 \|x_{k_i} - x_{m_i}\|_2, \quad m_i \leq k_i \leq l_i.$$

Since $\{f(x_k)\}$ converges and $\{x_{l_i}\}$ converges to x_*, $\{x_{m_i}\}$ converges to x_*. From $\{\nabla f_{l_i}^T g_{l_i}\}$ converging to zero, $\nabla f_*^T g_* = 0$. Applying Lemma 2, $\{\nabla f_{m_i}^T g_{m_i}\}$ converges to zero. Hence $|\nabla f_{m_i}^T g_{m_i}| < \epsilon_2$ for sufficiently large i. This contradicts (33) since $\epsilon_2 < \epsilon_1$. Therefore $\{\nabla f_k^T g_k\}$ converges to zero and the complementarity conditions are satisfied at every limit point of $\{x_k\}$.

(b) From (a), $\lim_{k\to\infty} \nabla f_k^T g_k = 0$. Assume now that (AS.2) is satisfied for sufficiently large k. Using Lemma 6, $\liminf_{k\to\infty} \nabla f_k^T \tilde{g}_k = 0$. The proof for $\lim_{k\to\infty} \nabla f_k^T \tilde{g}_k = 0$ is very similar to the proof of (a). We include it here for completeness. Let ϵ_1 in $(0, 1)$ be given and assume that there is a sequence $\{m_i\}$ such that $\|\nabla f_{m_i}^T \tilde{g}_{m_i}\|_2 \geq \epsilon_1$. Lemma 6 guarantees that for any ϵ_2 in $(0, \epsilon_1)$ there is a subsequence of $\{m_i\}$ (without loss of generality we assume that it is the full sequence) and a subsequence $\{l_i\}$ such that

$$\left|\nabla f_k^T \tilde{g}_k\right| \geq \epsilon_1, \quad m_i \leq k < l_i, \quad \left|\nabla f_{l_i}^T \tilde{g}_{l_i}\right| < \epsilon_2. \tag{36}$$

Using (19), we have

$$\left\|\nabla f_k - A\tilde{\lambda}_k\right\|_2^2 + \left\|\tilde{D}_k^{\frac{1}{2}} \tilde{\lambda}_k\right\|_2^2 \geq \epsilon_1, \quad m_i \leq k < l_i,$$

where $\tilde{\lambda}_k$ is defined in (17).

If the k-th iteration is successful, using Lemma 5 and Lemma 3, and noting that $\tilde{g}_k = \nabla f_k - A\tilde{\lambda}_k$,

$$f(x_k) - f(x_{k+1}) \geq \frac{\mu\epsilon_1\beta_{df}\theta_0^2}{2\chi_{\tilde{g}}} \min\left\{\Delta_k, \frac{\epsilon_1}{\chi_\varsigma \chi_{\tilde{g}}}, \chi_\beta\sqrt{\epsilon_1}\right\}, \quad m_i \leq k < l_i. \tag{37}$$

Since $f(x)$ is bounded below on $\mathcal{L}$, $\{f(x_k)\}$ converges and $\{f(x_k) - f(x_{k+1})\}$ converges to zero. From $\|x_{k+1} - x_k\|_2 \leq \beta_s \Delta_k$, it follows that, for sufficiently large i,

$$f(x_k) - f(x_{k+1}) \geq \epsilon_3 \|x_{k+1} - x_k\|_2, \quad m_i \leq k < l_i, \tag{38}$$

where $\epsilon_3 = \frac{\mu\epsilon_1\beta_{df}\theta_0^2}{2\chi_{\tilde{g}}\beta_s}$.

Consider a subsequence of $\{l_i\}$ such that $\{x_{l_i}\}$ converges to x_* and $\{\nabla f_{l_i}^T \tilde{g}_{l_i}\}$ converges to zero. Without loss of generality, the subsequence is still denoted by $\{l_i\}$. Using (38) and the triangle inequality,

$$f(x_{m_i}) - f(x_{k_i}) \geq \epsilon_3 \|x_{k_i} - x_{m_i}\|_2, \quad m_i \leq k_i \leq l_i.$$

Since $\{f(x_k)\}$ converges and $\{x_{l_i}\}$ converges to x_*, $\{x_{m_i}\}$ converges to x_*. From $\{\nabla f_{l_i}^T \tilde{g}_{l_i}\}$ converging to zero, applying Lemma 2, there exists λ_* such that $\nabla f_* = A^T\lambda_*$, $D_*\lambda_* = 0$ and $\lambda_* \geq 0$. From definition (10) of $\tilde{D}$, $\lim_{i\to\infty} \tilde{D}_{m_i}\lambda_{m_i} = 0$ and $\lim_{i\to\infty}(\nabla f_{m_i} - A^T\lambda_{m_i}) = 0$. Since $[A, -\tilde{D}_*]$ has full row rank, $\lim_{i\to\infty} \tilde{\lambda}_{m_i} = \lambda_*$ and $\{\nabla f_{m_i}^T \tilde{g}_{m_i}\}$ converges to zero. Hence $|\nabla f_{m_i}^T \tilde{g}_{m_i}| < \epsilon_2$ for sufficiently large i. This contradicts (36) since $\epsilon_2 < \epsilon_1$. Therefore $\{\nabla f_k^T \tilde{g}_k\}$ converges to zero. Thus the first order necessary conditions are satisfied at every limit point.

The proof is completed.

□

The results of Theorem 1 imply that, under the stated assumptions, the first order necessary conditions can be satisfied. Before we investigate satisfaction of the second order necessary conditions, several technical lemmas are required. First, we quote Lemma (4.10) in [23] below.

Lemma 7. *Let x_* be an isolated limit point of a sequence $\{x_k\}$ in $\Re^n$. If $\{x_k\}$ does not converge then there is a subsequence $\{x_{l_j}\}$ converging to x_*, and an $\epsilon > 0$ such that $\|x_{l_j+1} - x_{l_j}\|_2 \geq \epsilon$.*

Consequences of (AS.3) are subsequently examined.

Lemma 8. *Assume that $f(x) : \mathcal{F} \to \Re$ is twice continuously differentiable. Let the columns of Z_k form an orthonormal basis for the null space of $[A, -D_k^{\frac{1}{2}}]$ and p_k be a solution to the trust region subproblem (12) with $S_k = D_k$ and $B_k = \nabla^2 f_k$. Let p_k^* denote the damped step $p_k^* = \alpha_k p_k$ where $\alpha_k = \theta_k \alpha_k^*$ and α_k^* is defined in (14) with $d_k = p_k$. Then*

$$-\left(\psi_k(p_k^*) + \frac{1}{2} p_k^{*T} A^T D_k^{-1} C_k A p_k^*\right) \geq \frac{\theta_0^2 \min\{1, \beta_k^2\}}{2} (\nu_k \Delta_k^2 + \|R_k Z_k^T (p_k; \hat{p}_k)\|_2^2),$$

where β_k is the stepsize to the boundary of $\mathcal{F}$ along p_k, $\nu_k \geq 0$ is defined by (8), and R_k is defined by (9).

Proof. Let $\phi(\alpha) = \psi_k(\alpha p_k) + \frac{\alpha^2}{2} p_k^T A^T D_k^{-1} C_k A p_k$ and $\alpha \in [0, \min\{1, \beta_k\}]$ where β_k is the stepsize along p_k to the boundary of $\mathcal{F}$. Let α_k^* be the minimizer of $\phi(\alpha)$ in $[0, \min\{1, \beta_k\}]$ as defined in (14). Since p_k solves (12), $\alpha_k^* = \min\{1, \beta_k\}$. In addition,

$$\begin{aligned}
\phi(\alpha) &= \alpha \nabla f_k^T p_k + \frac{1}{2} \alpha^2 (p_k; \hat{p}_k)^T \hat{H}_k (p_k; \hat{p}_k) \\
&= -\alpha (p_k; \hat{p}_k)^T (\hat{H}_k + \nu_k I)(p_k; \hat{p}_k) + \frac{1}{2} \alpha^2 (p_k; \hat{p}_k)^T \hat{H}_k (p_k; \hat{p}_k) \qquad \text{(from (8))} \\
&= -\alpha \|R_k Z_k^T (p_k; \hat{p}_k)\|_2^2 + \frac{1}{2} \alpha^2 \|R_k Z_k^T (p_k; \hat{p}_k)\|_2^2 - \frac{1}{2} \alpha \nu_k \Delta_k^2, \qquad \text{(from (9))}.
\end{aligned}$$

From $\alpha_k^{*2} \leq \alpha_k^*$, $p_k^* = \theta_k \alpha_k^* p_k$, and $0 < \theta_0 \leq \theta_k < 1$,

$$-\left(\psi_k(p_k^*) + \frac{1}{2} p_k^{*T} A^T D_k^{-1} C_k A p_k^*\right) \geq \frac{\theta_0^2 \min\left\{1, \beta_k^2\right\}}{2} (\nu_k \Delta_k^2 + \|R_k Z_k^T (p_k; \hat{p}_k)\|_2^2).$$

The proof is completed.

□

The following lemma provides lower bounds for the stepsize sequence $\{\beta_k\}$ along $\{p_k\}$. The results in Lemma 9 hold for any subsequence generated by TRAM (consequently, it holds for the entire sequence as well).

Lemma 9. *Assume that (AS.0) holds, $f(x) : \mathcal{F} \to \Re$ is twice continuously differentiable. Let p_k be a solution to the trust region subproblem (12) with $S_k = D_k$ and $B_k = \nabla^2 f_k$, $\{x_k\}$ be any subsequence generated by TRAM and β_k be the stepsize to the boundary of $\mathcal{F}$ along p_k. Then there exist $\chi_0, \chi_1, \chi_2 > 0$ such that, for sufficiently large k,*

$$\beta_k \geq \frac{\nu_k}{\chi_0(\chi_1 + (\chi_2 + \nu_k)\Delta_k)}, \tag{39}$$

where $\nu_k \geq 0$ is defined in (8). Assume further that $\{x_k\}$ converges to x_* at which strict complementarity and dual feasibility are satisfied, and $\{(p_k; \hat{p}_k)\}$ converges to zero. Then $\liminf_{k\to\infty} \beta_k > 0$.

Proof. By definition,

$$\beta_k = \min\left\{-\frac{a_i^T x_k - b_i}{a_i^T p_k} : -\frac{a_i^T x_k - b_i}{a_i^T p_k} > 0\right\}.$$

From $\hat{p}_k = D_k^{-\frac{1}{2}} A p_k$ and (8), there exists λ_{k+1}^p such that

$$a_i^T p_k = \left(a_i^T x_k - b_i\right)^{\frac{1}{2}} (\hat{p}_k)_i = -\frac{\left(a_i^T x_k - b_i\right)\left(\lambda_{k+1}^p\right)_i}{\nu_k + |(\lambda_k)_i|}.$$

Hence, there exists $1 \leq j \leq m$ such that,

$$\beta_k \geq \frac{\nu_k + |(\lambda_k)_j|}{\left|\left(\lambda_{k+1}^p\right)_j\right|} \geq \frac{\nu_k}{\left\|\lambda_{k+1}^p\right\|_\infty}. \tag{40}$$

From (8),

$$\begin{bmatrix} A^T \\ -D_k^{\frac{1}{2}} \end{bmatrix} \lambda_{k+1}^p = \begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix} + (\hat{H}_k + \nu_k I)(p_k; \hat{p}_k). \tag{41}$$

Since $[A, -D_k^{\frac{1}{2}}]$ has full row rank in the compact set $\mathcal{L}$, $\{\lambda_k\}$ is bounded (using Lemma 1), and $f(x)$ is twice continuously differentiable, there exist $\chi_0, \chi_1, \chi_2 > 0$ such that

$$\left\|\lambda_{k+1}^p\right\|_\infty \leq \chi_0(\chi_1 + (\chi_2 + \nu_k)\Delta_k). \tag{42}$$

Substitute (42) into (40), (39) holds.

Assume now that $\{x_k\}$ converges to x_* satisfying strict complementarity, dual feasibility, and $\{(p_k; \hat{p}_k)\}$ converges to zero. Since (AS.0) holds, $\{\lambda_k\}$ converges to λ_*. From strict complementarity and dual feasibility, $(\lambda_*)_i > 0$ for any $a_i^T x_* - b_i = 0$. If there is no binding constraint at x_*, then $\beta_k = \infty$ for sufficiently large k and the result clearly holds. Assume that there is some i such that $a_i^T x_* - b_i = 0$. Since $\{p_k\}$ converges to zero, the index j defining β_k in (40) satisfies $a_j^T x_* - b_j = 0$ and $(\lambda_k)_j > 0$ for sufficiently large k.

Let the sequence $\{\Delta_k\}$ be partitioned into two subsequences: $\{k : \Delta_k \leq 1\}$ and $\{k : \Delta_k > 1\}$. For the subsequence $\{k : \Delta_k \leq 1\}$, using (40) and (42),

$$\beta_k \geq \frac{\nu_k + |(\lambda_k)_j|}{\chi_0(\chi_1 + (\chi_2 + \nu_k))}, \quad \text{for some } a_j^T x_* - b_j = 0.$$

Since $\lim_{k\to\infty}(\lambda_k)_i > 0$ for all $a_i^T x_* - b_i = 0$, $\liminf_{k\to\infty} \beta_k > 0$.

Consider the subsequence $\{k : \Delta_k > 1\}$. Using Lemma 8, since $\{(p_k; \hat{p}_k)\}$ converges to zero, $\|(p_k; \hat{p}_k)\|_2 < \Delta_k$ for sufficiently large k in the subsequence $\{\Delta_k > 1\}$. Thus $\nu_k = 0$ for sufficiently large k in the subsequence $\{k : \Delta_k > 1\}$. Using (41) and (AS.0), $\lim_{k\to\infty} \lambda_k = \lim_{k\to\infty} \lambda_k^p = \lambda_*$. From the first inequality in (40) and strict complementarity assumption, $\liminf_{k\to\infty} \beta_k > 0$. The proof is completed.

□

Lemma 10. *Assume that* $[A, D_*]$ *has full row rank at* $x_* \in \mathcal{F}$ *and the complementarity conditions are satisfied at* $(x_*; \lambda_*)$. *Let the columns of* Z_* *denote an orthonormal basis for the null space of* $[A, D_*]$. *If* $Z_*^T \hat{H}_* Z_*$ *is positive semidefinite, then* $d^T \nabla^2 f_* d \geq 0$ *for any* d *satisfying* $a_i^T d = 0$ *for all* i *with* $a_i^T x_* - b_i = 0$.

Proof. Let d satisfy $a_i^T d = 0$ for all i with $a_i^T x_* - b_i = 0$. Define $\hat{d}_i = (a_i^T x_* - b_i)^{-\frac{1}{2}} a_i^T d$ if $a_i^T x_* - b_i > 0$ and $\hat{d}_i = 0$ otherwise. Then $Ad - D_*^{\frac{1}{2}} \hat{d} = 0$. Hence $(d; \hat{d}) = Z_* w$ and $\hat{d}^T C_* \hat{d} = 0$. Since $Z_*^T \hat{H}_* Z_*$ is positive semidefinite, $d^T \nabla^2 f_* d = w^T Z_*^T \hat{H}_* Z_* w \geq 0$. □

The next theorem establishes that the projected Hessian becomes positive semidefinite asymptotically under the stated assumptions. Note that (AS.2) is only necessary for satisfaction of dual feasibility.

Theorem 2. *Assume (AS.0) holds and* $f(x) : \mathcal{F} \to \Re$ *is twice continuously differentiable. Let* $\{x_k\}$ *be the sequence generated by TRAM. If (AS.1) is satisfied at every iteration and (AS.3) holds for sufficiently large* k, *then*

(a) There is a limit point x_* *such that* $d^T \nabla^2 f_* d \geq 0$ *for any* d *satisfying* $a_i^T d = 0$ *for all* i *with* $a_i^T x_* - b_i = 0$;
(b) If x_* *is an isolated limit point, then* $d^T \nabla^2 f_* d \geq 0$ *for any* d *satisfying* $a_i^T d = 0$ *for all* i *with* $a_i^T x_* - b_i = 0$.

Proof. Using Lemma 10, in order to prove $d^T \nabla^2 f_* d \geq 0$ for any d satisfying $a_i^T d = 0$ for all i with $a_i^T x_* - b_i = 0$, we only need to show that the projected Hessian $Z_*^T \hat{H}_* Z_*$ is positive semidefinite.

(a) If zero is a limit point of $\{\nu_k\}$, the result immediately follows. Next we prove that zero is a limit point of $\{\nu_k\}$ by contradiction. Assume that $\nu_k \geq \epsilon > 0$ for sufficiently large k. First we show that this implies that $\{x_k\}$ converges, $\{\Delta_k\}$ converges to zero and $\liminf_{k \to \infty} \beta_k > 0$.
Using (39) in Lemma 9,

$$\beta_k \geq \frac{\nu_k}{\chi_0(\chi_1 + (\chi_2 + \nu_k)\Delta_k)}. \tag{43}$$

If $\{\Delta_k\}$ is bounded by $\chi_\Delta > 0$, then

$$\beta_k \geq \frac{\nu_k}{\chi_0(\chi_1 + (\chi_2 + \nu_k)\Delta_k)} \geq \frac{1}{\chi_0(\frac{\chi_1}{\epsilon} + (\frac{\chi_2}{\epsilon} + 1)\chi_\Delta)}. \tag{44}$$

Since (AS.3) holds for sufficiently large k, using Lemma 8, for sufficiently large k,

$$-\left(\psi_k(s_k) + \frac{1}{2} s_k{}^T A^T D_k^{-1} C_k A s_k\right) \geq \frac{\beta_q \theta_0^2}{2} \min\{1, \beta_k^2\} \epsilon \Delta_k^2. \tag{45}$$

Hence, for sufficiently large k and a successful step,

$$f(x_k) - f(x_{k+1}) \geq \frac{\beta_q \theta_0^2 \mu}{2} \min\left\{1, \beta_k^2\right\} \epsilon \Delta_k^2. \tag{46}$$

If there are finite number of successful steps, $\{x_k\}$ converges and $\{\Delta_k\}$ converges to zero. Moreover, using (44), $\liminf_{k\to\infty} \beta_k > 0$. Otherwise, let $\{k_i\}$ be the infinite sequence of successful iterations. Inequality (46), (43) and $\{f(x_k) - f(x_{k+1})\}$ converges to zero imply that $\{\Delta_k\}$ is bounded. This inequality, (46), and $\{f(x_k) - f(x_{k+1})\}$ converging to zero imply that

$$\sum_{i=1}^{\infty} \Delta_{k_i}^2 < \infty.$$

The trust region updating rules of TRAM specify that $\Delta_{k+1} \leq \gamma_1 \Delta_k$ for an unsuccessful iteration and $\Delta_{k+1} \leq \gamma_2 \Delta_k$ for a successful iteration. Hence

$$\sum_{k=1}^{\infty} \Delta_k^2 \leq \left(1 + \frac{\gamma_2^2}{1 - \gamma_1^2}\right) \sum_{i=1}^{\infty} \Delta_{k_i}^2.$$

Hence $\{x_k\}$ converges and $\{\Delta_k\}$ converges to zero. Since $\|s_k\|_2 \leq \beta_s \Delta_k$ and $\|p_k\|_2 \leq \Delta_k$, both $\{s_k\}$ and $\{p_k\}$ converge to zero. Moreover, using (44), $\liminf_{k\to\infty} \beta_k > 0$.

From (45), $\liminf_{k\to\infty} \beta_k > 0$, and $-\psi_k(s_k) \geq -(\psi_k(s_k) + \frac{1}{2} {s_k}^T A^T D_k^{-1} C_k A s_k)$, there exists $\chi_3 > 0$ such that

$$-\psi_k(s_k) \geq \chi_3 \Delta_k^2 \geq \frac{\chi_3}{\beta_s^2} \|s_k\|_2^2.$$

A standard estimate is that

$$|f(x_k + s_k) - f(x_k) - \psi_k(s_k)| \leq \|s_k\|_2^2 \max_{0 \leq \xi_k \leq 1} \left\|\nabla^2 f(x_k + \xi_k s_k) - \nabla^2 f(x_k)\right\|_2.$$

The last two inequalities, convergence of $\{x_k\}$ and $\{s_k\}$ converging to zero imply that $\{|\rho_k - 1|\}$ converges to zero. Therefore the entire sequence $\{\rho_k\}$ converges to unity. The trust region updating rules of TRAM imply that $\{\Delta_k\}$ cannot converge to zero, which is a contradiction. Hence there is a limit point with $Z_*^T \hat{H}_* Z_*$ positive semidefinite.

(b) If $\{x_k\}$ converges to x_*, the result follows from (a). If $\{x_k\}$ does not converge then Lemma 7 applies. Thus, if $\{x_{l_j}\}$ is the subsequence guaranteed by Lemma 7, then $\Delta_{l_j} \geq \frac{1}{\beta_s}\epsilon$. From Lemma 8,

$$-\left(\psi_{l_j}(s_{l_j}) + \frac{1}{2} {s_{l_j}}^T A^T D_{l_j}^{-1} C_{l_j} A s_{l_j}\right) \geq \frac{\beta_q \theta_0^2 \min\left\{1, \beta_{l_j}^2\right\}}{2} \nu_{l_j} \Delta_{l_j}^2.$$

Using Lemma 9, there exist $\chi_0, \chi_1, \chi_2 > 0$ such that, for sufficiently large j,

$$\beta_{l_j} \geq \frac{\nu_{l_j}}{\chi_0(\chi_1 + (\chi_2 + \nu_{l_j})\Delta_{l_j})}.$$

Since (AS.3) holds for sufficiently large k,

$$f(x_{l_j}) - f(x_{l_j+1}) \geq \frac{\mu \beta_q \theta_0^2}{2} \min \left\{ 1, \left(\frac{\nu_{l_j}}{\chi_0(\chi_1 + (\chi_2 + \nu_{l_j})\Delta_{l_j})} \right)^2 \right\} \nu_{l_j} \Delta_{l_j}^2.$$

From $\lim_{j\to\infty} f(x_{l_j}) - f(x_{l_j+1}) = 0$ and $\Delta_{l_j} \geq \frac{1}{\beta_s}\epsilon$, the above inequality implies that $\{\nu_{l_j}\}$ converges to zero. Thus $Z_*^T \hat{H}_* Z_*$ is positive semidefinite.

We have established (a)–(b).

□

4. Local convergence

Let $F(x, \lambda) = 0$ denote the optimality conditions (4) where

$$F(x, \lambda) \stackrel{\text{def}}{=} \begin{bmatrix} \nabla f(x) - A^T \lambda \\ D(x)\lambda \end{bmatrix},$$

and $D(x) = \text{diag}(Ax - b)$ as in (2). Then the Jacobian matrix $J(x, \lambda)$ of $F(x, \lambda)$ is

$$J(x, \lambda)^T \stackrel{\text{def}}{=} \begin{bmatrix} \nabla^2 f(x) & -A^T \\ \text{diag}(\lambda)A & D(x) \end{bmatrix}.$$

From (5), a Newton step p_k^N of the trust region subproblem (7) is a modified Newton step for $F(x, \lambda) = 0$.

To establish the quadratic convergence of the proposed TRAM, we prove that, asymptotically, the Newton step p_k^N of the trust region subproblem (7) is a sufficiently accurate approximation to the exact Newton step of $F(x, \lambda) = 0$. Moreover, asymptotically, the Newton step p_k^N is in the interior of the trust region (hence $p_k = p_k^N$) and (AS.1), (AS.2) and (AS.3) can be satisfied by a damped Newton step p_k^*.

First, we prove that the Jacobian matrix $J(x, \lambda)$ is nonsingular near a local minimizer satisfying strict complementarity and linear independence assumptions.

Theorem 3. *Assume that $f(x) : \mathcal{F} \to \Re$ is twice continuously differentiable and the second order sufficiency conditions of (1) are satisfied at $(x_*; \lambda_*)$. Assume further that strict complementarity holds at x_* and $\{a_i : a_i^T x_* - b_i = 0\}$ are linearly independent. Then*

(a) The Jacobian matrix $J(x_*, \lambda_*)$ is nonsingular;
(b) The symmetric matrix $A^T(\text{diag}(|\lambda|)D(x)^{-1})A + \nabla^2 f(x)$ is positive definite when $x \in \text{int}(\mathcal{F})$ and $(x; \lambda)$ is sufficiently close to $(x_*; \lambda_*)$.

Proof. Assume that there exists $(v; w)$, $v \in \Re^n$ and $w \in \Re^m$, such that

$$\begin{bmatrix} \nabla^2 f_* & -A^T \\ \mathrm{diag}(\lambda_*)A & D_* \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = 0.$$

Without loss of generality, partition $A = [A_0; A_1]$ where the rows of A_0 correspond to the indices $\{i : a_i^T x_* - b_i = 0\}$ and the rows of A_1 correspond to the indices $\{i : a_i^T x_* - b_i \neq 0\}$. Similarly partition w into w_0 and w_1. Since strict complementarity holds at $(x_*; \lambda_*)$, $A_0 v = 0$, $w_1 = 0$, and $\nabla^2 f_* v - A_0^T w_0 = 0$. Hence $v^T \nabla^2 f_* v = 0$. Since the second order sufficiency conditions are satisfied at $(x_*; \lambda_*)$ and $A_0 v = 0$, $v = 0$ holds. This implies that $A^T w = 0$. Therefore $A_0^T w_0 = 0$. Applying linear independence assumption of the columns of A_0^T, $w = 0$. Hence the Jacobian matrix $J(x_*, \lambda_*)$ is nonsingular.

Let the components of λ_0 denote the dual variables corresponding to the constraints $\{i : a_i^T x_* - b_i = 0\}$. Since the second order sufficiency conditions are satisfied at $(x_*; \lambda_*)$ with strict complementarity, there exist $\chi_0, \sigma_{\min}, \sigma_{\max} > 0$, such that when (x, λ) is in a neighborhood of $(x_*; \lambda_*)$, for any v satisfying $A_0 v = 0$,

$$\min(\lambda_0) \geq \chi_0, \quad v^T \nabla^2 f(x) v \geq \sigma_{\min} \|v\|_2^2, \quad \text{and} \quad \|\nabla^2 f(x)\|_2 \leq \sigma_{\max}. \tag{47}$$

Let $M = \nabla^2 f(x) + A^T(\mathrm{diag}(|\lambda|)D(x)^{-1})A$. In order to prove that $d^T M d > 0$ for any $d \neq 0$, without loss of generality, assume that $d = u + v$ where $\|u + v\|_2 = 1$ $u, v \in \Re^n$, with $A_0 v = 0$ and $u^T v = 0$. Using $\|v\|_2^2 = 1 - \|u\|_2^2$, $\|v\|_2 \leq 1$ and $\|u\|_2 \leq 1$,

$$\begin{aligned} (u + v)^T \nabla^2 f(x)(u + v) &= v^T \nabla^2 f(x) v + 2u^T \nabla^2 f(x) v + u^T \nabla^2 f(x) u \\ &\geq \sigma_{\min} \|v\|_2^2 - 2\sigma_{\max} \|u\|_2 \|v\|_2 - \sigma_{\max} \|u\|_2^2 \\ &= \sigma_{\min} \|v\|_2^2 - \sigma_{\max} \|u\|_2 (2\|v\|_2 + \|u\|_2) \\ &\geq \sigma_{\min} - (\sigma_{\min} + 3\sigma_{\max}) \|u\|_2. \end{aligned}$$

Let $x \in int(\mathcal{F})$ and (x, λ) be in the neighborhood of (x_*, λ_*) within which (47) holds. From $A_0 v = 0$, (47), and the above inequality,

$$\begin{aligned} (u + v)^T M(u + v) &= (u + v)^T \nabla^2 f(u + v) + (u + v)^T A^T (\mathrm{diag}(|\lambda|)D(x)^{-1})A(u + v) \\ &\geq (u + v)^T \nabla^2 f(u + v) + u^T A_0^T D_0^{-1} \mathrm{diag}(\lambda_0) A_0 u \\ &\geq \sigma_{\min} - (\sigma_{\min} + 3\sigma_{\max}) \|u\|_2 + \frac{\chi_0}{\|A_0 x - b_0\|_\infty} \|A_0 u\|_2^2. \end{aligned}$$

If $\|u\|_2 < \chi_1 \stackrel{\text{def}}{=} \frac{\sigma_{\min}}{2(\sigma_{\min} + 3\sigma_{\max})}$, then $(u + v)^T M(u + v) \geq \frac{1}{2}\sigma_{\min}$. Since A_0 is of full row rank and $u^T v = 0$ with $A_0 v = 0$, if $\|u\|_2 \geq \chi_1$, there exists $\chi_2 > 0$ such that $\|A_0 u\|_2 \geq \chi_2 \|u\|_2 \geq \chi_1 \chi_2$. Hence

$$(u + v)^T M(u + v) \geq \frac{\chi_0 \chi_1^2 \chi_2^2}{\|A_0 x - b_0\|_\infty} - (\sigma_{\min} + 3\sigma_{\max}).$$

When x is sufficiently close to x_* such that $\|A_0 x - b_0\|_\infty < \frac{2\chi_0 \chi_1^2 \chi_2^2}{\sigma_{\min} + 3\sigma_{\max}}$, $(u+v)^T M(u+v) > \sigma_{\min} + 3\sigma_{\max}$. Hence there exists a neighborhood of $(x_*; \lambda_*)$ within which the symmetric matrix M is positive definite. The proof is completed.

□

We have established in Theorem 1 and 2 that, assuming sufficient decrease conditions (AS.1) and (AS.3), there exists a limit point satisfying the complementarity and second order necessary conditions. We prove next that, assuming additionally the second order sufficiency condition and strict complementarity at this limit, convergence occurs and trust region size $\{\Delta_k\}$ is asymptotically bounded away from zero.

Theorem 4. *Assume that (AS.0) holds and $f(x) : \mathcal{F} \to \Re$ is twice continuously differentiable. Let $\{x_k\}$ and $\{s_k\}$ be the sequence generated by TRAM. Assume further that (AS.1) and (AS.3) hold for sufficiently large k and the second order sufficiency conditions and strict complementarity are satisfied at a limit x_* of $\{x_k\}$. Then $\{x_k\}$ converges to x_*, $Z_k^T \hat{H}_k Z_k$ is positive definite for sufficiently large k, all iterations are eventually successful, and $\{\Delta_k\}$ is bounded away from zero.*

Proof. For any $w \in \Re^m$, let $(d; \hat{d}) = Z_k^T w$ where $d \in \Re^n$, $\hat{d} \in \Re^m$ and the columns of Z_k form an orthonormal basis for the null space of $[A, D_k]$. Then $Ad - D_k^{\frac{1}{2}} \hat{d} = 0$, i.e., $\hat{d} = D_k^{-\frac{1}{2}} Ad$. Let $M_k = A^T(C_k D_k^{-1})A + \nabla^2 f_k$. Using Theorem 3 and $C_k = \mathrm{diag}(|\lambda_k|)$, when x_k is sufficiently close to x_*,

$$w^T Z_k^T \hat{H}_k Z_k w = (d; \hat{d})^T \hat{H}_k (d; \hat{d}) = d^T M_k d > 0, \quad \text{if } w \neq 0.$$

Hence $Z_k^T \hat{H}_k Z_k$ is positive definite when $(x_k; \lambda_k)$ is sufficiently close to $(x_*; \lambda_*)$.

Since the second order sufficiency conditions are satisfied at x_*, x_* is an isolated limit point. Let $\hat{s}_k = D_k^{-\frac{1}{2}} A s_k$. Then $Z_k Z_k^T (s_k; \hat{s}_k) = (s_k; \hat{s}_k)$. Since (AS.3) is satisfied for sufficiently large k, $\psi_k(s_k) + \frac{1}{2} s_k{}^T A^T D_k^{-1} C_k A s_k = \nabla f_k^T s_k + \frac{1}{2}(s_k; \hat{s}_k)^T \hat{H}_k (s_k; \hat{s}_k) < 0$ asymptotically. Hence

$$\begin{aligned} 0 \le \frac{1}{2}(s_k; \hat{s}_k)^T Z_k \big(Z_k^T \hat{H}_k Z_k\big) Z_k^T (s_k; \hat{s}_k) &< -[\nabla f_k, 0]^T Z_k Z_k^T (s_k; \hat{s}_k) \\ &\le \big\| Z_k^T (s_k; \hat{s}_k) \big\|_2 \big\| Z_k^T (\nabla f_k; 0) \big\|_2, \end{aligned}$$

whenever $Z_k^T \hat{H}_k Z_k$ is positive definite and k is sufficiently large. But

$$(s_k; \hat{s}_k)^T Z_k \big(Z_k^T \hat{H}_k Z_k\big) Z_k^T (s_k; \hat{s}_k) \ge \frac{1}{\big\| (Z_k^T \hat{H}_k Z_k)^{-1} \big\|_2} \big\| Z_k^T (s_k; \hat{s}_k) \big\|_2^2.$$

This implies that, for sufficiently large k, whenever $Z_k^T \hat{H}_k Z_k$ is positive definite,

$$\frac{1}{2} \big\| Z_k^T (s_k; \hat{s}_k) \big\|_2 \le \big\| \big(Z_k^T \hat{H}_k Z_k\big)^{-1} \big\|_2 \big\| Z_k^T (\nabla f_k; 0) \big\|_2, \tag{48}$$

From the result (a) in Theorem 1 and $\|Z_k Z_k^T (\nabla f_k; 0)\|_2^2 = -\nabla f_k^T g_k$, $\{Z_k^T (\nabla f_k; 0)\}$ converges to zero. Hence, for any subsequence $\{x_{l_j}\}$ converges to x_*, $\{(s_{l_j}; \hat{s}_{l_j})\}$ and $\{(p_{l_j}; \hat{p}_{l_j})\}$ converge to zero. Following Lemma 7, $\{x_k\}$ converges to x_*. Therefore $\{(p_k; \hat{p}_k)\}$ and $\{(s_k; \hat{s}_k)\}$ converge to zero. Moreover, $\{Z_k^T \hat{H}_k Z_k\}$ is positive definite for sufficiently large k.

Let κ be an upper bound on the condition number of $Z_k^T \hat{H}_k Z_k$. From $Z_k^T(\nabla f_k; 0) = -(Z_k^T \hat{H}_k Z_k) Z_k^T (p_k^N; \hat{p}_k^N)$ and (48),

$$\frac{1}{2}\|(s_k; \hat{s}_k)\|_2 \le \kappa \left\|(p_k^N; \hat{p}_k^N)\right\|_2, \tag{49}$$

when k is sufficiently large.

Since $\{x_k\}$ converges to x_* satisfying the second order sufficiency conditions with strict complementarity and $\{(p_k; \hat{p}_k)\}$ converges to zero, using Lemma 8 and 9, there exists $\chi_3 > 0$ such that, for sufficiently large k,

$$-\left(\psi_k(p_k^*) + \frac{1}{2} p_k^{*T} A^T D_k^{-1} C_k A p_k^*\right) \ge \frac{\theta_0^2 \chi_3}{2} \left\| R_k Z_k^T (p_k; \hat{p}_k) \right\|_2^2.$$

Since the eigenvalues of $\{Z_k^T \hat{H}_k Z_k\}$ are positive and bounded away from zero asymptotically, there exists $\chi_4 > 0$ such that, for sufficiently large k,

$$-\left(\psi_k(p_k^*) + \frac{1}{2} p_k^{*T} A^T D_k^{-1} C_k A p_k^*\right) \ge \chi_4 \min\left\{\Delta_k^2, \left\|(p_k^N; \hat{p}_k^N)\right\|_2^2\right\}.$$

Using (49), $\|s_k\| \le \beta_s \Delta_k$, and (AS.3), there exists $\chi_5 > 0$ such that

$$-\left(\psi_k(s_k) + \frac{1}{2} s_k^T A^T D_k^{-1} C_k A s_k\right) \ge \chi_4 \max\left\{\Delta_k^2, \frac{1}{2\kappa}\|(s_k; \hat{s}_k)\|_2^2\right\} \ge \chi_5 \|s_k\|_2^2. \tag{50}$$

Therefore $-\psi_k(s_k) \ge \chi_5 \|s_k\|_2^2$ for sufficiently large k. But $f(x)$ is twice continuously differentiable on $\mathcal{F}$, thus

$$\begin{aligned} &|f(x_k + s_k) - f(x_k) - \psi_k(s_k)| \\ \le\ & \|s_k\|_2^2 \max_{0 \le \xi_k \le 1} \left\|\nabla^2 f(x_k + \xi_k s_k) - \nabla^2 f(x_k)\right\|_2. \end{aligned}$$

Since $\{x_k\}$ converges and $\{s_k\}$ converges to zero, using (50), $\lim_{k\to\infty} \rho_k = 1$. This implies that $\rho_k > \eta$ for k sufficiently large. Therefore $\{\Delta_k\}$ is bounded away from zero.

□

Recall that $(p_k^N; \Delta\lambda_k^N)$ which satisfies (5), where $\lambda_{k+1}^N = \lambda_k + \Delta\lambda_k^N$, denotes the Newton step of the trust region subproblem. Lemma 11 shows that, under assumptions specified, the stepsize along p_k^N converges to one.

Lemma 11. *Assume that $f(x) : \mathcal{F} \to \Re$ is twice continuously differentiable, $\{x_k\}$ converges to x_*, and $\{\lambda_k\}$ is the least squares Lagrangian multiplier approximation (15). Moreover, $\{a_i : a_i^T x_* - b_i = 0\}$ are linearly independent and the second order sufficiency conditions and strict complementarity are satisfied at x_*. Let $(p_k^N; \lambda_{k+1}^N)$ be defined by (5). Then, $\{\lambda_k\}$ and $\{\lambda_k^N\}$ converge to λ_*, $\{p_k^N\}$ converges to zero, and*

$$\begin{aligned} &\|\lambda_k - \lambda_*\|_2 = O(\|x_k - x_*\|_2), \\ &\left\|\lambda_k - \lambda_k^N\right\|_2 = O\left(\left\|(x_k; \lambda_k^N) - (x_*; \lambda_*)\right\|_2\right), \\ &\left\|\lambda_{k+1}^N - \lambda_k\right\|_2 = O\left(\left\|(x_k; \lambda_k^N) - (x_*; \lambda_*)\right\|_2\right), \\ &\left|\alpha_k^N - 1\right| = O\left(\left\|(x_k; \lambda_k^N) - (x_*; \lambda_*)\right\|_2\right), \end{aligned}$$

where $\alpha_k^N = \min\{1, \beta_k^N\}$ *and* β_k^N *is the stepsize to the boundary of the feasible region* $\mathcal{F}$ *along* p_k^N.

Proof. By definition (15) of λ_k, $(AA^T + D_k)\lambda_k = A\nabla f_k$. Since $\{a_i : a_i^T x_* - b_i = 0\}$ are linearly independent, $[A, -D_*^{\frac{1}{2}}]$ has full row rank and $(AA^T + D_*)$ is nonsingular. In addition $\nabla f(x)$ is continuous in $\mathcal{F}$. Thus

$$\begin{aligned}\|\lambda_k - \lambda_*\|_2 &= \left\|\left(AA^T + D_k\right)^{-1} A\nabla f(x_k) - \left(AA^T + D_*\right)^{-1} A\nabla f(x_*)\right\|_2 \\ &= O(\|x_k - x_*\|_2).\end{aligned}$$

Since the coefficient matrix of (5) is equivalent to $J(x, \lambda)^T$ at $(x_*; \lambda_*)$, applying Theorem 3, the linear system (5) is nonsingular at $(x_*; \lambda_*)$. Since $\{x_k\}$ converges to x_* and complementarity is satisfied at x_*, $\{D_k\lambda_k\}$ and $\{\nabla f_k - A^T\lambda_k\}$ converge to zero. By definition (5) of (p_k^N, λ_{k+1}^N) and $\{\lambda_k\}$ converges to λ_*, $\{\lambda_k^N\}$ converges to λ_* and $\lim_{k\to\infty} p_k^N = 0$. Moreover, $\|\lambda_k - \lambda_k^N\|_2 \le \|\lambda_k - \lambda_*\|_2 + \|\lambda_k^N - \lambda_*\|_2 = O(\|(x_k; \lambda_k^N) - (x_*; \lambda_*)\|_2)$. From (5),

$$\begin{aligned}\left\|\lambda_{k+1}^N - \lambda_k\right\|_2 &= O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2) \\ &= O\left(\left\|\left(x_k; \lambda_k^N\right) - (x_*; \lambda_*)\right\|_2\right) + O\left(\left\|\lambda_k^N - \lambda_k\right\|_2\right) \\ &= O\left(\left\|\left(x_k; \lambda_k^N\right) - (x_*; \lambda_*)\right\|_2\right).\end{aligned}$$

If $\beta_k^N = \infty$, $\alpha_k^N = 1$. Consider the case when $\beta_k^N < \infty$. Using (5) again, for $1 \le i \le m$,

$$a_i^T p_k^N = -\frac{\left(a_i^T x_k - b_i\right)\left(\lambda_{k+1}^N\right)_i}{|(\lambda_k)_i|}.$$

Since $\{p_k^N\}$ converges to zero and strict complementarity is satisfied at x_*, for sufficiently large k,

$$\beta_k^N = \frac{|(\lambda_k)_j|}{\left(\lambda_{k+1}^N\right)_j}, \quad \text{for some } \left(\lambda_{k+1}^N\right)_j > 0 \text{ and } a_j^T x_* - b_j = 0.$$

From strict complementarity, $\lim_{k\to\infty}(\lambda_k^N)_j = (\lambda_*)_j > 0$ for any $a_j^T x_* - b_j = 0$. Hence

$$\left|1 - \alpha_k^N\right| = O\left(\left|1 - \beta_k^N\right|\right) = O\left(\left\|\lambda_{k+1}^N - \lambda_k\right\|_2\right) = O\left(\left\|\left(x_k; \lambda_k^N\right) - (x_*; \lambda_*)\right\|_2\right).$$

The proof is completed.

□

Theorem 5 states that, under the specified assumptions, Newton steps $(p_k^N; \hat{p}_k^N)$ can lead to successful steps and quadratic convergence.

Theorem 5. *Assume that* $0 < \mu < 1$ *and* $f(x) : \mathcal{F} \to \Re$ *is twice continuously differentiable on* $\mathcal{F}$. *Assume that* $\{x_k\}$ *converges to* x_*, *the second order sufficiency conditions and strict complementarity are satisfied at* $(x_*; \lambda_*)$ *and* $\{a_i : a_i^T x_* - b_i = 0\}$ *are linearly independent. Moreover, let* p_k^N *be defined by (5) and* θ_k *be the damping* *parameter with* $|\theta_k - 1| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$. *Then* $|1 - \alpha_k| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$, *and*

$$f\left(x_k + \alpha_k p_k^N\right) - f(x_k) < \mu \psi_k\left(\alpha_k p_k^N\right),$$

where $\alpha_k = \theta_k \alpha_k^N$, $0 < \theta_0 \le \theta_k < 1$, $\alpha_k^N = \min\{1, \beta_k^N\}$ *and* β_k^N *is the stepsize to the boundary of the feasible region* $\mathcal{F}$ *along* p_k^N. *In addition, if* $x_{k+1} = x_k + \alpha_k p_k^N$ *for sufficiently large* k, *then* $\{(x_k; \lambda_k^N)\}$ *converges quadratically to* $(x_*; \lambda_*)$.

Proof. Let $d_k = \alpha_k p_k^N$. Since $f(x)$ is twice continuously differentiable in $\mathcal{F}$, there exists $0 \le \xi_k \le 1$ such that

$$f(x_k + d_k) - f(x_k) = \psi_k(d_k) + \frac{1}{2} d_k{}^T (\nabla^2 f(x_k + \xi_k d_k) - \nabla^2 f_k) d_k.$$

From Lemma 11, $\{p_k^N\}$ and $\{d_k\}$ converge to zero. Therefore, there exists $\{\epsilon_k\}$ converging to zero such that, for sufficiently large k,

$$f(x_k + d_k) - f(x_k) = \psi_k(d_k) + \epsilon_k \|d_k\|_2^2.$$

From $0 < \mu < 1$,

$$\begin{aligned} & f(x_k + d_k) - f(x_k) - \mu \psi_k(d_k) \\ &= (1 - \mu) \left(\nabla f_k^T d_k + \frac{1}{2} d_k{}^T \nabla^2 f_k d_k \right) + \epsilon_k \|d_k\|_2^2 \\ &\le (1 - \mu) \left(\nabla f_k^T d_k + \frac{1}{2} (d_k; \hat{d}_k)^T \hat{H}_k (d_k; \hat{d}_k) \right) + \epsilon_k \|d_k\|_2^2. \end{aligned}$$

Since the columns of Z_k form an orthonormal basis for the null space of $[A, -D_k^{\frac{1}{2}}]$, and $A p_k^N - D_k^{\frac{1}{2}} \hat{p}_k^N = 0$, $(p_k^N; \hat{p}_k^N) = Z_k Z_k^T (p_k; \hat{p}_k^N)$. For sufficiently large k, $(p_k^N; \hat{p}_k^N)$ satisfies (8) with $\nu_k = 0$. Hence

$$\begin{aligned} \nabla f_k^T p_k^N &= -\left(p_k^N; \hat{p}_k^N\right)^T \hat{H}_k \left(p_k^N; \hat{p}_k^N\right) \\ &= -\left(p_k^N; \hat{p}_k^N\right)^T Z_k \left(Z_k^T \hat{H}_k Z_k\right) Z_k^T \left(p_k^N; \hat{p}_k^N\right). \end{aligned}$$

Substituting $d_k = \alpha_k p_k^N$ into the equations above,

$$\alpha_k \nabla f_k^T d_k = -(d_k; \hat{d}_k)^T Z_k \left(Z_k^T \hat{H}_k Z_k\right) Z_k^T (d_k; \hat{d}_k).$$

Hence

$$\begin{aligned} & f(x_k + d_k) - f(x_k) - \mu \psi_k(d_k) \\ &\le -\frac{(2 - \alpha_k)(1 - \mu)}{2\alpha_k} (d_k; \hat{d}_k)^T Z_k \left(Z_k^T \hat{H}_k Z_k\right) Z_k^T (d_k; \hat{d}_k) + \epsilon_k \|d_k\|_2^2. \end{aligned}$$

From Theorem 3, the projected Hessian $Z_k^T \hat{H}_k Z_k$ is positive definite for sufficiently large k. Hence there exists $\epsilon > 0$ such that, for k sufficiently large,

$$(d_k; \hat{d}_k)^T Z_k \left(Z_k^T \hat{H}_k Z_k\right) Z_k^T (d_k; \hat{d}_k) \ge \epsilon \left\| Z_k^T (d_k; \hat{d}_k) \right\|_2.$$

But $\|Z_k^T(d_k; \hat{d}_k)\|_2^2 = \|(d_k; \hat{d}_k)\|_2^2$. Hence

$$f(x_k + d_k) - f(x_k) - \mu\psi_k(d_k) \leq -\frac{2-\alpha_k}{2\alpha_k}(1-\mu)\epsilon\left\|(d_k; \hat{d}_k)\right\|_2^2 + \epsilon_k\|d_k\|_2^2.$$

Since $\{\theta_k\}$ converges to 1, $\alpha_k = \theta_k\alpha_k^N$ and $\{\alpha_k^N\}$ converges to 1 using Lemma 11, $\{\alpha_k\}$ converges to 1. But $\{\epsilon_k\}$ converges to zero. Hence, for sufficiently large k,

$$f(x_k + d_k) - f(x_k) < \mu\psi_k(d_k).$$

Let $w_k = \lambda_{k+1}^N - \lambda_k^N$. Similar to (5), $(d_k; w_k)$ satisfies

$$\tilde{J}_k^T(d_k; w_k) = -\left(\nabla f_k - A^T\lambda_k^N; D_k\lambda_k^N\right),$$

where

$$\tilde{J}_k^T = \begin{bmatrix} \frac{1}{\alpha_k}\nabla^2 f_k & -A^T \\ \frac{1}{\alpha_k}C_kA & D_k \end{bmatrix}. \tag{51}$$

Clearly,

$$\begin{aligned}&\left\|J(x_k, \lambda_k^N)^T - \tilde{J}_k^T\right\|_2 \\ \leq&\left\|(\text{diag}(\lambda_k^N) - C_k)A\right\|_2 + \left|1 - \frac{1}{\alpha_k}\right|\left\|\nabla^2 f_k\right\|_2 + \left|1 - \frac{1}{\alpha_k}\right|\|C_kA\|_2.\end{aligned}$$

Using Lemma 11, $|\alpha_k^N - 1| = O(\|(x_k; \lambda_k^N) - (x_*; \lambda_*)\|_2)$. In addition, $|\theta_k - 1| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$ and $0 < (1 - \theta_k\alpha_k^N) = (1 - \theta_k) + \theta_k(1 - \alpha_k^N)$. Hence $|\alpha_k - 1| = O(\|(x_k; \lambda_k^N) - (x_*; \lambda_*)\|_2)$. Moreover,

$$\left\|\text{diag}(\lambda_k^N) - C_k\right\|_2 = O\left(\left\|(x_k; \lambda_k^N) - (x_*; \lambda_*)\right\|_2\right).$$

Using (51), $|1 - \alpha_k| = O(\|(x_k; \lambda_k^N) - (x_*; \lambda_*)\|_2)$ and the equation above,

$$\left\|J(x_k, \lambda_k^N)^T - \tilde{J}_k^T\right\|_2 = O\left(\left\|(x_k; \lambda_k^N) - (x_*; \lambda_*)\right\|_2\right).$$

Applying Theorem 3.4 in [11], $\{(x_k; \lambda_k^N)\}$ converges quadratically to $(x_*; \lambda_*)$.

□

The assumption $|\theta_k - 1| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$ requires that the damping parameter converges to one sufficiently fast; this requirement is satisfied if $\theta_k = \max(\theta_0, 1 - \|D_k\lambda_k\|_\infty)$, for example. Finally Theorem 6 shows that quadratic convergence can be achieved with TRAM.

Theorem 6. *Assume that (AS.0) holds and $f(x) : \mathcal{F} \to \Re$ is twice continuously differentiable on $\mathcal{F}$. Assume (AS.1) is satisfied at every iteration, and (AS.2) and (AS.3) hold for sufficiently large k. Let $\{x_k\}$ be generated by TRAM in Fig. 1 and x_* be a limit point. Assume further that the second order sufficiency conditions are satisfied at $(x_*; \lambda_*)$ with strict complementarity and $|\theta_k - 1| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$. Let $\alpha_k = \theta_k \min\{1, \beta_k\}$ where β_k is the stepsize along p_k to the boundary of $\mathcal{F}$. Then, for sufficiently large k, $\alpha_k p_k$ satisfies (AS.1), (AS.2) and (AS.3). If $x_{k+1} = x_k + \alpha_k p_k$ when k is sufficiently large and $x_k + \alpha_k p_k$ yields a successful step, then $\{(x_k; \lambda_k^p)\}$ converges quadratically to $(x_*; \lambda_*)$.*

Proof. From Theorem 4, $\{x_k\}$ converges to x_* and $\{\Delta_k\}$ is bounded away from zero. Since p_k^N satisfies

$$(Z_k^T \hat{H}_k Z_k) Z_k^T (p_k^N; \hat{p}_k^N) = -Z_k^T (\nabla f_k; 0),$$

and $\{\nabla f_k^T g_k\}$ converges to zero, the sequence $\{(p_k^N; \hat{p}_k^N)\}$ converges to zero. Hence, for sufficiently large k, $\|(p_k^N; \hat{p}_k^N)\|_2 < \Delta_k$. This implies that $p_k = p_k^N$ and $\lambda_k^P = \lambda_k^N$ for sufficiently large k. Hence $\alpha_k = \theta_k \alpha_k^N$ where $\alpha_k^N = \min(1, \beta_k^N)$ and β_k^N is the stepsize along p_k^N to the boundary of $\mathcal{F}$.

Using Theorem 5, $\{\alpha_k\}$ converges to 1. From $0 \le \alpha_k \le 1$ and $(p_k; \hat{p}_k)^T \hat{H}_k (p_k, \hat{p}_k) \ge 0$ for sufficiently large k,

$$\begin{aligned}
&\psi_k(\alpha_k p_k) + \frac{\alpha_k^2}{2} {p_k}^T A^T D_k^{-1} C_k A p_k \\
&= \alpha_k \nabla f_k^T p_k + \frac{\alpha_k^2}{2} p_k^T \nabla^2 f_k p_k + \frac{\alpha_k^2}{2} {p_k}^T A^T D_k^{-1} C_k A p_k \\
&= \alpha_k \nabla f_k^T p_k + \frac{\alpha_k^2}{2} (p_k; \hat{p}_k)^T \hat{H}_k (p_k, \hat{p}_k) \\
&\le \alpha_k \left(\nabla f_k^T p_k + \frac{1}{2} (p_k; \hat{p}_k)^T \hat{H}_k (p_k, \hat{p}_k) \right) \\
&\le \alpha_k \left(\psi_k(p_k) + \frac{1}{2} {p_k}^T A^T D_k^{-1} C_k A p_k \right).
\end{aligned}$$

Since $0 < \beta_q < 1$ and $\{\alpha_k\}$ converges to 1, (AS.3) is satisfied with $\alpha_k p_k$ for sufficiently large k. From $0 < \beta_{cs} < 1$ and $\|g_k^*\| \le \Delta_k$, for k sufficiently large,

$$\psi_k(\alpha_k p_k) + \frac{\alpha_k^2}{2} p_k^T A^T D_k^{-1} C_k A p_k < \beta_{cs} \left(\psi_k(g_k^*) + \frac{1}{2} {g_k^*}^T A^T D_k^{-1} C_k A g_k^* \right),$$

i.e., $\alpha_k p_k$ satisfies (AS.1). Since the first order necessary conditions are satisfied at $(x_*; \lambda_*)$ with strict complementarity, $g_k = \tilde{g}_k$ for sufficiently large k. Thus $\alpha_k p_k$ satisfies (AS.2) asymptotically as well.

Following Theorem 5, $\alpha_k p_k$ yields a successful step for sufficiently large k. From the assumption that $x_{k+1} = x_k + \alpha_k p_k$ when $x_k + \alpha_k p_k$ yields a successful step and k is sufficiently large, $x_{k+1} = x_k + \alpha_k p_k$ asymptotically. Using Theorem 5 again, $\{(x_k; \lambda_k^P)\}$ converges quadratically to $(x_*; \lambda_*)$.

□

We have assumed strict complementarity in establishing the second order convergence. In the context of simple bound constrained minimization, it is shown that, with a proper modification on the scaling matrix, quadratic convergence can be achieved in the absence of the strict complementarity assumption [17]. It would be interesting to investigate whether similar results can be established here.

5. Conclusion

In this paper, global and local convergence properties of a trust region and affine scaling interior point method (TRAM) are established for minimizing a general nonlinear (nonconvex) function with linear inequality constraints.

Under a compactness and full rank assumption (AS.0), it is established that, if a computed step leads to a sufficient decrease with respect to the projected gradient g_k using affine scaling D_k, every limit point satisfies the complementarity conditions. If, in addition, (AS.2) is satisfied asymptotically, then the Kuhn-Tucker conditions are satisfied at every limit x_*. Moreover, if (AS.3) is satisfied asymptotically, then there exists a limit point at which both the first and second order necessary conditions are satisfied. Finally, if the second order sufficiency conditions and strict complementarity are satisfied at a limit point, then $\{(x_k; \lambda_k^p)\}$ converges quadratically.

In the proposed method, a 2-norm trust region subproblem can be approximately solved. This trust region subproblem uses an affine scaling so that a damped step of the solution of the trust region subproblem generates sufficient decrease of the objective function. Moreover, asymptotically, the solution of the trust region subproblem corresponds to a modified Newton step for the complementarity conditions. Preliminary computational results in [8] suggest that the proposed method often solves a large linear-inequality-constrained nonlinear minimization problem in a small number of iterations.

There is no theoretical difficulty in extending the proposed method for additional linear equality constraints; however, the use of the computational reflection technique cannot be directly applied. Using a penalty function for nonlinear constraints, we are currently extending the proposed trust region and affine scaling interior point method to more general nonlinear programming problems with additional equality constraints. In addition, we are investigating the role of dual multiplier updates and possibility of alternative multiplier updating schemes in the proposed algorithm context.

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