Total risk minimization using Monte-Carlo simulations

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Abstract

In an incomplete market, it is generally impossible to replicate an option exactly. In this case, total risk minimization chooses an optimal self-financing strategy that best approximates the option payoff by its final value. Total risk minimization is a dynamic stochastic programming problem, which is generally very challenging to solve; a direct approach may lead to very expensive computations.
We investigate total risk minimization using a piecewise linear criterion. We describe a method for computing the optimal hedging strategies for this stochastic programming problem using Monte Carlo simulation and spline approximations. We illustrate this method in the Black-Scholes and the stochastic volatility frameworks. We also compare the hedging performance of the strategies based on piecewise linear risk minimization, the traditional, quadratic risk minimizing strategies and the shortfall risk minimizing strategies. The numerical results show that piecewise linear risk minimization may lead to smaller hedging cost and significantly different, possibly better, hedging strategies. The values of the shortfall risk for the piecewise linear total risk minimizing strategies suggest that these strategies typically underhedge the options.

1. Introduction

Hedging is a method for reducing the sensitivity of a portfolio to market fluctuations. In particular, when hedging an option, one tries to construct a trading strategy that replicates the option payoff with no inflow or outflow of capital besides the initial costs. In the Black-Scholes framework, an option can be hedged by using only the underlying asset and a bond. However, the investor's position must be adjusted continuously, since it is only instantaneously risk-free. In practice, however, it is impossible to hedge continuously in time. In addition, one may want to hedge as little as possible due to transaction costs. If only discrete hedging times are allowed, achieving a risk-free position at each time is no longer possible since this instantaneous hedging will not last till the next rebalancing time. Moreover, presence of additional risks, e.g., jump risks, leads to an incomplete market. Under these conditions, it is not possible to totally hedge the intrinsic risk of an option that cannot be exactly replicated. There is much uncertainty regarding the choice of an optimal hedging strategy and in defining the fair price of an option.

El Karoui and Quenez ([10]) use the super-replication method for pricing and hedging in incomplete markets. The method consists in finding a self-financing strategy of minimum initial cost such that its final value is always larger than the option payoff. This minimum initial cost represents the ask price, or the seller's price of the option. Correspondingly, the method computes a bid price, or a buyer's price. However, only an interval of no-arbitrage prices is determined in this manner. Moreover, there are cases when using a super-replicating strategy for hedging an option is not interesting from a financial point of view. For example, in the Hull-White ([9]) stochastic volatility model, the super-replicating strategy for a call option is to hold the underlying asset (Frey [5]). In addition, the minimum initial cost of a super-replicating strategy may be undesirably large.

Another approach to pricing and hedging in incomplete markets is to compute an optimal strategy by minimizing a particular measure of the intrinsic risk of the option. Föllmer and Schweizer ([4]), Schäl ([14]), Schweizer ([15, 16]), Mercurio and Vorst ([12]), Heath, Platen and Schweizer ([6], [7]), Bertsimas et al. ([1]) study quadratic criteria for risk minimization. We only briefly describe them here, but they are presented in more detail in Section 2.

Suppose we want to hedge an option whose payoff is denoted by H and we only have a finite number of hedging times: $t_0,\ t_1, \ldots,\ t_M$. Suppose also that the financial market is modeled by a probability space $(\Omega, \mathcal{F}, P)$, with filtration $(\mathcal{F}_k)_{k=0,1,\ldots,M}$ and the discounted underlying asset price follows a square integrable process. Denote by V_k the value of the hedging strategy at time t_k and by C_k the cumulative cost of the hedging strategy up to time t_k (this includes the initial cost for setting up the hedging portfolio and the cost for rebalancing it at the hedging times $t_0, \ldots, t_k$).

Currently, there are two main quadratic hedging approaches for choosing an optimal strategy. One possibility is to control the total risk by minimizing the L^2-norm $E((H - V_M)^2)$, where $E(\cdot)$ denotes the expected value with respect to the probability measure P. This is the total risk minimization criterion. An optimal strategy for this criterion is self-financing, that is, its cumulative cost process is constant. A total risk minimizing strategy exists under the additional assumption that the discounted underlying asset price has a bounded mean-variance tradeoff. In this case, the strategy is given by an analytic formula. The existence and the uniqueness of a total risk minimizing strategy have been extensively studied by Schweizer ([15]).

Another possibility is to control the local incremental risk, by minimizing $E((C_{k+1} - C_k)^2|\mathcal{F}_k)$ for all $0 \leq k \leq M-1$. This is the local quadratic risk minimizing criterion. The same assumption that the discounted underlying asset price has a bounded mean-variance tradeoff is sufficient for the existence of an explicit local risk minimizing strategy (see Schäl [14]). This strategy is no longer self-financing, but it is mean-self-financing, i.e., the cumulative cost process is a martingale. In general, the initial costs for the local risk minimizing and total risk minimizing strategies are different. As Schäl noticed, the initial costs agree in the case when the discounted underlying asset price has a deterministic mean-variance tradeoff. He then suggests the interpretation of this initial cost as a *fair* *hedging price* for the option. However, as mentioned by Schweizer ([15]), this is not always appropriate.

The quadratic total and local risk minimizing hedging strategies have many theoretical properties, their existence and uniqueness have been extensively studied and, in the case of existence, they are given by analytic formula. However, the optimal hedging strategies hinge on the criteria for measuring the risk. Therefore, it is important to answer the natural question of how different hedging strategies are under different risk measures. Moreover, how should one choose a risk measure?

In the Black-Scholes framework, an option can be hedged completely, with no risk, i.e., zero in or out cashflows, besides the initial cost. When rebalancing can only be done at discrete times, a natural optimal hedging strategy is the one which minimizes the expected magnitude of the cashflows; this leads to the optimization problems, minimize $E(|H - V_M|)$, or minimize $E(|C_{k+1} - C_k| \, |\mathcal{F}_k)$, respectively.

Coleman, Li and Patron ([2]) investigate the piecewise linear criterion for local risk minimization. They illustrate the fact that piecewise linear local risk minimization may lead to very different, possibly better, hedging strategies. These strategies have a larger probability of small hedging cost and risk, although a very small probability of larger cost and risk than the traditional quadratic risk minimizing strategies. Although there is no analytic solution to the piecewise linear local risk minimization problem, the optimal hedging strategies can be computed very easily.

In this paper, we investigate hedging strategies based on piecewise linear total risk minimization. Minimizing the piecewise linear risk, $E(|H - V_M|)$, and minimizing the quadratic risk, $E((H - V_M)^2)$ are also likely to yield significantly different solutions. Assume that $p(S)$ is the conditional density function of the underlying price at time T. Minimizing $E((H - V_M))$ puts more emphasis on reducing the largest value of $\sqrt{p(S)}|H - V_M|$, whereas minimizing $E(|H - V_M|)$ attempts to reduce the density weighted incremental cashflow, $p(S)|H - V_M|$ for each underlying value S equally.

To illustrate the above discussion in more detail, consider the following comparison between the piecewise linear risk minimization with respect to the total risk measure $E(|H - V_M|)$, and the quadratic risk minimization with respect to $E((H - V_M)^2)$. Suppose the price of the underlying asset satisfies the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$$

where Z_t is a Wiener process. Let the initial value of the asset $S_0 = 100$, the instantaneous expected return $\mu = .2$, the volatility $\sigma = .2$ and the riskless rate of return $r = .1$. Suppose we want to statically hedge a deep in-the-money and a deep out-of-money put option with maturity $T = 1$; we only have one hedging opportunity, at time 0. At the maturity T we compare the payoff of the options with the hedging portfolio values of the strategies obtained by the piecewise linear and quadratic local risk minimization. The payoff and the hedging portfolio values at time T are multiplied by the density function of the asset price and are discounted to time 0. The first plot in Figure 1 shows the density weighted payoff and the density weighted values of the hedging portfolios at the maturity T for the in-the-money put option. The second plot presents the corresponding data for the out-of-money put option.

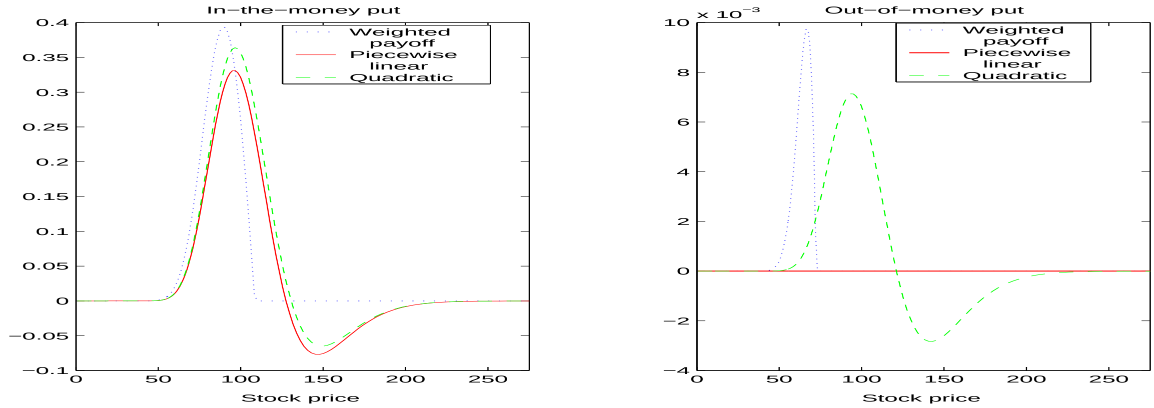

Figure 1: Best fitting of the option payoff

In the case of the in-the-money put option, the weighted payoff, closer to lognormal, is much easier to fit. We remark that in this case both criteria generate similar plots of the hedging strategy values and they fit the option payoff relatively well. However, the weighted payoff for the out-of-money put option seems more difficult to match. Despite the small values (of order 10^{-3}), it is important to note that the relative differences between the weighted payoff and the weighted values of the hedging portfolios are large. (The cost of an out-of-money put is much smaller than the cost of the in-the-money put.) We have illustrated the hedging of only one out-of-money put option; if we want to hedge 100 put options identical to the one considered, the absolute differences between the weighted payoff and the weighted hedging portfolio values will also be significant. The hedging styles of the two strategies are very different. The L^2-norm (i.e., quadratic) attempts to penalize large residuals excessively and this actually leads to a worse fit under most scenarios. Indeed, the probability that the put option expires out of money is very large, around .97, but the L^2-hedging strategy either over or under replicates the option payoff. On the other hand, the L^1-strategy hedges exactly the option payoff when it expires out of money. Suppose we short the out-of-money put option. At the maturity of the option, our possible losses are never greater than the strike price. Assume now that we want to hedge our position by buying the L^2-hedging strategy. We can see from the figure that, by excessively trying to reduce the risk in the unlikely event that the option expires in the money, the L^2-strategy actually introduces the very small probability of unlimited losses. This is not the case if we try to hedge the short position using the L^1-strategy.

The main difficulty in computing the optimal strategies under the piecewise linear total risk minimization criterion is that, because these strategies are self-financing, the total risk, $H - V_M$, depends on the entire path of the stock price. Total risk minimization is a dynamic stochastic programming problem which is computationally challenging to solve. Using a tree method to model the future uncertainties may lead to very expensive computations for solving this stochastic programming problem, since the number of tree nodes increases exponentially as the number of trading opportunities increases. We propose a method for computing the piecewise linear total risk minimizing hedging strategies using Monte Carlo simulation and approximating the holdings in the hedging portfolios by unknown cubic splines which are determined as the solution to an optimization problem.

The key insight underlying our method is similar to the idea behind the Longstaff-Schwartz method for valuing American options ([11]). Essentially, the optimal exercise strategy for an American option is determined by the conditional expected value of the payoff from continuing to keep the option alive. Longstaff and Schwartz compute the optimal exercise strategy for American options using Monte Carlo methods and approximating the conditional expected values of the payoff from continuation by functions of the state variables.

The method we propose for computing the optimal piecewise linear total risk minimizing strategies may also be useful in computing the quadratic total risk minimizing strategies, for example, in the case of the stochastic volatility models. Schweizer ([15]) establishes an analytical formula for the computation of the quadratic risk minimizing strategies when the stock price has a bounded mean-variance tradeoff and Bertsimas et al. ([1]) present a formula based on dynamic programming under the additional assumption of vector-Markov price processes. However, the numerical implementation of these formula may be quite involved in the stochastic volatility framework.

We illustrate our method in the Black-Scholes and stochastic volatility framework. We also investigate the differences between the hedging styles of the trading strategies based on piecewise linear and quadratic risk minimization. The behavior of the different hedging strategies for total risk minimization is similar to the one observed in the case of the local risk minimization (see Coleman et al. [2]). Piecewise linear total risk minimization generally leads to smaller hedging cost and risk than the corresponding quadratic criterion, although there is a very small probability of larger cost and risk.

Both quadratic and piecewise linear risk minimization are symmetric risk measures, since they penalize losses as well as gains. However, when hedging an option, one may be more interested in penalizing only the losses of his position. This leads to minimizing the shortfall risk, $E((H - V_M)^+)$. We remark that, while total risk minimization can be used for both hedging and pricing an option, shortfall risk minimization can only be used for hedging purposes. We investigate criteria for shortfall risk minimization and compare the optimal hedging strategies for these criteria with the quadratic and piecewise linear total risk minimizing strategies. The optimal hedging strategy performances depend on the moneyness of the options and the number of rebalancing opportunities. Analyzing the values of the shortfall risk for the optimal total risk minimizing strategies, suggests that, while quadratic total risk minimization shows no trend for either overhedging, or underhedging, the corresponding piecewise linear criterion typically underhedges the options.

To summarize the main contributions of this paper, we firstly propose a computational method to approximate optimal hedging strategies for total risk minimization under the L^1-risk measure. Secondly, we compare the total risk minimizing hedging strategies for the L^1, L^2 and shortfall risk measures.

Section 2 of the paper describes the different risk minimization criteria for discrete hedging. In Section 3 we present our method for computing the piecewise linear total risk minimizing strategies. We illustrate this method in the Black-Scholes framework and compare the different criteria for total risk minimization in this framework. Section 4 has a similar analysis for a stochastic volatility framework. In Section 5 we investigate criteria for shortfall risk minimization and compare the performance of the hedging strategies for shortfall, piecewise linear and quadratic total risk minimization. We conclude in Section 6.

2. Discrete hedging criteria

Consider a financial market where a risky asset (called stock) and a risk-free asset (called bond) are traded. Let $T > 0$ and assume we only have a finite number of hedging dates over the time horizon $[0, T]$. Let $0 = t_0 < t_1 < \ldots < t_M = T$ denote these discrete hedging times. Suppose the financial market is modeled as a filtered probability space $(\Omega, \mathcal{F}, P)$, with filtration $(\mathcal{F}_k)_{k=0,1,\ldots,M}$, where $\mathcal{F}_k$ corresponds to the hedging time t_k and w.l.o.g. $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is trivial. Suppose, moreover, that the stock price follows a stochastic process $S = (S_k)_{k=0,1,\ldots,M}$, with S_k being $\mathcal{F}_k$-measurable for all $0 \leq k \leq M$. We can set the bond price $B \equiv 1$ by assuming the discounted stock price process $X = (X_k)_{k=0,1,\ldots,M}$, where $X_k = \frac{S_k}{B_k}, \forall 0 \leq k \leq M$.

Assume that we want to hedge a European option with maturity T and payoff given by a $\mathcal{F}_M$-measurable random variable H. For example, $H = (K - X_M)^+$ for a European put with maturity T and discounted strike price K.

A *trading strategy* is given by two stochastic processes $(\xi_k)_{k=0,1,\ldots,M}$ and $(\eta_k)_{k=0,1,\ldots,M}$, where ξ_k is the number of shares held at time t_k and η_k is the amount invested in the bond at time t_k. We assume ξ_k, η_k are $\mathcal{F}_k$-measurable, for all $0 \leq k \leq M$ and $\xi_M = 0$. Consider the portfolio consisting of the combination of the stock and bond given by the trading strategy. The condition $\xi_M = 0$ corresponds to the fact that at time M we liquidate the portfolio in order to cover for the option payoff.

The value of the portfolio at any time t_k, $0 \leq k \leq M$, is given by:

$$V_k = \xi_k X_k + \eta_k.$$

For all $0 \leq j \leq M - 1$, denote by $\Delta X_j = X_{j+1} - X_j$. With this notation, $\xi_j \Delta X_j$ represents the change in value due to the change in the stock price at time t_{j+1} before any changes in the portfolio. Therefore, the *accumulated gain* G_k is given by:

$$G_k(\xi) = \sum_{j=0}^{k-1} \xi_j \Delta X_j, \quad 1 \leq k \leq M$$

and $G_0 = 0$.

The *cumulative cost* at time t_k, C_k, is defined by:

$$C_k = V_k - G_k, \quad 0 \leq k \leq M.$$

A strategy is called *self-financing* if its cumulative cost process $(C_k)_{k=0,1,\ldots,M}$ is constant over time, i.e. $C_0 = C_1 = \ldots = C_M$. This is equivalent to $(\xi_{k+1} - \xi_k)X_{k+1} + \eta_{k+1} - \eta_k = 0$ (a.s.), for all $0 \leq k \leq M - 1$. In other words, any fluctuations in the stock price can be neutralized by rebalancing ξ and η with no inflow or outflow of capital. The value of the portfolio for a self-financing strategy is then given by $V_k = V_0 + G_k$ at any time $0 \leq k \leq M$.

A market is complete if any claim H is attainable, that is, there exists a self-financing strategy with $V_M = H$ (a.s.). If the market is incomplete, for instance in the case of discrete hedging, a claim is, in general, non-attainable and a hedging strategy has to be chosen based on some optimality criterion.

One approach to hedging in an incomplete market is to first impose $V_M = H$. Since such a strategy cannot be self-financing, we should then choose the optimal trading strategy to minimize the incremental cost incurred from adjusting the portfolio at each hedging time. This is the *local risk minimization.* The traditional criterion for local risk minimization is the quadratic criterion, given by minimizing:

$$E((C_{k+1}-C_k)^2|\mathcal{F}_k)\ ,\ 0\le k\le M-1. \tag{1}$$

This criterion is discussed in detail in Föllmer and Schweizer ([4]), Schäl ([14]), Schweizer ([15, 16]).

A quadratic local risk minimizing strategy is guaranteed to exist under the assumptions that H is a square integrable random variable, X is a square integrable process with bounded mean-variance tradeoff, that is:

$$\frac{(E(\Delta X_k|\mathcal{F}_k))^2}{\mathrm{Var}(\Delta X_k|\mathcal{F}_k)} \text{ is P-a.s. uniformly bounded.}$$

Moreover, this hedging strategy is given explicitly by:

$$\begin{cases} \xi_M^{(l)}=0,\ \eta_M^{(l)}=H \\ \xi_k^{(l)}=\dfrac{\mathrm{Cov}(\xi_{k+1}^{(l)}X_{k+1}+\eta_{k+1}^{(l)},X_{k+1}|\ \mathcal{F}_k)}{\mathrm{Var}(X_{k+1}|\mathcal{F}_k)}, & 0\le k\le M-1 \\ \eta_k^{(l)}=E((\xi_{k+1}^{(l)}-\xi_k^{(l)})X_{k+1}+\eta_{k+1}^{(l)}|\mathcal{F}_k), & 0\le k\le M-1. \end{cases} \tag{2}$$

The choice of the quadratic criterion for risk minimization is, however, subjective. Alternatively, one can choose to minimize:

$$E(|C_{k+1}-C_k)|\ |\mathcal{F}_k)\ ,\ 0\le k\le M-1. \tag{3}$$

As illustrated by Coleman et al. ([2]), even if there is no analytic solution to the above piecewise linear risk minimization problem, an optimal hedging strategy can be easily computed. Criterion (3) for piecewise linear local risk minimization leads to significantly different hedging strategies and possibly better hedging results.

Another approach to hedging in an incomplete market is to consider only self-financing strategies. An optimal self-financing strategy is then chosen which best approximates H by its terminal value V_M. The quadratic criterion for this *total risk minimization* is given by minimizing the L^2-norm:

$$E((H-V_M)^2)=E((H-V_0-\sum_{j=0}^{M-1}\xi_j\Delta X_j)^2). \tag{4}$$

By solving the total risk minimization problem (4), we obtain the initial value of the portfolio, V_0, and the number of shares, $(\xi_0,\ldots,\xi_{M-1})$. The amount invested in the bond, $(\eta_0,\ldots,\eta_M)$, is then uniquely determined since the strategy is self-financing. If the discounted stock price is given by a square integrable process with bounded mean-variance tradeoff and if the payoff is given by a square integrable random variable, then problem (4) has a unique solution. The existence and uniqueness of a total risk minimizing strategy under the quadratic criterion have been extensively studied by Schweizer ([15]).

Schweizer gives an analytic formula which relates the holdings and the hedging portfolio values for the quadratic total risk minimizing strategy to the holdings and the portfolio values for the quadratic local risk minimizing strategy:

$$\begin{cases} V_0^{(t)} = \frac{E(H\prod_{j=0}^{M-1}(1-\beta_j\Delta X_j))}{E(\prod_{j=0}^{M-1}(1-\beta_j\Delta X_j))} \\ \xi_M^{(t)} = 0 \\ \xi_k^{(t)} = \xi_k^{(l)} + \beta_k(V_k^{(l)} - V_0^{(t)} - G_k(\xi^{(t)})) + \gamma_k, \quad 0 \le k \le M-1. \end{cases} \tag{5}$$

where the processes $(\beta_k)_{k=0,\dots,M-1}$ and $(\gamma_k)_{k=0,\dots,M-1}$ are given by the formula:

$$\beta_k = \frac{E(\Delta X_k \prod_{j=k+1}^{M-1}(1-\beta_j\Delta X_j)|\mathcal{F}_k)}{E(\Delta X_k^2 \prod_{j=k+1}^{M-1}(1-\beta_j\Delta X_j)^2|\mathcal{F}_k)}$$

$$\gamma_k = \frac{E((V_T^{(l)} - G_T(\xi^{(l)}) - V_k^{(l)} + G_k(\xi^{(l)}))\Delta X_k \prod_{j=k+1}^{M-1}(1-\beta_j\Delta X_j)|\mathcal{F}_k)}{E(\Delta X_k^2 \prod_{j=k+1}^{M-1}(1-\beta_j\Delta X_j)^2|\mathcal{F}_k)}$$

Bertsimas et al. ([1]) also obtains a formula for the quadratic total risk minimizing strategy, using dynamic programming, in the case of vector-Markov price processes.

The corresponding piecewise linear total risk minimization criterion is given by the L^1-norm:

$$E(|H - V_M|) = E(|H - V_0 - \sum_{j=0}^{M-1} \xi_j \Delta X_j|). \tag{6}$$

We are interested in computing optimal hedging strategies given by the piecewise linear total risk minimization problem (6). This is a dynamic stochastic programming problem that is, in general, very difficult to solve. Since $H - V_0 - \sum_{j=0}^{M-1} \xi_j \Delta X_j$ depends on the entire path of the stock price, a direct approach to problem (6) can be very expensive computationally. In order to see this, assume that we use Monte Carlo simulation and we generate L independent scenarios for the stock price. The total risk minimization problem (6) corresponds, in this case, to minimizing the expected total risk over all the scenarios:

$$\min_{\substack{V_0, \xi_0, \xi_j^{(k)} \\ \xi_j : \mathcal{F}_j-\text{measurable}}} \sum_{k=1}^{L} \left| H^{(k)} - V_0 - \xi_0 \Delta X_1^{(k)} - \sum_{j=1}^{M-1} \xi_j^{(k)} \Delta X_j^{(k)} \right| \tag{7}$$

The notation $^{(k)}$ means that the option payoff, the stock price and the holdings correspond to the k^{th} scenario. We remark that at time 0, the stock price is deterministic and, therefore, the holdings in the hedging portfolio at time 0 have to be the same for all the scenarios.

The number of unknowns in problem (7) is of order $L \cdot M$, where L is the number of scenarios and M is the number of rebalancing times. Therefore, trying to solve this problem directly is computationally very challenging when the number of scenarios is large and the rebalancing is frequent.

In order to reduce the complexity of problem (7) we try to approximate the holdings ξ_j. Spline functions have been extensively used for function approximations, since they are very attractive from a computational point of view. We choose to approximate the holdings ξ_j by unknown cubic splines.

The number of unknowns at each hedging time in the problem formulation (7) is equal to the number of scenarios; after approximating the holdings by cubic splines, the number of unknowns at each hedging time is reduced to the number of parameters in the cubic splines, which is typically very small.

An important issue to be considered when approximating the holdings in a hedging strategy by cubic splines is that the optimal hedging strategy has to be path dependent. Indeed, the total risk,

$$H - V_M = H - V_0 - \sum_{j=0}^{M-1} \xi_j \Delta X_j,$$

minimized by the optimal hedging strategy, depends on the entire path of the stock price. Although the holdings $(\xi_j)_{j=0,\ldots,M-1}$ are computed at time 0 and any measurable $(\xi_j)_{j=0,\ldots,M-1}$ is an admissible hedging strategy, intuitively, at any time t_j, $0 \leq j \leq M-1$, the optimal holdings ξ_j will have an intrinsic information about the past history of the stock price and the optimal holdings up to time t_j.

In this paper, we describe a method for solving the total risk minimization problem (6) by approximating the holdings in the optimal hedging strategy with unknown cubic splines and trying to capture the path dependency of the strategy by a simple spline formulation. The unknown cubic splines are determined as solutions of an optimization problem that consists in minimizing the total risk over a set of scenarios for the stock price. Since the strategy computed in this way is suboptimal we have to analyze its degree of optimality. We also compare the hedging strategies based on the piecewise linear total risk minimization criterion, to the traditional strategies based on quadratic total risk minimization.

3. Total risk minimization in the Black-Scholes framework

We will first describe our method in the Black-Scholes framework. We suppose that the stock price is given by the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t, \tag{8}$$

where Z_t is a Brownian motion. We also assume that the writer of a European option with maturity T has only M hedging opportunities at $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M := T$ to hedge his position using the underlying stock and a bond.

Using Monte Carlo simulation, we generate L independent samples for the stock price, based on equation (8). We want to determine the holdings in the hedging strategy such that the expected total risk over all the scenarios is minimized.

The total risk minimization problem for the piecewise linear criteria becomes:

$$\min_{\substack{V_0, \xi_0, \xi_j^{(k)} \\ \xi_j : \mathcal{F}_j - \text{measurable}}} \sum_{k=1}^{L} \left| H^{(k)} - V_0 - \xi_0 \Delta X_1^{(k)} - \sum_{j=1}^{M-1} \xi_j^{(k)} \Delta X_j^{(k)} \right|. \tag{9}$$

As before, the notation $^{(k)}$ refers to the k^{th} scenario.

3.1. First formulation

We want to reduce the complexity of the above problem by approximating the holdings ξ_k. We first choose to ignore the fact that the hedging strategy is path dependent and assume that the amount $\xi_j X_j$ invested in the stock at any time t_j depends only on the stock price at time t_j. We will investigate the degree of optimality that can be achieved under this assumption and we will pursue subsequent refinement of this assumption. This is a natural assumption, since one should take into account the current value of the stock price, X_j, when rebalancing the portfolio at time t_j. Thus, we can assume:

$$\xi_j = D_j(X_j), \qquad \forall j = 1, \ldots, M-1, \tag{10}$$

with D_j unknown functions. Let us suppose that the holdings depend continuously on the stock price, that is, D_j is a continuous function, $\forall j = 1, \ldots, M-1$. We denote by D_0 the constant function identically equal to ξ_0. The total risk minimization problem under the piecewise linear criterion becomes:

$$\min_{V_0, D_0, \ldots, D_{M-1}} \sum_{k=1}^{L} \left| H^{(k)} - V_0 - \sum_{j=0}^{M-1} D_j(X_j^{(k)}) \Delta X_j^{(k)} \right|. \tag{11}$$

In order to make the above problem computationally attractive, we assume that each function D_j is a cubic spline with fixed end conditions and spline knots placed with respect to the stock price. The function D_j is then uniquely determined by its values at the spline knots. Note that D_j is a linear function of its knot values. In this way, problem (11) becomes an L^1-optimization problem with unknowns V_0, D_0 and the values of the cubic splines D_j, $j \geq 1$ at their knots.

The number of knots for each spline in our implementation is typically very small (around 8) and independent of the number of scenarios. Therefore, the number of unknowns in the L^1-optimization problem (11) is of order M, where M is the number of rebalancing times. We can now solve this problem and compute the piecewise linear risk minimizing strategy that satisfies assumption (10) on the special form of the holdings ξ_j.

The question that arises is how good assumption (10) is. In order to answer this question, we will investigate the quadratic total risk minimization problem (4). We can compute the quadratic risk minimizing strategy either by solving an optimization problem similar to (11), or by using the theoretical formula (5). By comparing the hedging strategies obtained by these two methods, we will try to assert the quality of assumption (10).

We can modify the quadratic risk minimization problem (4), using an approach similar to the one described above for piecewise linear risk minimization. Under the assumption, $\xi_j = D_j(X_j)$, $\forall j = 1, \ldots, M-1$ and with the notation $D_0 \equiv \xi_0$, the problem becomes:

$$\min_{V_0, D_0, \ldots, D_{M-1}} \sum_{k=1}^{L} (H^{(k)} - V_0 - \sum_{j=0}^{M-1} D_j(X_j^{(k)}) \Delta X_j^{(k)})^2 \tag{12}$$

We obtain, therefore, the optimal quadratic risk minimizing hedging strategy which satisfies assumption (10).

Another method for solving problem (4) is to use Schweizer's analytic solution (5) and compute the optimal quadratic risk minimizing strategy, in the general case, with no assumption on the form of the holdings. In the Black-Scholes model, the mean-variance of the stock price is not only bounded, but also deterministic. As mentioned in Schweizer's paper ([15]), formula (5) reduces in this case to:

$$\begin{cases} V_0^{(t)} = V_0^{(l)} \\ \xi_M^{(t)} = 0 \\ \xi_k^{(t)} = \xi_k^{(l)} + \alpha_k (V_k^{(l)} - V_0^{(l)} - G_k(\xi^{(t)})), \quad 0 \le k \le M-1. \end{cases} \tag{13}$$

where the process $(\alpha_k)_{k=0,\ldots,M-1}$ is given by:

$$\alpha_k = \frac{E(\Delta X_k|\mathcal{F}_k)}{E(\Delta X_k^2|\mathcal{F}_k)}$$

We first compute the quadratic local risk minimizing strategy, as given by formula (2). The details of this computation are given in Coleman et al. ([2]). We then use formula (13) to obtain the holdings in the total risk minimizing hedging portfolio for each scenario.

The total risk minimizing hedging strategy computed from the analytical formula (13) in the above manner, is used as a benchmark for the solution of the quadratic risk minimization problem (12), in order to evaluate the validity of the assumption (10).

We also want to compare the effectiveness of the hedging strategies based on piecewise linear risk minimization and, respectively, quadratic risk minimization.

The numerical results presented below refer to hedging put options with maturity $T = 1$ and different strike prices. The initial stock price is $S_0 = 100$. The instantaneous expected return of the stock price is $\mu = .15$, the volatility, $\sigma = .2$ and the riskless rate of return, $r = .04$. The number of scenarios in the Monte Carlo simulation of the stock price is $L = 40000$ and the number of time steps in this simulation is 600.

We have computed three risk minimizing hedging strategies:

- Strategy 1: Piecewise linear risk minimizing strategy satisfying (10)
- Strategy 2: Quadratic risk minimizing strategy satisfying (10)
- Strategy 3: Quadratic risk minimizing strategy given by the analytical formula (13)

For each of these strategies and each scenario, we compute the following:

- *Total cost*:

$$H - \sum_{k=0}^{M-1} \xi_k \Delta X_k \tag{14}$$

  This is the total amount of money necessary for the writer to implement the self-financing hedging strategy and honor the option payoff at expiry. Since the hedging strategy is self-financing, there are no intermediate costs for rebalancing the hedging portfolio.

- *Total risk*:

$$|H - V_M| \tag{15}$$

This measures the difference between the final value of the hedging portfolio and the option payoff. The strategy being self-financing, it is the only unplanned cost or income.

Tables 1 and 2 show the average cumulative cost and average total risk over 40000 simulated scenarios, for different number of time steps per rebalancing time. The last column in these tables correspond to the case of the static hedge, when we only have one hedging opportunity at time 0.

Table 1: Average value of the total cost over 40000 scenarios

		# of time steps per rebalancing time				
Strike	Strategy	25	50	100	300	600
90	1	2.2194	1.9764	1.0876	0.9398	0.9398
	2	2.4540	2.4033	2.3155	2.0400	1.7421
	3	2.4838	2.4387	2.3474	2.0429	1.7454
95	1	3.7878	3.6356	3.2435	1.6648	1.6648
	2	3.9512	3.8830	3.7647	3.4006	2.9735
	3	3.9770	3.9188	3.8022	3.4018	2.9745
100	1	5.8421	5.7082	5.5074	4.0392	2.7269
	2	5.9183	5.8396	5.6983	5.2566	4.6948
	3	5.9413	5.8773	5.7399	5.2565	4.6928
105	1	8.3549	8.2549	8.1113	7.2494	5.5301
	2	8.3613	8.2809	8.1280	7.6307	6.9449
	3	8.3866	8.3221	8.1724	7.6303	6.9392
110	1	11.2609	11.1988	11.0950	10.6364	9.2160
	2	11.2566	11.1789	11.0264	10.4994	9.7148
	3	11.2858	11.2221	11.0713	10.5007	9.7072

Average total cost for put options with $T = 1$, different strike prices and number of timesteps per rebalancing time, for strategies: 1 - piecewise linear with (10), 2 - quadratic with (10) and 3 - quadratic given by analytical formula; $S_0 = 100$, $\mu = .15$, $\sigma = .2$, $r = .04$.

We remark that, in the case of Strategy 1, the average values of the cumulative cost in Table 1 and total risk in Table 2 are equal for some of the put options considered, as for example, the out-of-money put options with 1 or 2 hedging opportunities. This happens because the holdings in the optimal hedging portfolio of Strategy 1 are zero. Therefore, if the put option is not in-the-money and the number of rebalancing opportunities is sufficiently small, the optimal hedging Strategy 1 is not to hedge at all. This is intuitively quite reasonable since the likelihood of the option expiring out-of-money is large and one has no opportunity of further adjusting the hedging portfolio. The optimal hedging strategies 2 and 3, on the other hand, still choose to hedge these particular put options. We remark that out-of-money put options with more hedging opportunities are hedged by Strategy 1. Experiments show that out-of-money put options which are closer to expiry will be hedged by Strategy 1.

When the rebalancing is infrequent, the average values of the total risk for the quadratic risk minimizing strategies 2 and 3 are very close. The same can be observed for the cumula-

Table 2: Average value of the total risk over 40000 scenarios

Strike	Strategy	# of time steps per rebalancing time 25	50	100	300	600
90	1	0.6031	0.7822	0.9276	0.9398	0.9398
	2	0.6312	0.8410	1.1212	1.5727	1.7707
	3	0.5336	0.7450	1.0377	1.5799	1.7759
95	1	0.7761	1.0419	1.3829	1.6648	1.6648
	2	0.7918	1.0771	1.4687	2.1945	2.6222
	3	0.6885	0.9641	1.3592	2.1993	2.6251
100	1	0.9790	1.2921	1.7293	2.5544	2.7269
	2	0.9877	1.3144	1.7784	2.7944	3.5117
	3	0.8295	1.1636	1.6479	2.7914	3.5119
105	1	1.1000	1.4535	1.9668	3.1622	3.9566
	2	1.1068	1.4677	2.0051	3.2892	4.3184
	3	0.9465	1.3180	1.8694	3.2774	4.3170
110	1	1.1240	1.5192	2.0798	3.4688	4.7912
	2	1.1308	1.5344	2.1240	3.6189	4.9366
	3	1.0147	1.4171	2.0036	3.6027	4.9355

Average total risk for the hedging of put options with different strike prices and different number of time steps per rebalancing time, for the three strategies and in the setup described in Table 1.

tive cost. However, as the rebalancing becomes frequent enough, the total risk for Strategy 2 becomes larger than the total risk for Strategy 3. The results maintain the same trend even if we increase the number of spline knots or change their position. This suggests that the constraint (10), on the form of the holdings leads to supplementary risk and a better assumption has to be found.

The numerical results in Tables 1 and 2 illustrate that the hedging strategies based on the piecewise linear and, respectively, quadratic risk minimization perform differently in terms of average cumulative cost and risk. In the case of the in-the-money put options, the values of the average cumulative cost are very close for all the three strategies. However, as the option becomes out-of-money and the rebalancing is less frequent, the average cumulative cost for Strategy 1 is almost half the average cumulative cost of the quadratic strategies. The average total risk has the same trend. Nevertheless, since it may be possible to eliminate part of the total risk for Strategies 1 and 2, by using a less restrictive constraint than (10), the above results do not show very clearly the difference between the piecewise linear and the quadratic risk minimizing strategies. The numerical results obtained with a better assumption on the form of the holdings will allow further discussion on this subject.

3.2. Second formulation

As illustrated above, the constraint that the holdings at any time t_j depend only on the current stock price, $\xi_j = C_j(X_j)$, may be too restrictive. In order to obtain a better formulation, let us analyze in more detail the holdings satisfying assumption (10). Consider the particular case of the at-the-money put options with 6 hedging opportunities. Figure 2 shows the number of shares in the optimal hedging portfolio after the third rebalancing opportunity, for the quadratic risk minimizing Strategies 2 and 3.

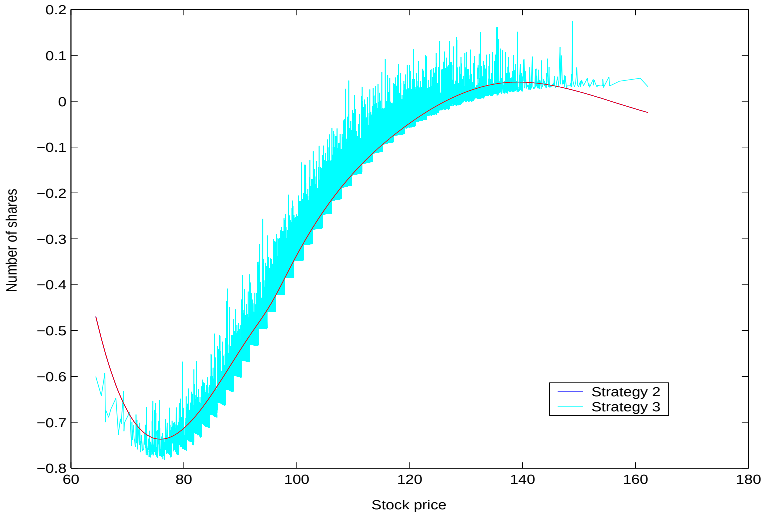

Figure 2. Number of shares in the hedging portfolio after the third rebalancing time for the at-the-money put option with 6 rebalancing opportunities

We can see that in the case of Strategy 3, for the same value of the current stock price, we may have different number of shares in the hedging portfolio for the different scenarios. This is because this hedging strategy depends not only on the current value of the stock price, but also on the path of the stock price up to the current time. However, since the holdings for Strategy 2 satisfy assumption (10), they only depend on the current stock price and this assumption can become too restrictive. Note, however, that the holdings obtained under (10) capture quite well the trend of the optional holdings. To further reduce the risk, we have to incorporate the dependence on the path of the stock price in the assumption on the form of the holdings.

Strategy 2 considers only the hedging strategies for which the amount invested in the stock at any time t_j depends only on the stock price X_j at time t_j. It may be more natural to assume, however, that the investment in the stock at time t_j also depends on the cumulative gain up to time t_j. We assume that the holdings depend linearly on the past gain, specifically:

$$\xi_j = D_j(X_j) + \frac{1}{X_j}\sum_{i=0}^{j-1} \xi_i \Delta X_i, \qquad \forall j = 1, \ldots, M-1$$

with D_j unknown cubic splines. As before, we make the convention $D_0 \equiv \xi_0$. After some algebraic manipulation and ignoring the higher order terms containing products $\Delta X_{i_1} \Delta X_{i_2}$, we obtain:

$$\xi_j = D_j(X_j) + \frac{1}{X_j}\sum_{i=0}^{j-1} D_i(X_i) \Delta X_i, \qquad \forall j = 0, \ldots, M-1.$$

We introduce more degrees of freedom in the above formulation by allowing the effect of the current stock price, X_j, on the holdings at time t_j to be different from the effect of the past stock prices, $X_0, \ldots, X_{j-1}$.

The assumption on the form of the holdings ξ_j becomes:

$$\xi_j = D_j(X_j) + \frac{1}{X_j} \sum_{i=0}^{j-1} \tilde{D}_i(X_i) \Delta X_i, \qquad \forall j = 0, \ldots, M-1, \tag{16}$$

where for $j \geq 1$, D_j and $\tilde{D}_j$ are unknown cubic splines with fixed end conditions and spline knots, while D_0, $\tilde{D}_0$ are constant functions. With this formulation, the piecewise linear optimization problem (6) becomes:

$$\min_{V_0, D_j, \tilde{D}_j} \sum_{k=1}^{L} \left| H^{(k)} - V_0 - \sum_{j=0}^{M-1} \left(D_j(X_j^{(k)}) + \sum_{i=0}^{j-1} \tilde{D}_j(X_j^{(k)}) \frac{\Delta X_i^{(k)}}{X_j^{(k)}} \right) \Delta X_j^{(k)} \right| \tag{17}$$

Problem (17) can be interpreted, similarly to problem (11), as a L^1-optimization problem with unknowns V_0, D_0, $\tilde{D}_0$ and the values of the cubic splines D_j, $\tilde{D}_j$, $j \geq 1$ at their knots.

The corresponding formulation for the quadratic risk minimization criterion is:

$$\min_{V_0, D_j, \tilde{D}_j} \sum_{k=1}^{L} \left(H^{(k)} - V_0 - \sum_{j=0}^{M-1} \left(D_j(X_j^{(k)}) + \sum_{i=0}^{j-1} \tilde{D}_j(X_j^{(k)}) \frac{\Delta X_i^{(k)}}{X_j^{(k)}} \right) \Delta X_j^{(k)} \right)^2 \tag{18}$$

We note that the number of knots for each spline is usually small (around 8). The number of unknowns in the above problems is approximately double to the number of unknowns in the previous formulation.

The optimization problems (17) and (18) allow us to compute the optimal piecewise linear and, respectively, quadratic risk minimizing strategies satisfying assumption (16) on the form of the holdings in the hedging portfolio. We can now investigate the quality of this assumption using the three strategies:

- Strategy 1: Piecewise linear risk minimizing strategy satisfying (16)
- Strategy 2: Quadratic risk minimizing strategy satisfying (16)
- Strategy 3: Quadratic risk minimizing strategy given by the analytical formula (13)

We first re-examine the case considered in Figure 3 of the at-the-money put option with 6 hedging opportunities. The number of shares in the optimal hedging portfolio for Strategies 2 and 3, after the third rebalancing time is shown in Figure 3. We remark that the values of the holdings for the optimal quadratic Strategy 2 satisfying constraint (16) follow closely the values of the holdings for the theoretical quadratic Strategy 3.

Tables 3 and 4 show the average values over 40000 scenarios of the cumulative cost and total risk, as defined before, for the above hedging strategies and different numbers of hedging opportunities.

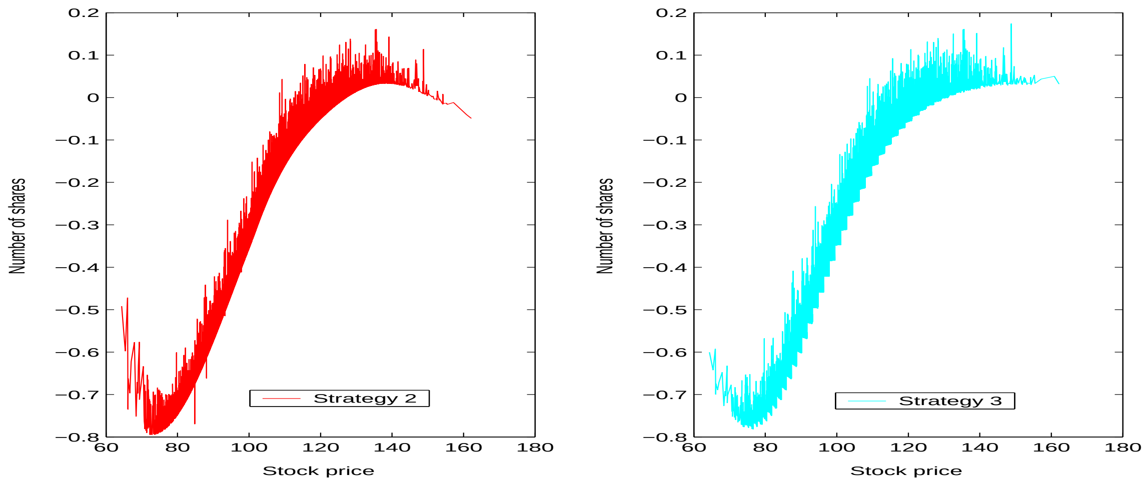

Figure 3. Number of shares in the hedging portfolio after the third rebalancing time for the at-the-money put option with 6 rebalancing opportunities

We remark that in the case of one hedging opportunity, the assumptions (10) and (16) on the form of the holdings coincide and, therefore, the last column in Tables 3 and 4 has the same results as the last column in Tables 1 and, respectively 2.

As noticed before, the optimal hedging Strategy 1 for some of the put options which are not in-the-money and have very few rebalancing opportunities, is not to hedge at all. This is shown by the fact that the holdings in the hedging portfolios for these options are zero, which implies that the average cumulative cost and the average total risk are equal.

In contrast with the numerical results presented earlier, the quadratic strategies 2 and 3 now yield very close values for the average cumulative cost in Table 3 and, respectively, the average total risk in Table 4. We conclude that imposing the constraint (16) on the form of the holdings in the hedging portfolio does not affect significantly the optimal value of the average total hedging risk over the 40000 simulated scenarios.

Table 3: Average value of the total cost over 40000 scenarios

		# of time steps per rebalancing time				
Strike	Strategy	25	50	100	300	600
90	1	2.2728	2.1093	1.5031	0.9398	0.9398
	2	2.4504	2.4086	2.3224	2.0388	1.7421
	3	2.4838	2.4387	2.3474	2.0429	1.7454
95	1	3.7964	3.6640	3.4080	1.6648	1.6648
	2	3.9443	3.8885	3.7741	3.3983	2.9735
	3	3.9770	3.9188	3.8022	3.4018	2.9745
100	1	5.8223	5.6896	5.5067	4.0644	2.7269
	2	5.9118	5.8455	5.7119	5.2530	4.6948
	3	5.9413	5.8773	5.7399	5.2565	4.6928
105	1	8.2982	8.1835	8.0393	7.2893	5.5301
	2	8.3584	8.2882	8.1412	7.6261	6.9449
	3	8.3866	8.3221	8.1724	7.6303	6.9392
110	1	11.2072	11.1146	10.9945	10.6934	9.2160
	2	11.2569	11.1881	11.0413	10.4945	9.7148
	3	11.2858	11.2221	11.0713	10.5007	9.7072

Average total cost for put options with different strike prices and number of time steps per rebalancing time, for the three strategies: 1 - piecewise linear with (16), 2 - quadratic with (16) and 3 - quadratic given by analytical formula; same setup as in Table 1.

Table 4: Average value of the total risk over 40000 scenarios

		# of time steps per rebalancing time				
Strike	Strategy	25	50	100	300	600
90	1	0.5033	0.6819	0.8874	0.9398	0.9398
	2	0.5450	0.7497	1.0325	1.5722	1.7707
	3	0.5336	0.7450	1.0377	1.5799	1.7759
95	1	0.6575	0.9062	1.2512	1.6648	1.6648
	2	0.6952	0.9662	1.3551	2.1908	2.6222
	3	0.6885	0.9641	1.3592	2.1993	2.6251
100	1	0.8246	1.1269	1.5635	2.5524	2.7269
	2	0.8563	1.1789	1.6518	2.7843	3.5117
	3	0.8295	1.1636	1.6479	2.7914	3.5119
105	1	0.9380	1.2800	1.7897	3.1551	3.9566
	2	0.9722	1.3319	1.8802	3.2738	4.3184
	3	0.9465	1.3180	1.8694	3.2774	4.3170
110	1	1.0140	1.3806	1.9099	3.4619	4.7912
	2	1.0460	1.4279	2.0079	3.6025	4.9366
	3	1.0147	1.4171	2.0036	3.6027	4.9355

Average total risk for put options with different strike prices and number of time steps per rebalancing time, for the three strategies and in the setup described in Table 3.

The numerical results suggest that assumption (16) leads to smaller average total hedging risk than assumption (10). In the case of the quadratic risk minimization, the average total hedging risk is very close to optimal. Therefore, we use the optimization problems (17) and (18) to compute the optimal hedging strategies under the piecewise linear and the quadratic risk minimizing criteria.

Tables 3 and 4 allow a clearer comparison of the hedging strategies based on the two criteria for risk minimization. We remark that the performance of these strategies depends on the moneyness of the options and on the number of rebalancing opportunities. The piecewise linear risk minimizing strategy yields a smaller average cumulative cost and risk for almost all the options considered. However, for in-the-money put options the values for the average cumulative cost and, respectively, total risk are close for all three strategies. The differences tend to increase as the put options are out-of-money and the rebalancing is less frequent. For the out-of-money put options with only 1 or 2 hedging opportunities the average cumulative cost for Strategy 1 is almost half the average cumulative cost for Strategies 2 and 3. The same happens for the average total risk.

Even if the market is incomplete due to the discrete hedging, many practitioners are still using delta hedging in order to hedge an option in the current framework. They choose a self-financing strategy such that the initial value of the hedging portfolio, V_0, is given by the value of the option at t_0, as computed by the Black-Scholes formula and the number of shares, ξ_k, at any hedging time t_k is equal to the delta of the option at t_k,

$$\xi_k = \left(\frac{\partial V}{\partial S}\right)_{t_k},$$

where V denotes the value of the option as given by the Black-Scholes formula. However, delta hedging insures a risk-free replication of the option only if the hedging is continuous. In the case of discrete rebalancing, delta hedging is no longer optimal since the corresponding portfolio is only instantaneously risk-free and the risk-free position does not last till the next rebalancing time. Tables 5 and 6 show the average values of the cumulative cost and risk over the 40000 generated scenarios for the delta hedging strategy in comparison to the piecewise linear and quadratic risk minimizing strategies satisfying assumption (16) - Strategies 1 and 2, respectively.

Table 5: Average value of the total cost over 40000 scenarios

		# of time steps per rebalancing time				
Strike	Strategy	25	50	100	300	600
90	1	2.2728	2.1093	1.5031	0.9398	0.9398
	2	2.4504	2.4086	2.3224	2.0388	1.7421
	Delta	2.5583	2.5859	2.6454	2.8838	3.2819
95	1	3.7964	3.6640	3.4080	1.6648	1.6648
	2	3.9443	3.8885	3.7741	3.3983	2.9735
	Delta	4.0702	4.1028	4.1763	4.4830	4.9793
100	1	5.8223	5.6896	5.5067	4.0644	2.7269
	2	5.9118	5.8455	5.7119	5.2530	4.6948
	Delta	6.0483	6.0897	6.1734	6.5382	7.1098
105	1	8.2982	8.1835	8.0393	7.2893	5.5301
	2	8.3584	8.2882	8.1412	7.6261	6.9449
	Delta	8.5011	8.5505	8.6407	9.0457	9.6607
110	1	11.2072	11.1146	10.9945	10.6934	9.2160
	2	11.2569	11.1881	11.0413	10.4945	9.7148
	Delta	11.4019	11.4537	11.5484	11.9712	12.5952

Average total cost for put options with different strike prices and number of time steps per rebalancing time, for the three strategies: 1 - piecewise linear with (16), 2 - quadratic with (16) and 3 - delta hedging; same setup as in Table 3.

Table 6: Average value of the total risk over 40000 scenarios

Strike	Strategy	# of time steps per rebalancing time 25	50	100	300	600
90	1	0.5033	0.6819	0.8874	0.9398	0.9398
	2	0.5450	0.7497	1.0325	1.5722	1.7707
	Delta	0.6366	0.8935	1.2681	2.2099	3.2836
95	1	0.6575	0.9062	1.2512	1.6648	1.6648
	2	0.6952	0.9662	1.3551	2.1908	2.6222
	Delta	0.8042	1.1325	1.6160	2.8786	4.2846
100	1	0.8246	1.1269	1.5635	2.5524	2.7269
	2	0.8563	1.1789	1.6518	2.7843	3.5117
	Delta	0.9481	1.3385	1.9128	3.4582	5.1359
105	1	0.9380	1.2800	1.7897	3.1551	3.9566
	2	0.9722	1.3319	1.8802	3.2738	4.3184
	Delta	1.0576	1.4881	2.1282	3.8736	5.7216
110	1	1.0140	1.3806	1.9099	3.4619	4.7912
	2	1.0460	1.4279	2.0079	3.6025	4.9366
	Delta	1.1144	1.5725	2.2450	4.0892	5.9833

Average total risk for put options with different strike prices and number of time steps per rebalancing time, for the three strategies and in the setup described in Table 5.

We remark that when the rebalancing is frequent, the values of the total hedging cost and risk for the delta hedging strategy are very close, though slightly larger than the corresponding values for the piecewise linear and quadratic total risk minimizing strategies. However, as the number of rebalancing opportunities decreases, delta hedging an option leads to much larger hedging cost and risk than hedging the option by any of the two optimal hedging strategies for total risk minimization.

Next we analyze the distributions of the cumulative cost and total risk for the at-the-money put option with 6 hedging opportunities. The average cumulative cost from Table 3 is 5.5067 for Strategy 1, 5.7119 for Strategy 2 and 5.7399 for Strategy 3. The histograms for each strategy of the cumulative cost over the 40000 simulated scenarios are presented in Figure 4. We mention that all three strategies have very few values of the cumulative cost larger than the range of values illustrated in Figure 4, however, we chose this range in order to make the figure clearer.

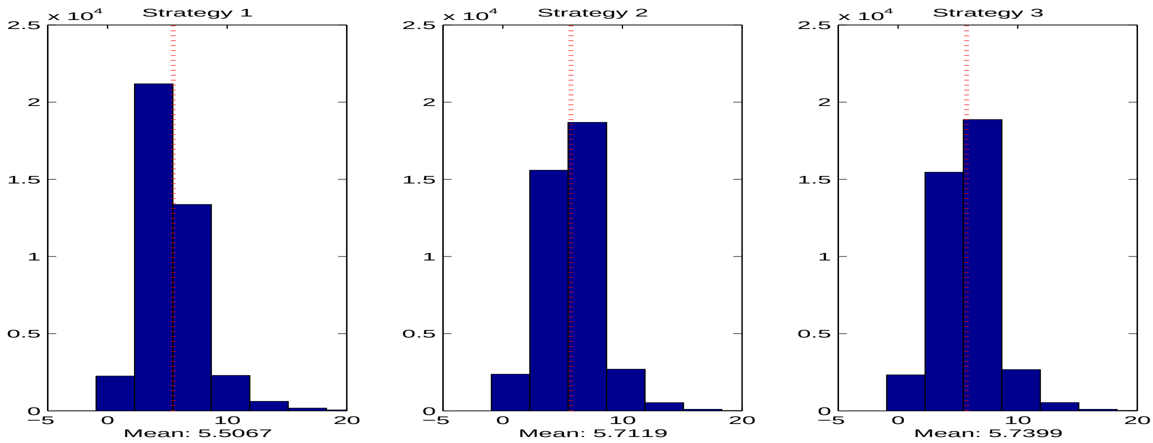

Figure 4. Histograms of the total hedging cost over 40000 scenarios

The distribution of the cumulative cost for Strategy 1 is more asymmetric about its mean compared to the distributions for Strategies 2 and 3. About 60% of the cumulative costs for Strategy 1 are less than the mean, while in the case of the quadratic strategies 2 and 3 the median is almost equal to the mean. The skewness of the distributions, which is another indication of the asymmetry of the data, is equal to 1.9012 for Strategy 1, 0.8017 for Strategy 2 and 0.8394 for Strategy 3.

We remark, however, that while Strategy 1 has a larger probability of smaller hedging cost, it also has a small probability of larger hedging costs than Strategies 2 and 3.

The next figure, Figure 5, shows the histograms of the total risk over the simulated scenarios, for each hedging strategy. As in the case of Figure 4, the range of values in Figure 5 was chosen for clarity, though all the strategies lead to very few values of the total risk larger than the values in the chosen interval.

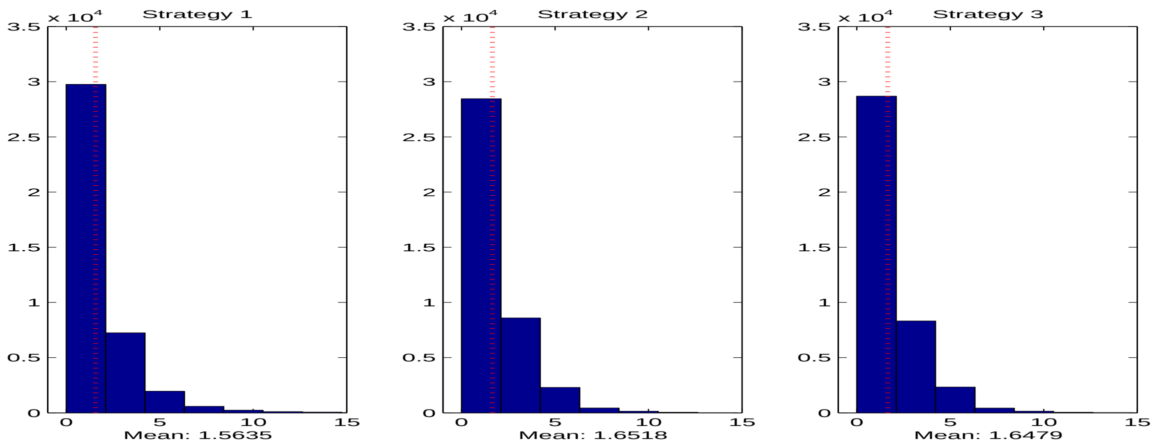

Figure 5. Histograms of the total hedging risk over 40000 scenarios

The distributions of the total hedging risk for the three strategies have similar shapes. However, the mean for Strategy 1 is smaller than the mean for the quadratic strategies. The mean values of the total risk, as given in Table 4, are 1.5635, 1.6518 and, respectively, 1.6479. 65% of the total risk for Strategy 1 is less than the mean, while this happens 62% of the time for Strategies 2 and 3. The skewness in the case of Strategy 1 is 3.4414, larger than the skewness for Strategy 2, 2.0153, and Strategy 2, 2.1058. We note, however, that, as in the case of the total hedging cost, Strategy 1 has also a small probability of larger risk than Strategies 2 and 3. We remark that the distributions of Strategies 2 and 3, for both cumulative cost and risk, are very similar, another indication that (16) is sufficiently flexible to capture the optimal risk performance.

A similar behavior of the strategies based on the piecewise linear and quadratic criteria has been observed in the case of the local risk minimization, as shown by Coleman et al. ([2]). Table 7 presents, for comparison, the average cumulative cost over the same 40000 scenarios for the optimal piecewise linear (Strategy 1) and quadratic (Strategy 2) local risk minimizing hedging strategies. We do not include the results for the average risk, since the risk measure has different meanings in the case of the local risk minimization and the total risk minimization.

Table 7: Average value of the total cost over 40000 scenarios for local risk minimization

		# of time steps per rebalancing time				
Strike	Strategy	25	50	100	300	600
90	1	2.1933	2.1043	1.8592	1.1690	0.9398
	2	2.4846	2.4377	2.3487	2.0424	1.7454
95	1	3.7284	3.6485	3.3907	2.1243	1.6648
	2	3.9785	3.9178	3.8036	3.4008	2.9745
100	1	5.7803	5.7698	5.5225	4.2964	2.7269
	2	5.9433	5.8765	5.7414	5.2550	4.6928
105	1	8.3483	8.4152	8.1908	7.4178	5.5301
	2	8.3889	8.3220	8.1738	7.6285	6.9392
110	1	11.3760	11.5276	11.3383	11.0652	9.2160
	2	11.2883	11.2226	11.0723	10.4989	9.7072

Average total for the hedging of put options with different strike prices and number of rebalancing opportunities, for the two strategies: 1 - piecewise linear local risk minimization, 2 - quadratic local risk minimization; same setup as in Table 3.

As mentioned by Schäl ([14]), when the stock price has a deterministic mean-variance trade-off, the expected total hedging cost for the optimal quadratic local risk minimizing strategy is equal to the expected total hedging cost for the optimal quadratic total risk minimizing strategy. We remark that the average cumulative cost for the quadratic local risk minimizing Strategy 2 in Table 7 is very close to the average cumulative cost for the quadratic total risk minimizing strategies 2 and 3 in Table 3 for all the put options considered. Schäl ([14]) suggests the interpretation of the total hedging cost as a *fair hedging price* for the option. However, an example given by Mercurio and Vorst ([12]), shows that this is not always appropriate.

We note that, in the case of static hedging, that is only one hedging opportunity, the local risk minimization and the total risk minimization criteria coincide. This is why the numerical results for the piecewise linear and the theoretical quadratic risk minimizing strategies in the last column of Table 3 are the same as the corresponding results in Table 7.

In the case of the local risk minimization the hedging performance of the strategies also depends on the moneyness of the options and on the number of rebalancing opportunities, with the average cumulative cost for the piecewise linear local risk minimizing strategy being the smaller for the out-of-money and at-the-money put options. However, for in-the-money put options, the quadratic local risk minimizing strategy is slightly better, even though the values are close. The total risk minimization shows an improvement in terms of total hedging cost for the piecewise linear criterion, especially in the case of in-the-money put option. As a result, the average cumulative cost for the piecewise linear total risk minimizing strategy is the smallest for almost all the put options considered.

As shown by Coleman et al. ([2]), the values of the optimal hedging portfolios for local risk minimization satisfy discrete hedging put-call parity. This is also true in the case of the total risk minimization, the proof being very similar.

Suppose that we have computed the optimal holdings ξ^p, η^p in the portfolio for hedging a put option with maturity T, discounted strike price K and M hedging opportunities at $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M := T$. We can derive a relation between these holdings and the corresponding optimal holdings ξ^c, η^c for the call option on the same underlying asset and with the same maturity, strike price and hedging opportunities. We have the following property:

$$\begin{cases} \xi_k^c = \xi_k^p + 1 \\ \eta_k^c = \eta_k^p - K \end{cases}$$

for all $0 \leq k \leq M - 1$.

Moreover, the discounted values of the portfolios for hedging the put and the call options, V_k^p and V_k^c, satisfy the following put-call parity relation for all $0 \leq k \leq M$:

$$V_k^c - V_k^p = X_k - K.$$

Similarly, the relation between the cumulative costs for the call and put options is given by:

$$C_k^c = C_k^p + X_0 - K,$$

for all $0 \leq k \leq M$.

Therefore, if we know the optimal strategy for hedging the put option, we can compute the optimal strategy for the call, directly, without solving any optimization problems.

4. Total risk minimization in a stochastic volatility framework

In this section we assume that the stock price follows a Heston type stochastic volatility model ([8]). The discounted stock price X and its volatility Y satisfy a stochastic differential equation of the form:

$$\frac{dX_t}{X_t} = \alpha Y_t dt + Y_t dZ_t \qquad (19)$$
$$dY_t = \left(\frac{4\beta\theta - \delta^2}{8Y_t} - \frac{\beta}{2}Y_t\right) dt + \frac{\delta}{2} dZ'_t$$

where Z_t and Z'_t are Brownian motions with instantaneous correlation ρ.

In the Heston type model, the square of the volatility, $F := Y^2$ is a Cox-Ingersoll-Ross type process satisfying the stochastic differential equation:

$$dF_t = \beta(\theta - F_t)dt + \delta\sqrt{F_t}dZ'_t \qquad (20)$$

As in the previous section, we assume the writer of a European option wants to hedge his position using only the underlying stock and a bond, but he only has a finite number of hedging opportunities.

Formula (5) given by Schweizer ([15]), or the formula presented by Bertsimas et al. ([1]), can be used to compute the optimal quadratic total risk minimizing strategy. We compute both the piecewise linear and quadratic risk minimizing strategies as given by the optimization problems (17) and (18) using Monte Carlo implementation.

Since the formulation of problems (17) and (18) depends on the entire stock price path, we are interested in generating strongly convergent discrete path approximations to the stochastic differential equations (19) and (20). We use Euler's method for equations (19) and (20) to generate scenarios for the stock price and volatility.

The parameters for our numerical experiments are chosen as in Heath et al. ([6],[7]), in which the authors investigate continuous hedging under the total and local quadratic risk minimizing criteria and provide comparative numerical results for a class of stochastic volatility models. The values of the parameters are $\alpha = 0.5$, $\beta = 5$, $\theta = 0.04$, $\delta = 0.6$ and $\rho = 0$. As emphasized by Heath et al. ([6],[7]), these parameters satisfy Feller's test for explosions: $\beta\theta \geq \frac{1}{2}\delta$, which insures a positive solution for F_t in the stochastic differential equation (20). We generate 10000 scenarios using 1024 time steps in Euler's method. We have also performed numerical experiments for 20000 simulated scenarios, the results being very close in value to the results presented below. The initial stock price and volatility are $X_0 = 100$ and $Y_0 = 0.2$. The riskless rate of return is $r = 0.04$. As before, we want to hedge put options with maturity $T = 1$ and different strike prices.

We first assume that the holdings in the hedging portfolio depend on the current stock price and the past gains, their form being given by the constraint (16):

$$\xi_j = D_j(X_j) + \frac{1}{X_j}\sum_{i=0}^{j-1} \tilde{D}_i(X_i)\Delta X_i, \qquad \forall j = 0, \ldots, M-1.$$

We remark that this constraint assumes the holdings are independent of the current volatility. This is attractive, since the volatility is not observable in the market. On the other hand, since the volatility is no longer constant in the current framework, it may be reasonable to assume that it also affects the form of the holdings. We will investigate later a different constraint on the form of the holdings which takes into account the volatility. However, the new formulation, while being computationally more expensive to implement, does not improve significantly the average total hedging cost and risk.

We compute the total risk minimizing strategies satisfying assumption (16):

- Strategy 1: Piecewise linear risk minimizing strategy
- Strategy 2: Quadratic risk minimizing strategy

Tables 8 and 9 present the average values of the total hedging cost and risk over the 10000 simulated scenarios. We remark that the last column in these tables corresponds to the static hedging, when we only have one rebalancing opportunity, at time 0.

Table 8: Average value of the total cost over 10000 scenarios

		# of time steps per rebalancing time				
Strike	Strategy	16	64	128	512	1024
90	1	1.9433	1.5446	1.1637	1.0199	1.0199
	2	2.3366	2.2365	2.2137	1.9469	1.7340
95	1	3.4682	3.2307	3.0079	1.7710	1.7738
	2	3.7234	3.6141	3.5726	3.2049	2.9003
100	1	5.4967	5.2277	5.1111	3.8699	2.8902
	2	5.5977	5.4786	5.4225	4.9567	4.5512
105	1	7.9197	7.7034	7.6502	7.0733	5.8629
	2	8.0112	7.8777	7.8106	7.2709	6.7681
110	1	10.8262	10.7099	10.6651	10.5153	9.5471
	2	10.9231	10.7769	10.7051	10.1219	9.5382

Average total cost for the hedging of put options with $T = 1$, different strike prices and number of rebalancing opportunities, for the two strategies satisfying (16): 1 - piecewise linear, 2 - quadratic; $X_0 = 100$, $Y_0 = 0.2$, $r = 0.04$, $\alpha = 0.5$, $\beta = 5$, $\theta = 0.04$, $\delta = 0.6$ and $\rho = 0$.

The above numerical results follow the trend observed in the Black-Scholes framework. For out-of-money and at-the-money put options the average cumulative cost and risk for the piecewise linear risk minimizing strategy are much smaller than the corresponding values for the quadratic risk minimizing strategy. The differences increase as the rebalancing is less frequent. For the deep out-of-money put options with very few hedging opportunities the values for the piecewise linear risk minimizing strategy are almost half the values for the quadratic risk minimizing strategy.

Table 9: Average value of the total risk over 10000 scenarios

		# of time steps per rebalancing time				
Strike	Strategy	16	64	128	512	1024
90	1	0.8395	0.9099	0.9901	1.0199	1.0199
	2	0.9727	1.0942	1.2546	1.7399	1.8985
95	1	1.1469	1.2728	1.4854	1.7737	1.7738
	2	1.2599	1.4251	1.6598	2.4190	2.7518
100	1	1.4342	1.5745	1.8701	2.7670	2.8902
	2	1.5274	1.7164	2.0283	3.1000	3.6495
105	1	1.6076	1.7925	2.1315	3.4513	4.1089
	2	1.7032	1.9303	2.3000	3.6521	4.4442
110	1	1.7004	1.9156	2.2754	3.7799	4.8597
	2	1.7915	2.0499	2.4533	4.0204	5.0373

Average total risk for the hedging of put options with different strike prices and number of rebalancing opportunities, for the two strategies and in the setup described in Table 8.

In the case of the in-the-money put options, the two strategies yield close values for the average cumulative cost and risk, with the piecewise linear risk minimizing strategy being better in most of the cases.

We can also analyze the distributions of the total hedging cost and risk for the two hedging strategies. Figure 6 shows the histograms of the total hedging cost over the 10000 simulated scenarios for each strategy, in the case of the at-the-money put option with 8 hedging opportunities.

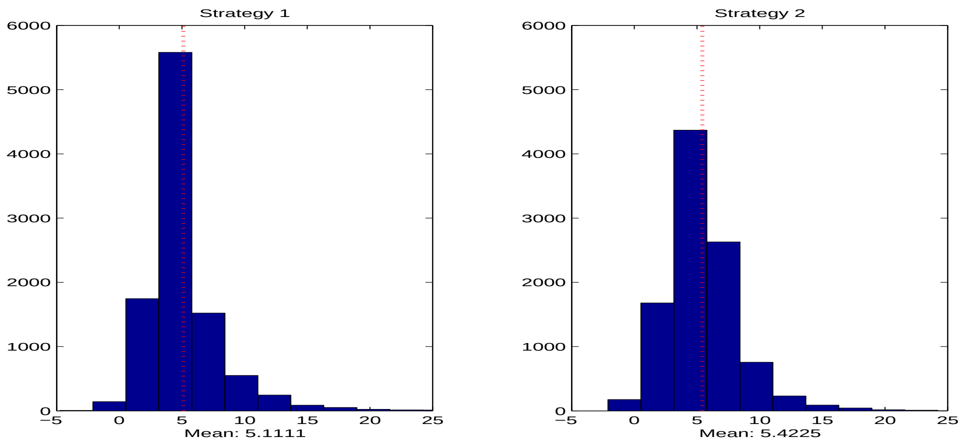

Figure 6. Histograms of the total hedging cost over 10000 scenarios

The average cumlative costs, given in Table 8, are 5.1111 for Strategy 1 , and 5.4225 for Strategy 2. As in the Black-Scholes framework, the distribution of the cumulative cost for the piecewise linear risk minimizing strategy is more asymmetric about its mean than the distribution of the quadratic risk minimizing strategy. In the case of Strategy 1, 65% of the cumulative costs for Strategy 1 are less than the mean, while this happens only 55% of the time for Strategy 2. The skewness is 2.7526 for Strategy 1 and 1.3711 for strategy 2. However, we remark again that piecewise linear risk minimization may lead, with a very small probability, to larger total hedging cost than the quadratic risk minimization.

Figure 7 presents the histograms of the total hedging risk for the same at-the-money put option with 8 hedging opportunities. As shown in Table 9, the average total hedging risk is 1.8701 in the case of Strategy 1 and 2.0283 in the case of Strategy 2.

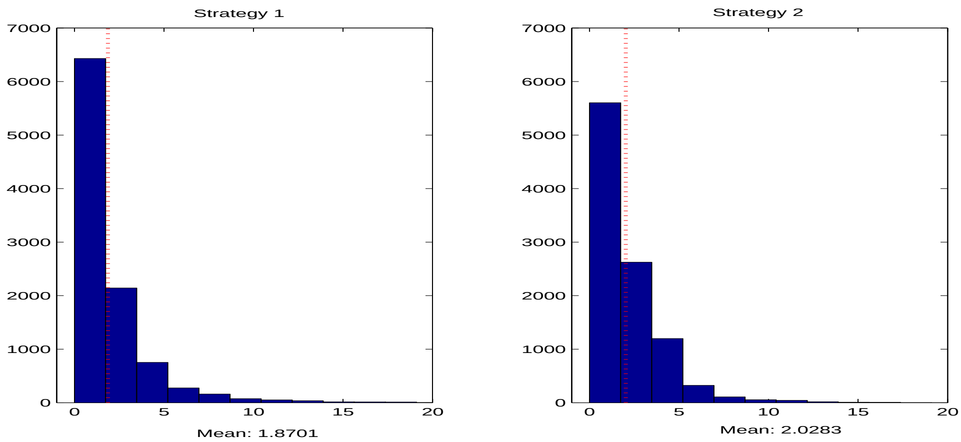

Figure 7. Histograms of the total hedging risk over 10000 scenarios

The distributions of the total risk for both strategies are asymmetric about their mean. However, the mean for Strategy 1 is smaller than the mean for Strategy 2. Strategy 1 yields smaller than the mean total risk 67% of the time, while this happens for 62% of the total risk for Strategy 2. The skewness is 4.0549 for Strategy 1 and 2.5346 for Strategy 2. The total risk for the piecewise linear risk minimizing strategy has a very small probability of larger values than the cumulative cost for the quadratic risk minimizing strategy.

We mention that the range of values illustrated in Figures 6 and 7 was chosen for clarity, but both strategies can lead to values of the cumulative cost and risk larger than the values in the selected interval.

It is interesting to analyze the quadratic risk minimizing Strategy 2 as the number of hedging opportunities increases and compare it to the quadratic risk minimizing strategy for continuous trading. Such an analysis requires, however, a very thorough investigation and the simulation of a larger number of scenarios. For a very brief comparison, we illustrate the case of the in-the-money put option with maturity $T = 1$ and strike price $100 * \exp(r \cdot T)$, where r is the riskless rate of return. Heath et al. ([6], [7]) compute the expected cumulative hedging cost and the expected squared net loss $E((H - V_M)^2)$ for the continuous hedging of this option under the quadratic risk measure. They obtain an expected hedging cost of 7.691 and an expected squared net loss of 3.685. Table 10 shows the average over 10000 of the cumulative hedging cost and squared net loss for the quadratic risk minimizing Strategy 2 as the number of time steps per rebalancing time decreases. We remark that, as the number of hedging opportunities increases, the average values of the cumulative hedging cost and squared net loss in Table 10, approach the values given by Heath et al. ([6], [7]).

Table 10: Average total hedging cost and squared net loss for Strategy 2 over 10000 scenarios

Time steps	Cost	Net loss
1024	6.3182	32.9453
512	6.8041	22.5713
128	7.2715	9.5157
64	7.3155	6.4883
16	7.3260	3.7875

Average total cost and squared net loss for the hedging of the put option with $T = 1$ and strike price $100 * \exp(r \cdot T)$, for the quadratic risk minimizing strategy; same setup as described in Table 8.

We have remarked earlier in this section that the constraint (16) on the form of the holdings does not take into account the volatility Y_t. It may be reasonable to include the effect of the volatility on the holdings in the hedging portfolio and use the following constraint:

$$\xi_j = D_j(X_j, Y_j) + \frac{1}{X_j} \sum_{i=0}^{j-1} \tilde{D}_i(X_i, Y_i) \Delta X_i, \qquad \forall j = 0, \ldots, M-1. \tag{21}$$

The unknown functions D_j, $\tilde{D}_j$, $j = 1, \ldots, M-1$, are now bicubic splines with fixed end conditions and knots placed with respect to the stock price and volatility. For each $j = 1, \ldots, M-1$, D_j and $\tilde{D}_j$ depend on the stock price and the volatility at time t_j. We assume, as before, that D_0 and $\tilde{D}_0$ are constant functions.

Solving an L^1-optimization problem similar to (17) and, respectively, an L^2-optimization problem similar to (18), we compute the total risk minimizing strategies satisfying assumption (21):

- Strategy 1: Piecewise linear risk minimizing strategy
- Strategy 2: Quadratic risk minimizing strategy

Since the assumption (21) involves bicubic splines, computing the above optimal strategies is much more expensive than computing the optimal strategies satisfying (16).

The average values of the cumulative hedging cost and risk for these two strategies over the 10000 simulated scenarios are presented below, in Tables 11 and 12, respectively. In order to make the comparison easier, we also reproduce the corresponding results from Tables 8 and 9. We remark that in the case of the static hedging, assumptions (16) and (21) coincide. This is why, in Tables 11 and 12, the columns for 1024 time steps per rebalancing time, which correspond to static hedging in our implementaton, coincide.

Table 11: Average value of the total cost over 10000 scenarios

		# of time steps per rebalancing time					
		With assumption (18)			With assumption (13)		
Strike	Strategy	128	512	1024	128	512	1024
90	1	1.2055	1.0186	1.0199	1.1637	1.0199	1.0199
	2	2.1913	1.9507	1.7340	2.2137	1.9469	1.7340
95	1	2.9708	1.7715	1.7738	3.0079	1.7710	1.7738
	2	3.5455	3.2149	2.9003	3.5726	3.2049	2.9003
100	1	5.0478	3.9188	2.8902	5.1111	3.8699	2.8902
	2	5.3959	4.9729	4.5512	5.4225	4.9567	4.5512
105	1	7.6027	7.0814	5.8629	7.6502	7.0733	5.8629
	2	7.7734	7.2961	6.7681	7.8106	7.2709	6.7681
110	1	10.6063	10.5019	9.5471	10.6651	10.5153	9.5471
	2	10.6836	10.1544	9.5382	10.7051	10.1219	9.5382

Average total cost for the hedging of put options with different strike prices and number of rebalancing opportunities, for the two strategies satisfying (21): 1 - piecewise linear, 2 - quadratic; same setup as described in Table 8.

Table 12: Average value of the total risk over 10000 scenarios

		# of time steps per rebalancing time					
		With assumption (21)			With assumption (16)		
Strike	Strategy	128	512	1024	128	512	1024
90	1	0.9464	1.0193	1.0199	0.9901	1.0199	1.0199
	2	1.2028	1.7407	1.8985	1.2546	1.7399	1.8985
95	1	1.4197	1.7726	1.7738	1.4854	1.7737	1.7738
	2	1.6107	2.4152	2.7518	1.6598	2.4190	2.7518
100	1	1.8175	2.7492	2.8902	1.8701	2.7670	2.8902
	2	1.9414	3.0880	3.6495	2.0283	3.1000	3.6495
105	1	2.1104	3.4145	4.1089	2.1315	3.4513	4.1089
	2	2.2950	3.6276	4.4442	2.3000	3.6521	4.4442
110	1	2.2754	3.7335	4.8597	2.2754	3.7799	4.8597
	2	2.4102	3.9797	5.0373	2.4533	4.0204	5.0373

Average total risk for the hedging of put options with different strike prices and number of rebalancing opportunities, for the two strategies and in the setup described in Table 11.

Computing the optimal strategies satisfying the constraint (21) on the form of the holdings is expensive, however, these strategies do not lead to significantly better cumulative hedging cost or risk, as can be seen by comparing the values of the cumulative cost and risk for these strategies to the corresponding values for the optimal hedging strategies satisfying the constraint (16). Moreover, assumption (21) relies on the values of the volatility, which are not directly observable in the market. In conclusion, it seems reasonable to compute the optimal hedging strategies in this framework by solving the optimization problems (17) and (18), even if their formulation takes into account only the dependence of the holdings in the hedging portfolio on the stock price path.

The numerical results presented in this section refer to hedging put options. However, as mentioned at the end of Section 3, hedging call options is closely related to hedging put options on the same underlying asset and with the same maturity and strike price. The optimal hedging portfolio values satisfy discrete hedging put-call parity. Moreover, if the holdings in the optimal portfolio for hedging the put options are known, the optimal holdings for the call options can be computed directly, without solving any optimization problems.

5. Shortfall risk minimization

An important criticism of the quadratic risk minimizing criterion, which is also valid for the piecewise linear risk measure, is the fact that it penalizes symmetrically losses, as well as gains.

It has been argued (see Bertsimas et al. [1]) that, in the case of pricing an option, a symmetric risk measure is the natural choice, since we do not know a priori if the option is being sold or purchased. However, when hedging an option, one tries to replicate the option payoff by constructing a hedging portfolio and he may be interested in penalizing only the costs and not also the profits from his position.

We will investigate here only the perspective of the writer of an option. When using a self-financing strategy to hedge an option with payoff H and maturity T, the total risk for the writer of the option is given by the difference between the payoff H and the final value of the hedging strategy, V_M. Even if V_M does not match exactly H, if $V_M \geq H$ the writer is still on the safe side, that is, he can cover the option payoff with no suplementary inflow of capital. Therefore, the writer of the option may prefer to choose a hedging strategy that minimizes only the shortfall risk, $E((H - V_M)^+)$:

$$\min E((H - V_M)^+), \tag{22}$$

and not the total risk, $E(|H - V_M|)$ or $E((H - V_M)^2)$.

A self-financing hedging strategy such that $V_M \geq H$, a.s., is called a super-replicating strategy. Unfortunately, the minimum initial cost of a super-replicating strategy is often too high. Moreover, in practice, one may be inclined not to use a super-replicating hedging strategy if he can make higher profits by accepting the risk of a loss.

In order to see that it can be quite expensive to super-replicate an option, we compare the minimum initial cost of a super-replicating strategy - obtained by minimizing $E((H-V_M)^+)$ - with the initial cost of the total risk minimizing strategies described in Section 3.1 - computed by minimizing $E(|H - V_M|)$ and $E((H - V_M)^2)$, respectively. The numerical results refer to the hedging put options with maturity $T = 1$ and different strike prices when we only have a finite number of hedging opportunities at $0 = t_0 < t_1 < \ldots < t_M := T$. The stock price follows a Black-Scholes model with instantaneous expected return $\mu = .15$ and volatility $\sigma = .2$. The initial stock price is $S_0 = 100$. We generate 40000 scenarios for the stock price using Monte Carlo simulation. The riskless rate of return is $r = .04$.

An optimal super-replicating strategy for (22) can be obtained in a similar way to the computation of a total risk minimizing strategy described in Section 3.1, by assuming that the optimal holdings have the special form given by (16). Moreover, since,

$$(H - V_M)^+ = \frac{1}{2}(H - V_M + |H - V_M|) \tag{23}$$

problem (22) can be implemented as a linear programming problem.

Table 13 shows the minimum initial cost for a super-replicating strategy satisfying assumption (16), in comparison with the initial cost of the piecewise linear total risk minimizing strategy - Strategy 1 - and the quadratic total risk minimizing strategy - Strategy 2 - satisfying the same assumption.

Table 13: Initial portfolio cost

		# of time steps per rebalancing time			
Strike	Strategy	50	100	300	600
90	Super-replicate	7.4806	10.3742	19.5669	28.1378
	1	1.9022	1.0070	0.0000	0.0000
	2	2.4086	2.3224	2.0388	1.7421
95	Super-replicate	9.7100	12.7437	22.2787	32.2861
	1	3.5152	3.0875	0.0000	0.0000
	2	3.8885	3.7741	3.3983	2.9735
100	Super-replicate	12.3146	15.3754	24.9454	35.7017
	1	5.5279	5.2248	2.7110	0.0000
	2	5.8455	5.7119	5.2530	4.6948
105	Super-replicate	15.3226	18.1656	27.7273	39.3592
	1	8.0693	7.8209	6.5076	3.2595
	2	8.2882	8.1412	7.6261	6.9449
110	Super-replicate	18.9710	21.4535	30.8217	43.1454
	1	11.0098	10.9945	10.1126	7.6382
	2	11.1881	11.0413	10.4945	9.7148

Initial portfolio cost for put options with different strike prices and number of time steps per rebalancing time, for the three strategies: super-replicating, 1 - piecewise linear and 2 - quadratic; same setup as in Table 1.

We can see from Table 13 that buying the initial portfolio for super-hedging is much more expensive than buying the initial portfolio for total risk minimization. Therefore, computing a hedging strategy by simply minimizing the shortfall risk $E((H - V_M)^+)$ is not very attractive from a practical point of view, even if a super-replicating strategy prevents the risk of any loss at the maturity of the option. In these conditions, an investor who still wants to penalize only the shortfall risk, but has a given initial capital and is willing to accept some risk of loss, may choose an optimal self-financing hedging strategy in the following way:

$$\begin{aligned} \min \quad & E((H - V_M)^+) \\ \text{s.t.} \quad & V_0 \text{ given} \end{aligned} \tag{24}$$

The above criteria for minimizing the shortfall risk has been studied by Föllmer and Leukert ([3]), and Runggaldier ([13]).

Alternative to penalizing the positive values of $H - V_M$, by minimizing $E((H - V_M)^+)$, one may try to penalize those values which are above the mean. This corresponds to minimizing:

$$E\left((H - V_M - E(H - V_M))^+\right). \tag{25}$$

However, note that, since for a self-financing strategy, $V_M = V_0 + \sum_{k=0}^{M-1} \xi_k \Delta X_k$, we have:

$$H - V_M - E(H - V_M) = H - E(H) - \sum_{k=0}^{M-1} \xi_k \Delta X_k + E(\sum_{k=0}^{M-1} \xi_k \Delta X_k)$$

Therefore, the initial value of the hedging portfolio, V_0 cannot be determined by minimizing (25). In these conditions, a natural idea is to impose the constraint:

$$E(H - V_M) = 0 \Leftrightarrow V_0 = E(H - \sum_{k=0}^{M-1} \xi_k \Delta X_k), \tag{26}$$

that is, the initial value of the hedging portfolio is equal to the expected value of the difference between the option payoff and the cumulative gain of the portfolio. With this constraint, criterion (25) becomes:

$$\begin{aligned} \min \quad & E((H - V_M)^+) \\ \text{s.t.} \quad & E(H - V_M) = 0 \end{aligned} \tag{27}$$

By (23), this criterion is equivalent to:

$$\begin{aligned} \min \quad & E(|H - V_M|) \\ \text{s.t.} \quad & E(H - V_M) = 0 \end{aligned} \tag{28}$$

Assuming that the holdings have the special form given by (16):

$$\xi_j = D_j(X_j) + \frac{1}{X_j} \sum_{i=0}^{j-1} \tilde{D}_i(X_i) \Delta X_i, \qquad \forall j = 0, \ldots, M-1,$$

an optimal strategy for the above problem can be computed is a similar way to the piecewise linear total risk minimization problem (6).

We remark that the shortfall risk minimization problem (27) is not equivalent to problem (24), since (27) imposes a relation between the optimal holdings ξ_k and the initial value of the hedging portfolio, V_0.

In order to investigate the two shortfall risk minimization criteria (24) and (27), we first compute the optimal hedging strategy for the second criterion, (27), then using the initial value of this hedging strategy as given value for V_0, we calculate the optimal holdings for the strategy based on the first criterion, (24).

We denote by Strategy 3, the optimal strategy solving the first shortfall risk minimization problem, (24), and by Strategy 4, the optimal strategy for the second problem, (27). We remark the the initial portfolio values, V_0, are the same for both strategies, however, the holdings, ξ_k, are different. For comparison with the minimum initial cost super-replicating strategy, Table 14 illustrates the values of V_0 for Strategy 4, for the same put options as in Table 13.

Table 14: Initial portfolio cost

	# of time steps per rebalancing time			
Strike	50	100	300	600
90	2.2065	2.0562	1.4911	1.2486
95	3.7236	3.5833	2.8130	2.3168
100	5.7089	5.5760	4.8485	3.9832
105	8.1940	8.0620	7.5616	6.3626
110	11.1355	11.0106	10.7330	9.5202

Initial portfolio cost for hedging put options with different strike prices and number of time steps per rebalancing time, for the optimal shortfall risk minimizing strategy solving (27); same setup as in Table 1.

We remark that the initial portfolio values for strategies 3 and 4 are much smaller than the initial values for the minimal cost super-replicating strategy and they are comparable to the initial portfolio values for the total risk minimizing strategies.

Strategies 3 and 4 have a reasonable initial cost compared to the super - replicating strategy. However, this reduction in the initial cost has been achieved by allowing a nonzero probability of a loss. While a super - replicating strategy prevents any loss, Strategies 3 and 4 have a nonzero shortfall risk. Table 15 illustrates the average values of the shortfall risk, $(H - V_M)^+$, over 40000 scenarios for the hedging strategies 3 and 4. We note that the shortfall risk increases as the options become more in-the-money and we rebalance less frequently. Moreover, since Strategy 3 minimizes the shortfall risk for a given initial portfolio, this strategy yields smaller values of the shortfall risk than Strategy 4, which has the same initial investment.

Table 15: Average value of the shortfall risk over 40000 scenarios

		# of time steps per rebalancing time			
Strike	Strategy	50	100	300	600
90	3	0.2826	0.4280	0.6768	0.7578
	4	0.3437	0.4735	0.6904	0.7632
95	3	0.3638	0.5571	0.9822	1.1761
	4	0.4570	0.6391	1.0279	1.1957
100	3	0.4349	0.6782	1.2353	1.6182
	4	0.5597	0.7847	1.3547	1.6781
105	3	0.4944	0.7534	1.3846	1.9844
	4	0.6498	0.9008	1.6037	2.1252
110	3	0.5288	0.8007	1.4518	2.1905
	4	0.7035	0.9693	1.7426	2.4653

Average value of the shortfall risk for hedging put options with different strike prices and number of time steps per rebalancing time, for the optimal shortfall risk minimizing strategies solving (24) and (27); same setup as in Table 1.

We can also compute the average values of the shortfall risk, $(H - V_M)^+$, over the same 40000 paths, for the piecewise linear and quadratic total risk minimizing Strategies 1 and 2, respectively. These values will certainly be larger than the corresponding values for Strategies 3 and 4, which are shortfall risk minimizing strategies. However, the results provide interesting information about the behavior of the total risk minimizing strategies. The average shortfall risk for Strategies 1 and 2 is illustrated in Table 16.

We remark from Table 16 that the quadratic total risk minimizing Strategy 2 always yields smaller average shortfall risk than the piecewise linear risk minimizing Strategy 1. Using the relation:

$$|H - V_M| = (H - V_M)^+ + (V_M - H)^+, \tag{29}$$

we can analyze the average values of the shortfall risk, $(H - V_M)^+$, from Table 16, in comparison with the average values of the total hedging risk, $|H - V_M|$, from Table 4. While in the case of Strategy 2, the average shortfall risk is approximately half the average total risk, in the case of Strategy 1, these values are much closer, especially for out-of-money put options. By (29), it follows that Strategy 1 typically underhedges the options, while Strategy 2 shows no trend for either underhedging, or overhedging.

Table 16: Average value of the shortfall risk over 40000 scenarios for the piecewise linear and quadratic total risk minimizing strategies

		# of time steps per rebalancing time			
Strike	Strategy	50	100	300	600
90	1	0.4446	0.7533	0.9398	0.9398
	2	0.3920	0.5299	0.7868	0.8854
95	1	0.5306	0.8057	1.6648	1.6648
	2	0.4905	0.6850	1.0966	1.3111
100	1	0.6222	0.9088	1.9555	2.7269
	2	0.5868	0.8259	1.3938	1.7558
105	1	0.7043	0.9979	1.9898	3.1136
	2	0.6690	0.9420	1.6385	2.1592
110	1	0.7650	1.0773	2.0424	3.1845
	2	0.7214	1.0160	1.8026	2.4683

Average value of the shortfall risk for hedging put options with different strike prices and number of time steps per rebalancing time, for the optimal total risk minimizing strategies 1 and 2; same setup as in Table 1.

We will now investigate the cumulative hedging cost. Table 17 illustrates the average values of the cumulative hedging cost over 40000 scenarios for the shortfall risk minimizing Strategies 3 and 4. For comparison we include the corresponding values from Table 3 for the piecewise linear and quadratic total risk minimizing strategies satisfying (16).

As illustrated in Table 17, even if the two shortfall risk minimizing strategies start with the same investment in the hedging portfolio, Strategy 4, which has to satisfy the constraint $E(H - V_M) = 0$, yields larger values of the average cumulative hedging cost than Strategy 3. We also note that using a quadratic criterion for minimizing the risk leads to the largest hedging cost, as can be seen by comparing the cost for Strategy 2 to the cost of the other three strategies. The performance of the hedging strategies also depends on the moneyness of the options and the number of rebalancing opportunities: the piecewise linear total risk minimization has the smallest average cost when the put options are out-of-the-money and the rebalancing in infrequent, however, as the options become in-the-money or the number of hedging opportunities increases, the shortfall risk minimization criterion (24) is the least expensive on average.

Table 17: Average value of the total cost over 40000 scenarios

		# of time steps per rebalancing time			
Strike	Strategy	50	100	300	600
95	1	3.6640	3.4080	1.6648	1.6648
	2	3.8885	3.7741	3.3983	2.9735
	3	3.2156	3.1629	2.5151	2.1555
	4	3.7236	3.5833	2.8130	2.3168
100	1	5.6896	5.5067	4.0644	2.7269
	2	5.8455	5.7119	5.2530	4.6948
	3	5.0506	5.0474	4.2289	3.6124
	4	5.7089	5.5760	4.8485	3.9832
105	1	8.1835	8.0393	7.2893	5.5301
	2	8.2882	8.1412	7.6261	6.9449
	3	7.3748	7.3405	6.5371	5.6365
	4	8.1940	8.0620	7.5616	6.3626

Average total hedging cost for put options with different strike prices and number of time steps per rebalancing time, for the total risk minimizing strategies: 1 - piecewise linear and 2 - quadratic and the shortfall risk minimizing strategies: 3 - strategy solving (24), 4 - strategy solving (27); same setup as in Table 1.

As in Section 3.1, we investigate the distributions of the shortfall risk and cumulative cost for the shortfall risk minimizing Strategies 3 and 4, in the particular case of the at-the-money put options with 6 hedging opportunities. The histograms of the shortfall risk, $(H - V_M)^+$, over the 40000 simulated scenarios are presented in Figure 8. We mention that the strategies have a few values of the shortfall risk outside the represented interval, however, we chose this range to make the figure clearer.

From Table 15 the average values of the shortfall risk are 0.6782 for Strategy 3 and 0.7847 for Strategy 4. The distributions of the shortfall risk for the two strategies are very similar in the chosen interval. However, Strategy 3 has a longer right tail, outside the interval. This can be seen from the values of the skewness: 6.6456 for Strategy 3, compared to 4.2451 for Strategy 4.

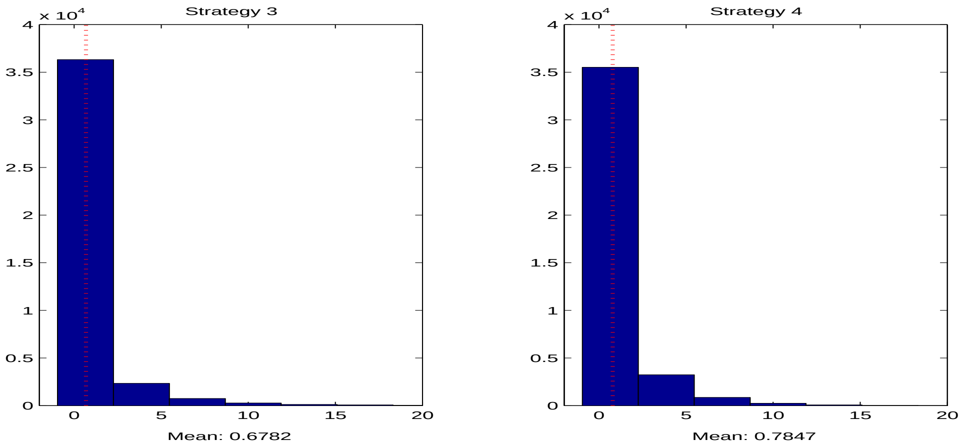

Figure 8. Histograms of the shortfall risk over 40000 scenarios

The histograms of the cumulative cost for the shortfall risk minimizing Strategies 3 and 4 are illustrated in Figure 9. As before, the range of the figure has been chosen for clarity.

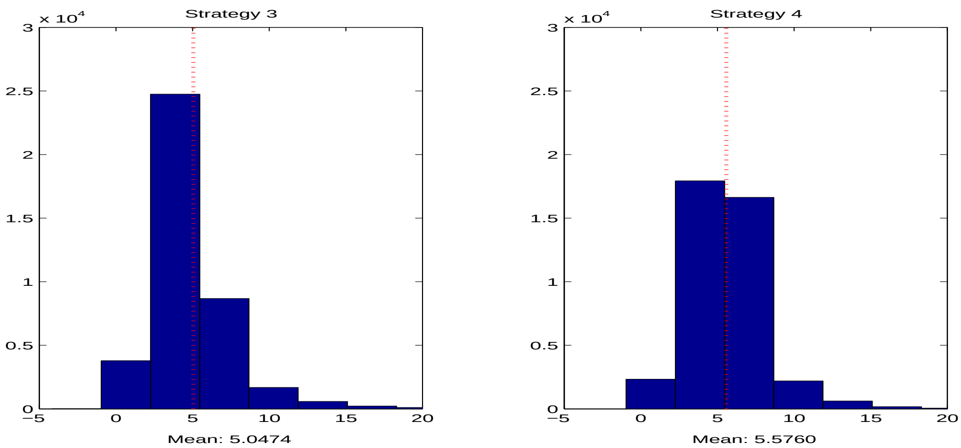

Figure 9. Histograms of the cumulative cost over 40000 scenarios

The average values of the cumulative cost for Strategies 3 and 4 are 5.0474 and 5.5760, respectively, as can be seen from Table 17. We remark that Strategy 3 is more asymmetric than Strategy 4. The values of the skewness, 3.2878 for Strategy 3 and 1.7171 for Strategy 4, show that Strategy 3 has also a longer right tail.

As illustrated in this section, the shortfall risk minimizing strategies have attractive features, they have smaller average loss and, possibly, smaller cumulative hegding cost than the total risk minimizing strategies. However, when choosing between shortfall and total risk minimization, one has to take into account the fact that shortfall risk minimization can only be used for hedging purposes, while total risk minimization can be used for both hedging and pricing, since the initial value of a total risk minimizing strategy may be considered as a "fair value" of the option. Moreover, when hedging an option, the shortfall risk measure is appropriate if one is inclined to penalize only the costs, and not the profits of his position; if one prefers to penalize both losses and gains, he has to choose a symmetric risk measure, such as the total risk measure.

5. Conclusions

In a complete market, there exists a unique self-financing strategy that exactly replicates the option payoff. Market completeness is not, however, a realistic assumption. For example, introducing stochastic volatility or volatility with jumps in the Black-Scholes model in order to explain the market data, or allowing for discrete hedging, leads to an incomplete market. If the market is incomplete, the optimal hedging strategy for an option depends on the criterion for measuring the risk. The traditional strategies found in the literature are based on quadratic risk measures.

We investigate alternative piecewise linear risk minimizing criteria for total-risk minimization. Unfortunately, there are no analytic solutions to the piecewise linear risk minimization problem. Since a direct approach to this dynamic stochastic programming problem may be computationally very expensive, we obtain the optimal piecewise linear risk minimizing strategies using Monte Carlo simulations and approximating the holdings in the hedging portfolio by cubic splines. We analyze this approach in the Black-Scholes and stochastic volatility frameworks.

The numerical results illustrate that, as in the case of the local risk minimization, the piecewise linear total risk minimization criterion typically leads to smaller average hedging cost and risk. We remark that the hedging performance of the optimal strategies depends on the moneyness of the options and on the number of rebalancing opportunities. The hedging strategies based on piecewise linear risk minimization have quite different, and often preferable, properties compared to the traditional, quadratic risk minimizing strategies. The distributions of the cumulative cost and risk show that these new strategies have a larger probability of small cost and risk, though they also have a very small probability of larger cost and risk. We also remark that in the stochastic framework analyzed in this paper, the volatility does not significantly affect the average total cost and risk of the hedging strategies.

Comparing the hedging performance of the optimal strategies for piecewise linear and quadratic total risk minimization to the performance of the shortfall risk minimizing strategies, we note that the quadratic criterion yields the largest values of the average cumulative hedging cost. Shortfall risk minimization may lead to smaller average cumulative hedging cost than piecewise linear risk minimization, depending on the moneyness of the options and the number of hedging opportunities.

By analyzing the values of the shortfall risk for the piecewise linear and quadratic total risk minimizing hedging strategies, we infer that the piecewise linear criterion typically leads to options being underhedged, while quadratic total risk minimization shows no trend for either overhedging, or underhedging the options.

A shortfall risk measure may be more attractive than a total risk measure when one tries to hedge an option and he is inclined to penalize only the costs of his position. However, shortfall risk minimization cannot be used for pricing the option, while total risk minimization can be used for both hedging and pricing. Moreover, when one prefers to penalize both losses and gains, a shortfall risk measure is no longer appropriate.

References

[1] D. Bertsimas, L. Kogan, and A. Lo. Hedging derivative securities and incomplete markets: An ϵ-arbitrage approach. *Operations Research*, 49:372–397, 2001.

[2] T. F. Coleman, Y. Li, and M. Patron. Discrete hedging under piecewise-linear risk-minimization. *Journal of Risk*, 5(3), 2003.

[3] H. Föllmer and P. Leukert. Efficient hedging: Cost versus shortfall risk. *Finance and Stochastics*, 4:117–146, 2000.

[4] H. Föllmer and M. Schweizer. Hedging by sequential regression: An introduction to the mathematics of option trading. *The ASTIN Bulletin*, 1:147–160, 1989.

[5] Rüdiger Frey. Derivative asset analysis in models with level-dependent and stochastic volatility. *CWI Quaterly*, 10(1):1–34, 1997.

[6] D. Heath, E. Platen, and M. Schweizer. A comparison of two quadratic approaches to hedging in incomplete markets. *Mathematical Finance*, 11:385–413, 2001a.

[7] D. Heath, E. Platen, and M. Schweizer. Numerical comparison of local risk-minimisation and mean-variance hedging. In *Option pricing, interest rates and risk management*, pages 509–537. (ed. E. Jouini, J. Cvitanic and, M. Musiela), Cambridge Univ. Press, 2001b.

[8] S.L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

[9] J. C. Hull and A. White. The pricing of options on assets with stochastic volatilities. *Journal of Finance*, 42:281–300, 1987.

[10] N. El Karoui and M. C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM Journal on Control and Optimization*, 33(1):27–66, 1995.

[11] F. Longstaff and E. S. Schwartz. Valuing american options by simulation: A simple least-squares approach. *The Review of Financial Studies*, 14(1):113–147, 2001.

[12] F. Mercurio and T. C. F. Vorst. Option pricing with hedging at fixed trading dates. *Applied Mathematical Science*, 3:135–158, 1996.

[13] W. J. Runggaldier. Adaptive and robust control procedures for risk minimization under uncertainty. In *Optimal Control and Partial Differential Equations, Volume in Honour of Prof. Alain Bensoussan's 60th Birthday*, pages 549–557. (J.L.Menaldi, E.Rofman, A.Sulem, eds.), IOS Press, 2001.

[14] M. Schäl. On quadratic cost criteria for option hedging. *Mathematics of Operation Research*, 19(1):121–131, 1994.

[15] M. Schweizer. Variance-optimal hedging in discrete time. *Mathematics of Operation Research*, 20:1–32, 1995.

[16] M. Schweizer. A guided tour through quadratic hedging approaches. In *Option pricing, interest rates and risk management*, pages 538–574. (ed. E. Jouini, J. Cvitanic and, M. Musiela), Cambridge Univ. Press, 2001.