Min-Max Robust and CVaR Robust Mean-Variance Portfolios *

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Abstract

This paper investigates robust optimization methods for mean-variance (MV) portfolio selection problems under the estimation risk in mean returns. We show that, with an ellipsoidal uncertainty set based on the statistics of the sample mean estimates, the portfolio from the min-max robust MV model equals the portfolio from the standard MV model based on the nominal mean estimates but with a larger risk aversion parameter. We demonstrate that the min-max robust portfolios can vary significantly with the initial data used to generate uncertainty sets. In addition, min-max robust portfolios can be too conservative and unable to achieve a high return. Adjustment of the conservatism in the min-max robust model can only be achieved by excluding poor mean return scenarios, which runs counter to the principle of min-max robustness. We propose a Conditional Value-at-Risk (CVaR) robust portfolio optimization model to address estimation risk. We show that, using CVaR to quantify the estimation risk in mean return, the conservatism level of the portfolios can be more naturally adjusted by gradually including better scenarios; the confidence level β can be interpreted as an estimation risk aversion parameter. We compare min-max robust portfolios with an interval uncertainty set and CVaR robust portfolios in terms of actual frontier variation, efficiency, and asset diversification. We illustrate that the maximum worst-case mean return portfolio from the min-max robust model typically consists of a single asset, no matter how an interval uncertainty set is selected. In contrast, the maximum CVaR mean return portfolio typically consists of multiple assets. In addition, we illustrate that, for the CVaR robust model, the distance between the actual mean-variance frontiers and the true efficient frontier is relatively insensitive for different confidence levels as well as different sampling techniques.

Keywords: Mean-variance portfolio optimization, robust optimization, CVaR, VaR

1 Introduction

Financial portfolio selection attempts to maximize return and minimize risk. In the mean-variance (MV) model introduced by Markowitz (1952), assets are allocated to maximize the expected rate of the portfolio return as well as to minimize the variance. A portfolio allocation is considered to be *efficient* if it has the minimum risk for a given level of expected return.

Despite its theoretical importance to modern finance, the MV model is known to suffer severe limitations in practice. One of the basic problems that limits the applicability of the MV model is the inevitable estimation error in the asset mean returns and the covariance matrix. Best and Grauer (1991) analyze the effect of changes in mean returns on the MV efficient frontier and compositions of optimal portfolios. Broadie (1993) investigates the impact of errors in parameter estimates on the *actual frontiers*, which are obtained by applying the true parameters on the portfolio weights derived from their estimated parameters. Thus the actual frontier represents

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the actual performance of optimal portfolios based on estimated model parameters. Both of these studies show that different input estimates to the MV model can result in large variations in the composition of efficient portfolios. Unfortunately accurate estimation of mean returns is notoriously difficult. Since estimation of the covariance matrix is relatively easier, we focus, in this paper, on the estimation error in mean return only and investigate appropriate ways to take this estimation risk into account in the MV model.

Recently min-max robust portfolio optimization has been an active research area, see, e.g., Garlappi et al. (2007), Goldfarb and Iyengar (2003), Tütüncü and Koenig (2004). Min-max robust optimization yields the optimal portfolio that has the best worst-case performance within the given uncertainty sets of the input parameters. The uncertainty set typically corresponds to some confidence level β. In this regard, min-max robust optimization can be considered as a quantile-based approach, similar to the Value-at-Risk (VaR) measure. One drawback of the min-max approach is that, similar to VaR, it entirely ignores the severity of the tail scenarios which occur with a probability of $1 - \beta$. In addition, the dependence on a single large loss scenario makes a min-max robust portfolio quite sensitive to the initial data used to generate uncertainty sets. In practice, it can be difficult to choose appropriate uncertainty sets.

One of the main objectives of this paper is to propose a Conditional Value-at-Risk (CVaR) robust portfolio optimization model, which selects a portfolio under CVaR measure for the estimation risk in mean return. Instead of focusing on the worst-case scenario in the uncertainty set, an optimal portfolio is selected based on the tail of the large mean loss scenarios specified by a confidence level. The conservatism level can be controlled by adjusting the confidence level. Therefore the model parameter uncertainty is considered as a special type of risk. The CVaR of a portfolio's mean loss is used as a performance measure of this portfolio. In addition to minimizing the variance of the portfolio return, the CVaR robust model determines the optimal portfolio by maximizing the average over the tail of the worst mean returns with respect to an assumed distribution. The proposed CVaR robust formulation provides robustness by considering the average of the tail of poor mean return scenarios. As the confidence level β approaches 1, the CVaR robust measure in mean return uncertainty also becomes focused on the worst scenario. Decreasing the confidence level, however, leads to the consideration of better mean return scenarios and thus is less dependent on the worst case. When $\beta = 0$, CVaR robust measure in mean return uncertainty takes all possible mean returns into consideration. This may be appropriate when an investor has complete tolerance to estimation risk. Thus the confidence level β in the CVaR robust model can be used as an estimation risk aversion parameter. The proposed CVaR robust mean-variance portfolio formulation is described in §3.

Before introducing CVaR robust model, in §2, we first review the min-max robust portfolio optimization framework and highlight its potential problems. We show that, with an ellipsoidal uncertainty set based on the statistics of the sample mean estimates, the robust portfolio from the min-max robust MV model equals the portfolio from the standard MV model based on the nominal mean estimate but with a larger risk aversion parameter. We also illustrate the characteristics of min-max robust portfolios with an interval uncertainty set. If the uncertainty interval for mean return contains the worst sample scenario, the min-max robust model often produces portfolios with very low return. Portfolios with higher return can be generated in a min-max robust model by choosing the uncertainty interval to correspond to a smaller confidence interval. Unfortunately, this is at the expense of ignoring worse sample scenarios.

In §4, we compare min-max robust and CVaR robust methods from the following perspectives: the ease of adjusting the robustness level according to an investor's aversion to estimation risk, the variations in actual frontiers and the closeness of the actual frontiers to the true efficient frontier, and the diversification level of the resulting robust portfolios. Diversification is an important way to reduce the overall portfolio return risk by spreading the investment across a wide variety of asset classes. We show that, for the min-max robust formulation with interval uncertainty sets, the maximum worst-case expected return portfolio (corresponding to $\lambda = 0$ in the min-max model) always consists of a single asset; while using CVaR to measure estimation risk in mean return, the resulting robust portfolio which maximizes the CVaR of mean return are more diversified. We show computationally, in addition, that the diversification level decreases as the estimation risk aversion parameter decreases. We also consider two different distributions to characterize uncertainty in mean return estimation, and compare the diversification level of CVaR robust portfolios between two different sampling techniques.

One way of computing CVaR robust portfolios is to discretize, via simulation, the CVaR robust optimization problem. The CVaR function is approximated by a piecewise linear function and the discretized CVaR optimization problem can be formulated as a quadratic programming (QP) problem. However, the QP approach becomes inefficient when the number of simulations or the number of assets becomes large. In §5, a smoothing technique is proposed to compute CVaR robust portfolios. In contrast to the QP approach, the smoothing method uses a continuously differentiable piecewise quadratic function to approximate the CVaR function. We illustrate that, when the computation of CVaR robust portfolios becomes a large scale optimization problem, the smoothing approach is computationally more efficient than the QP approach. We conclude the paper in §6.

2 Min-max Robust Actual Frontiers

Let $\mu \in R^n$ be the vector of the mean returns of n risky assets and Q be the n-by-n positive semi-definite covariance matrix. Let x_i, $1 \leq i \leq n$, denote the percentage holding of the i^{th} asset. A MV efficient portfolio x solves the following quadratic programming (QP) problem:

$$
\begin{array}{ll}
\min\limits_{x} & -\mu^T x + \lambda x^T Q x \\
\text{s.t.} & x \in \Omega,
\end{array}
\tag{1}
$$

where $\lambda \geq 0$ is the risk-aversion parameter and Ω denotes the feasible portfolio set. Unless otherwise stated, in this paper, $\Omega = \{x \in \mathbb{R}^n \mid e^T x = 1, x \geq 0\}$, where e denotes the n-by-1 vector of all ones.

Let $x^*(\lambda)$ denote the optimal mean-variance portfolio (1) with the risk aversion parameter $\lambda \geq 0$. The curve $\{(\sqrt{x^*(\lambda)^T Q x^*(\lambda)}, \mu^T x^*(\lambda)),\ \lambda \geq 0\}$ in the space of standard deviation and mean is the *efficient frontier*. When $\lambda = 0$, $x^*(0)$ is the maximum-return portfolio, which ignores the risk. When $\lambda = \infty$, problem (1) yields the minimum-variance portfolio.

In practice, the mean return μ and the covariance matrix Q are not known. Estimates $\bar{\mu}$ and $\bar{Q}$ are typically computed from empirical return observations. Unfortunately, MV optimal portfolios can be very sensitive to estimation errors, which can be quite large in practice.

Recent development in efficient computational methods for robust optimization problems has generated great interest in min-max robust portfolio optimization. In robust optimization, uncertainty sets specify most or all of possible realizations for the input parameters, which typically corresponds to a confidence level under an assumed distribution. Assume that the uncertainty sets for the mean vector μ and the covariance matrix Q are S_μ and S_Q, respectively. The min-max robust formulation for (1) can be formulated as follows:

$$
\begin{array}{ll}
\min\limits_{x} & \max\limits_{\mu \in S_\mu, Q \in S_Q} -\mu^T x + \lambda x^T Q x \\
\text{s.t.} & x \in \Omega\ .
\end{array}
\tag{2}
$$

Robust portfolios depend heavily on specification of uncertainty sets. Goldfarb and Iyengar (2003) use ellipsoidal uncertainty sets and formulate problem (2) as a second-order cone programming (SOCP) problem. Tütüncü and Koenig (2004) consider intervals as uncertainty sets and solve problem (2) using a saddle-point method. In addition, Lobo and Boyd (1999) show that an optimal portfolio that minimizes the worst-case risk under each or a combination of the above uncertainty structures can be computed efficiently using analytic center cutting plane methods.

Assuming that the covariance matrix Q is known, Garlappi et al. (2007) consider the ellipsoidal uncertainty set based on the following statistical properties of the mean estimates. Assume that asset returns have a joint normal distribution and mean estimate $\bar{\mu}$ is computed from T samples of n assets. If the covariance matrix Q is known, then the quantity

$$
\frac{T(T-n)}{(T-1)n}(\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu) \tag{3}
$$

has a χ_n^2 distribution with n degrees of freedom. Specifically, Garlappi et al. (2007) consider the following ellipsoidal uncertainty set for the min-max robust portfolio optimziation,

$$(\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu) \le \chi, \tag{4}$$

where $\chi = \frac{(T-1)n}{T(T-n)} q \ge 0$ and q is a quantile for some confidence level based on (3).

How does the min-max robust MV portfolio differ from the MV portfolio based on nominal estimates? In order to analyze the precise relationship between the min-max robust portfolio and the standard MV portfolio, instead of (1), we first consider a mean-standard deviation formulation (5) below,

$$\begin{array}{ll} \min\limits_{x} & -\mu^T x + \lambda\sqrt{x^T Q x} \\ \text{subject to} & e^T x = 1, \quad x \ge 0, \end{array} \tag{5}$$

which generates the same MV efficient frontier as (1).

Using the same ellipsoidal uncertainty set (4), the robust min-max optimization problem for (5) becomes

$$\begin{array}{ll} \min\limits_{x} & \max\limits_{\mu} -\mu^T x + \lambda\sqrt{x^T Q x} \\ \text{s.t.} & (\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu) \le \chi \\ & e^T x = 1, \quad x \ge 0\ . \end{array} \tag{6}$$

Theorem 2.1 below [1] shows that the min-max robust portfolio from (6) always corresponds to the optimal mean-standard deviation portfolio (5) based on nominal estimates $\bar{\mu}$ and Q but with the larger risk aversion parameter $\lambda + \sqrt{\chi}$. The proof is presented in Appendix A.

Theorem 2.1. *Assume that Q is symmetric positive definite and $\chi \ge 0$. The min-max robust portfolio for (6) is an optimal portfolio of the mean-standard deviation problem (5) with nominal estimates $\bar{\mu}$ and Q for the larger risk aversion parameter $\lambda + \sqrt{\chi}$.*

From Theorem 2.1, the min-max robust mean-standard deviation model adds robustness by increasing the risk aversion parameter from λ to $\lambda + \sqrt{\chi}$. Thus frontiers from the min-max robust mean-standard deviation model, with the uncertainty set based on (3), are squeezed segments of the frontiers from the mean-standard deviation model based on the nominal estimates, see Figure 1.

In terms of the mean-variance optimal portfolio, the relationship between the risk aversion parameters is not as explicit. It can be shown that the min-max robust mean variance portfolio, which solves

$$\begin{array}{lll} \min\limits_{x} & \max\limits_{\mu} & -\mu^T x + \lambda x^T Q x \\ \text{s.t.} & (\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu) \le \chi & \\ & e^T x = 1, \quad x \ge 0\ , & \end{array} \tag{7}$$

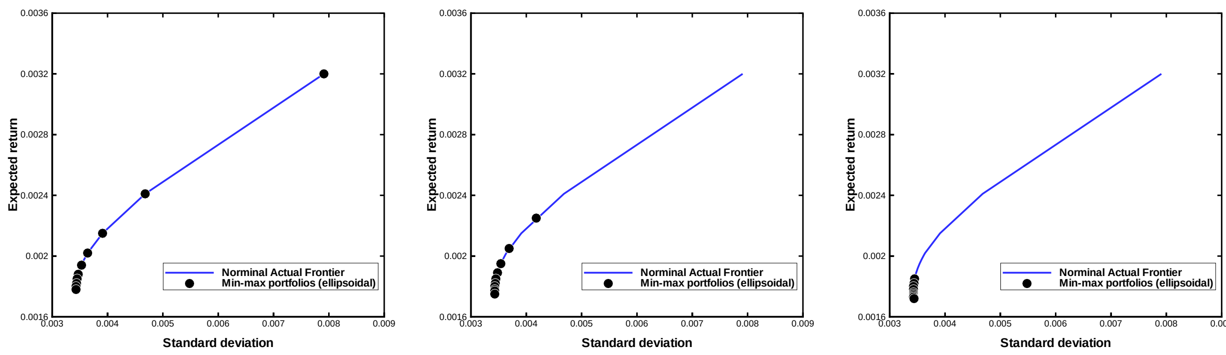
is a standard mean variance optimal portfolio (1) with the nominal estimates $\bar{\mu}$ and Q for some larger risk aversion parameter. This is formally stated in Theorem 2.2. The proof is given in Appendix A.

Theorem 2.2. *Assume that Q is symmetric positive definite and $\chi \ge 0$. Any min-max robust mean-variance portfolio (7) is an optimal mean-variance portfolio* (1) *based on nominal estimates $\bar{\mu}$ and Q with a risk aversion parameter $\hat{\lambda} \ge \lambda$.*

Note that Theorem 2.2 holds if constraint $x \ge 0$ is absent or additional linear constraints are imposed.

The interval uncertainty sets have also been used for robust MV portfolio optimization, e.g., in Tütüncü and Koenig (2004). For example, the uncertainty sets $S_\mu$and S_Q below can be considered,

[1] As is pointed out by a referee, this result has also been observed in Schöttle and Werner (2006) and Meucci (2005).



(a) $\chi = 0$ (b) $\chi = 5$ (c) $\chi = 50$

Figure 1: Min-max robust frontier: squeezed frontier from the nominal problem based on the estimate.

$$
\begin{aligned}
S_\mu &= \{\mu : \mu^L \le \mu \le \mu^U\} , \\
S_Q &= \{Q : Q^L \le Q \le Q^U, Q \succeq 0\} ,
\end{aligned}
$$

where μ^L, μ^U, Q^L and Q^U are lower and upper bounds, and $Q \succeq 0$ indicates that the covariance matrix Q is symmetric positive semi-definite. Tütüncü and Koenig (2004) show that, when $Q^U \succeq 0$, μ^L and Q^U are the optimal solutions for the problem

$$
\max_{\mu \in S_\mu, Q \in S_Q} -\mu^T x + \lambda x^T Q x , \quad \lambda \ge 0,
$$

regardless of the values of nonnegative λ and nonnegative vector x. When Q is assumed to be known, the min-max robust problem (2) with $\Omega = \{x : e^T x = 1, x \ge 0\}$ is reduced to the following standard MV optimization problem:

$$
\begin{array}{ll}
\min\limits_{x} & -(\mu^L)^T x + \lambda x^T Q x \\
\text{s.t.} & e^T x = 1, \ x \ge 0 .
\end{array} \tag{8}
$$

Thus, if the interval uncertainty set is obtained according to a quantile of mean returns, min-max robustness can be regarded as a quantile-based robustness approach. Note that the only difference between (8) and (1) is that μ is replaced by μ^L in (8). Thus the min-max robust MV portfolio now becomes sensitive to specification of μ^L. In practice, variations in μ^L when specified from return samples can be quite large. Moreover, portfolios based on the worst case of return scenario in an uncertainty set leads to very pessimistic performance and the maximum return portfolio typically concentrates on a single asset as in the standard MV portfolio case. Note that adjusting conservatism is done by eliminating the worst sample scenario, which runs counter to the robust objective.

3 CVaR Robust Mean-Variance Portfolios

We can regard uncertainty in mean portfolio return due to estimation error in asset mean returns, which can be considered as estimation risk. Based on statistical properties for the estimates, this estimation risk can be measured using different risk measures, e.g., VaR and CVaR.

We now propose a CVaR robust mean-variance portfolio optimization formulation, in which the return performance is measured by CVaR of the portfolio mean return, when the assets mean returns are uncertain. In contrast to the min-max robust model, which depends on the worst sample scenario of μ, the CVaR robust model produces a portfolio based on a tail of the mean loss distribution.

CVaR, as a risk measure, is based on VaR, which can be regarded as an extension to the notion of the worst case. Consider a specific risk denoted by a random variable L (which typically corresponds to loss). Assume that L has a density function $p(l)$. The probability of L not exceeding a threshold α is given by:

$$\Psi(\alpha) = \int_{l \leq \alpha} p(l)\, dl \; . \tag{9}$$

Here we assume that the probability distribution for L has no jumps; thus $\Psi(\alpha)$ is everywhere continuous with respect to α.

Given a confidence level $\beta \in (0,1)$, e.g., $\beta = 95\%$, the associated Value-at-Risk, VaR_β, is defined as:

$$\text{VaR}_\beta = \min \left\{ \alpha \in R : \Psi(\alpha) \geq \beta \right\} \; . \tag{10}$$

The corresponding CVaR, denoted by CVaR_β, is given by

$$\text{CVaR}_\beta = \mathbf{E}(L \mid L \geq \text{VaR}_\beta) = \frac{1}{1-\beta} \int_{l \geq \text{VaR}_\beta} l p(l) dl. \tag{11}$$

Thus CVaR_β is the expected loss conditional on the loss being greater than or equal to VaR_β.

In addition, CVaR has the following equivalent expression,

$$\text{CVaR}_\beta = \min_\alpha \left(\alpha + (1-\beta)^{-1} \mathbf{E}([L - \alpha]^+) \right) \; , \tag{12}$$

where $[z]^+ = \max(z, 0)$, see Rockafellar and Uryasev (2000).

In contrast to VaR, CVaR is a coherent risk measure and has additional attractive properties such as convexity, see, e.g., Artzner et al. (1997) and Rockafellar and Uryasev (2000). Note that, while VaR is a quantile, CVaR depends on the entire tail of the worst scenarios corresponding to a given confidence level.

We consider a CVaR robust MV optimization by replacing the *mean loss* by a CVaR measure of mean loss. We denote this measure of risk as CVaR^μ, where the superscript μ emphasizes that the risk measure is with respect to the uncertainty in μ. For a portfolio of n assets, we let the decision vector $x \in \Omega$ be the portfolio percentage weights, and $\mu \in R^n$ be the random vector of the mean returns. We assume that μ has a probability density function. Thus $\text{CVaR}^\mu_\beta(-\mu^T x)$ is the mean of the $(1-\beta)$-tail (worst-case) mean loss $-\mu^T x$. In other words,

$$\text{CVaR}^\mu_\beta(-\mu^T x) = \min_\alpha \left(\alpha + (1-\beta)^{-1} \mathbf{E}\left([-\mu^T x - \alpha]^+ \right) \right) \; . \tag{13}$$

Replacing the mean loss $-\mu^T x$ by $\text{CVaR}^\mu_\beta(-\mu^T x)$ in the MV model, a CVaR robust MV efficient portfolio is determined as the solution to the following problem:

$$\begin{array}{ll} \min\limits_x & \text{CVaR}^\mu_\beta(-\mu^T x) + \lambda x^T \bar{Q} x \\ \text{s.t.} & x \in \Omega \; , \end{array} \tag{14}$$

where $\bar{Q}$ is an estimate of the variance matrix Q. Recall that we ignore in this paper the estimation risk in the covariance matrix. Solving (14) with different values of λ ranging from 0 to ∞, we can generate a sequence of CVaR robust optimal portfolios.

Define the following auxiliary function

$$F_\beta(x, \alpha) = \alpha + \frac{1}{1-\beta} \int_{\mu \in R^n} [-\mu^T x - \alpha]^+ p(\mu)\, d\mu \; . \tag{15}$$

Assume that the distribution for μ is continuous, CVaR_β^μ is convex with respect to x, and $F_\beta(x, \alpha)$ is both convex and continuously differentiable. Therefore, for any fixed $x \in \Omega$, $\text{CVaR}_\beta^\mu(-\mu^T x)$ can be determined as follows:

$$\text{CVaR}_\beta^\mu(-\mu^T x) = \min_\alpha F_\beta(x, \alpha) \, . \tag{16}$$

Thus,

$$\min_x \left(\text{CVaR}_\beta^\mu(-\mu^T x) + \lambda x^T \bar{Q} x \right) \equiv \min_{x,\alpha} \left(F_\beta(x, \alpha) + \lambda x^T \bar{Q} x \right) \ , \tag{17}$$

where the objectives on both sides achieve the same minimum values, and a pair $(x^*, \ \alpha^*)$ is the solution of the right hand side if and only if x^* is the solution of the left hand side and $\alpha^* \in \text{argmin}_{\alpha \in R} F_\beta(x^*, \alpha)$.

While the min-max robust optimization neglects any probability information on the mean distribution, once the uncertainty set is specified, CVaR robust portfolios computed from (14) depend on the entire $(1-\beta)$-tail of the mean loss distribution. Using the CVaR robust MV model (14), adjusting the confidence level β of CVaR^μ naturally corresponds to adjusting an investor's tolerance to estimation risk. When the β value increases, the corresponding CVaR_β^μ of the mean loss increases. For a high confidence level (β close to 1), the optimization focuses on extreme mean loss scenarios; this corresponds to an investor who is highly averse to the estimation risk in μ. The resulting optimal portfolio tends to be more robust. Conversely, when the β value decreases, the resulting optimal portfolio becomes less robust. As $\beta \to 0$, all scenarios of the mean loss are considered; thus less emphasis is placed on the worst mean loss scenarios. Note that the choice of β (or portfolio's robustness) implicitly affects the portfolio's expected return: the maximum expected return achievable for a higher β is generally less than that for a lower β. The choice of β depends on an individual investor's risk-averse characteristics with respect to the estimation risk in μ.

Using Monte Carlo simulations, problem (14) can be solved as a quadratic programming problem. Given $\mu_1, \mu_2, \ldots, \mu_m$, where each μ_i is an independent sample of the mean return vector from its assumed distribution, a CVaR robust MV optimization problem (14) can be approximated by the following QP problem:

$$\begin{array}{ll} \min\limits_{x,z,\alpha} & \alpha + \dfrac{1}{m(1-\beta)} \displaystyle\sum_{i=1}^m z_i + \lambda x^T \bar{Q} x \\ \text{s.t.} & x \in \Omega \, , \\ & z_i \geq 0 \, , \\ & z_i + \mu_i^T x + \alpha \geq 0, \ i = 1, \ldots, m \, . \end{array} \tag{18}$$

This QP problem has $O(m+n)$ variables and $O(m+n)$ constraints, where m is the number of μ-samples and n is the number of assets.

Using concrete examples and the QP formulation (18), next we demonstrate properties of the CVaR robust portfolios and the impact of the β value.

4 Comparing Min-Max Robust and CVaR Robust MV Portfolios

In this section, we compare min-max robust portfolios with CVaR robust portfolios in terms of robustness, efficiency, and diversification properties. In the subsequent computational examples, we assume that return samples are drawn from a joint multi-normal distributions with a known mean returns μ and covariance matrix Q. We evaluate actual performance of the min-max robust and CVaR robust portfolios.

Both the CVaR robust model and the min-max robust model depend on the distribution assumption of μ, assuming the uncertainty interval for μ in the latter corresponds to a confidence level. Unfortunately, in general, this distribution may not be known. In practice, one can use the resampling technique (RS), see, e.g., Michaud (1998), to generate some possible/reasonable realizations. We implement this technique as follows. Assume that the initial 100 return samples are from the normal distribution with mean μ and covariance matrix Q. We then compute the mean $\bar{\mu}$ and covariance matrix estimate $\bar{Q}$ based on these return samples. Assuming that $\bar{\mu}$ and $\bar{Q}$

are representative of μ and Q, we simultaneously generate 10,000 sets of independent return samples, each set consisting of 100 return samples. Regarding each set of 100 samples as equally likely to be observed, we compute the mean of each sample set and obtain 10,000 estimates of mean returns as equally likely. These 10,000 estimates now form the uncertain set for the mean return. In addition, the boundary vectors μ^L and μ^U can be determined by selecting the lowest and highest values respectively from these estimates for mean returns.

Alternatively, we can generate samples which are consistent with the statistical property (3), i.e., $\frac{T(T-n)}{(T-1)n}(\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu)$ has a χ^2_n distribution with n degrees of freedom. This technique is subsequently referred to as the CHI technique.

Let GG^T be the Cholesky factorization for the symmetric positive semi-definite matrix Q, where G is a lower triangular matrix. Equation (3) specifies that the square of the 2-norm of $y = G^{-1}(\bar{\mu}-\mu)$ has a χ^2_n distribution. Given a sample ϕ from the χ^2_n distribution, we generate a sample y which is uniformly distributed on the sphere$\|y\|_2^2 = \frac{(T-1)n}{T(T-n)}\phi$. This can easily be done using the normal-deviate method, see, e.g., Muller (1995) and Marsaglia (1972), as follows: let $z = [z_1, z_2, \ldots, z_n]^T$ be $n \times 1$ independent standard normals and obtain y from $y = \sqrt{\frac{(T-1)n}{T(T-n)}\phi}\left(\frac{z}{\|z\|_2}\right)$

If we generate m independent samples from the χ^2_n distribution, then the described computation generates m independent samples of y uniformly distributed on the corresponding spheres. Thus we obtain m independent μ-samples via $\mu = \bar{\mu} + Gy$. We consider both RS and CHI sampling techniques for each example in the subsequent computational investigation.

To analyze the quality of efficient frontiers from robust optimization, similar to Broadie (1993), we consider the actual frontier , which demonstrates the actual performance of the portfolios based on estimates. The actual frontier is the curve $\{(\sqrt{x(\lambda)^T Q x(\lambda)}, \mu^T x(\lambda)),\ \lambda \geq 0\}$ in the space of standard deviation and mean of the portfolio return, where $x(\lambda)$ is the optimal portfolio with the risk aversion parameter λ. For example, if $x(\lambda)$ is obtained from min-max robust portfolio optimization, this is referred to as the actual min-max frontier.

We first consider a 10-asset example with data given in Table 6 in the appendix. We generate μ-samples using the RS sampling technique and the CHI sampling technique as described. For a set of 10,000 samples (which depends on the initial 100 return samples) of μ, we obtain a CVaR robust actual frontier by solving the CVaR robust problem (18) for different λ values. For the 10-asset example using CHI-sampling, Figure 2 compares the actual frontier from the CVaR robust formulation with the actual frontier from the standard MV optimization based on the nominal estimates. We note that, unlike min-max robust with the ellipsoidal uncertainty set based on the statistics (3), this CVaR actual frontier lies above the actual frontiers from the MV optimization based on the nominal estimates.

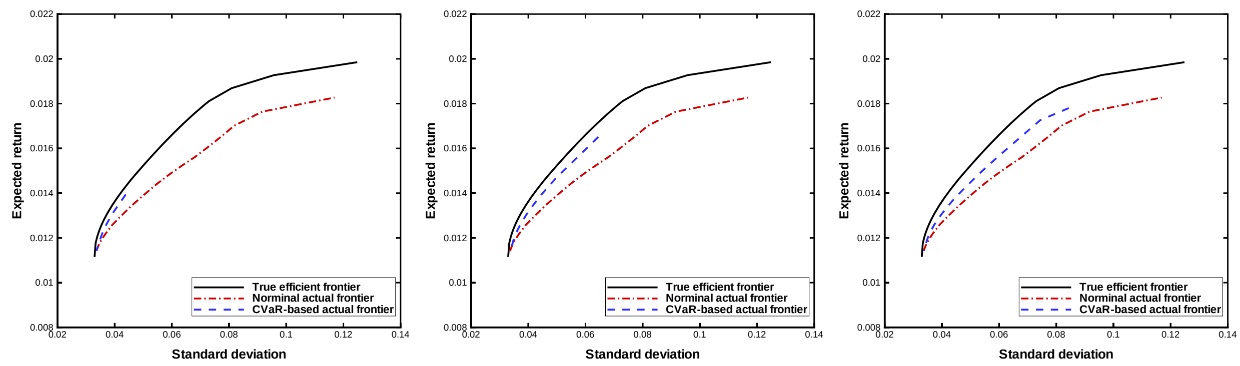
To illustrate characteristics of the actual frontier, we repeat this computation 100 times, each with different 100 random initial return samples. For each 10,000 μ-samples generated, we compute three separate actual frontiers for confidence levels $\beta = 90\%$, 60% and 30% respectively. The top plots (a)–(c) in Figure 3 are for the RS technique, and the bottom plots (d)–(f) are for the CHI sampling technique. Note that, the right-most points on actual frontiers correspond to the portfolios with the maximum-return achievable using the CVaR robust formulation.

We make the following three main observations regarding the CVaR Robust Portfolios.

CVaR robust actual frontiers vary with the initial data.

Similar to the min-max robust actual frontiers, the CVaR robust actual frontiers vary with the initial data used to generate sets of μ-samples. The variation of actual frontiers mainly comes from the variation in the estimate $\bar{\mu}$, computed from 100 initial return samples. Since only a limited number of return samples are available in practice, variations inevitably exist in robust MV models, whether min-max robust or CVaR robust is considered. The level of variation can be considered as an indicator of the level of estimation risk exposed by portfolios from a robust model. It can be observed that the variation in actual frontiers seems to increase as the confidence level β decreases.

A more risk averse investor who expects to take less estimation risk may choose a larger β. On the other hand, an investor, who is tolerant to estimation risk, may choose a smaller β. The plots in Figure 3 depict the



(a) CHI: 90% confidence level (b) CHI: 60% confidence level (c) CHI: 30% confidence level

Figure 2: CVaR robust actual frontiers and actual frontiers based on MV optimization with nominal estimates for the 10-asset example. Nominal actual frontiers are calculated by using the standard MV model with parameter $\bar{\mu}$ estimated based on 100 return samples (with data in Table 6).

positive association between β and a portfolio's conservatism level.

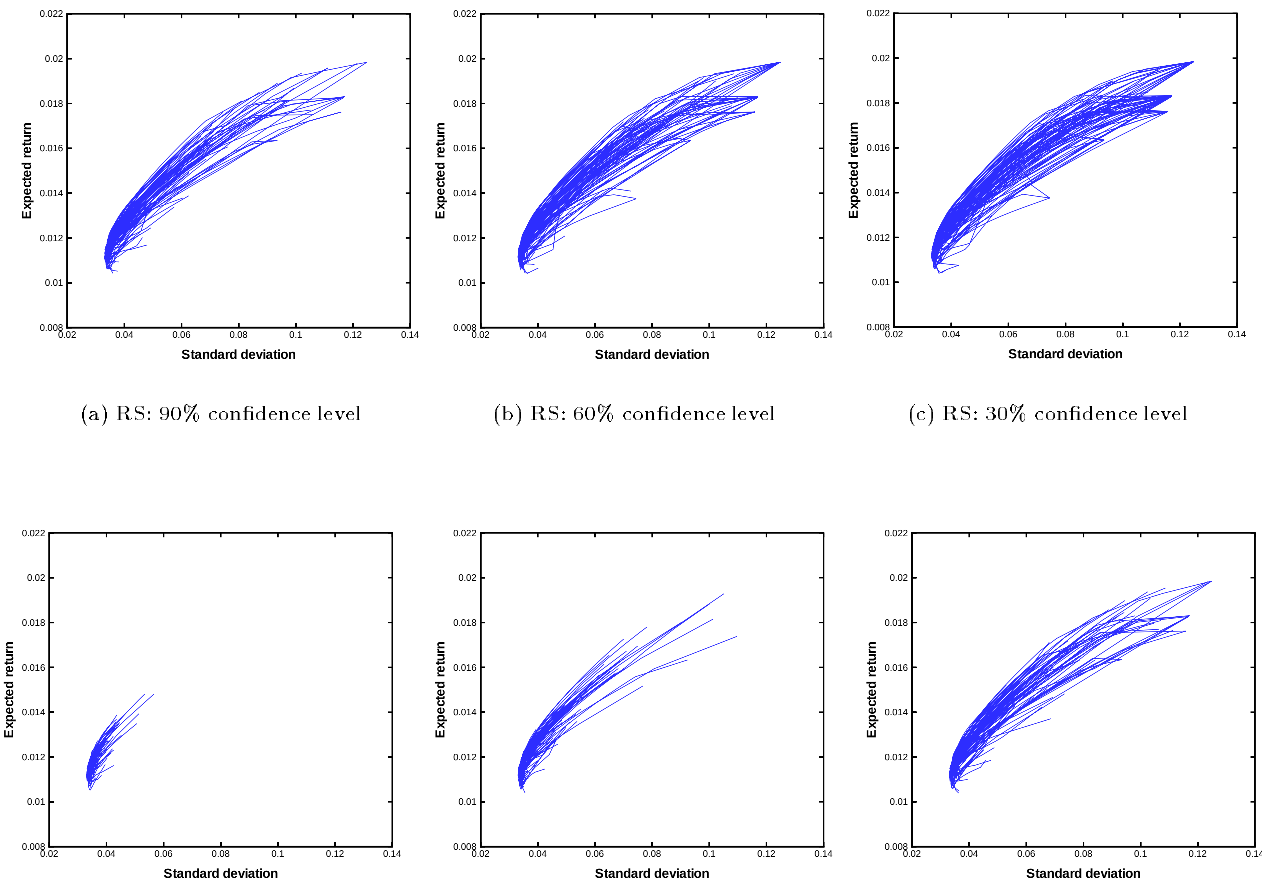
In addition, we note that the variations of the actual frontiers in Figure 3(a)-(c) are larger than the ones in Figure 3(d)-(f). Figure 9(a)-(h) in Appendix C compares the (marginal) distribution for each of the 8 assets in Appendix B generated by using the RS and CHI sampling techniques, with data provided in Table 5. As can be seen, the samples obtained from the CHI technique have larger variance, which may explain the difference in actual frontiers from the two sampling techniques.

Higher expected return can be achieved with a smaller confidence level β.

In addition to variation in actual frontiers, we also evaluate the "average" performance of these actual frontiers. We plot the"average" actual frontiers graphed in Figure 3 against the true efficient frontier in Figure 4. The true efficient frontier is used as a benchmark to assess the portfolio efficiency. The plots for the RS technique are on the top panel, while the ones for the CHI technique are on the bottom panel. As can be seen, when β approaches 1, CVaR robust actual frontiers become shorter on average; the maximum expected return achievable becomes lower. As expected, an investor, who is more averse to estimation risk, obtains smaller return; this confirms that it is reasonable to regard β as an indicator for the level of tolerance for estimation risk. On the other hand, an investor who is more tolerant towards estimation risk chooses a smaller β and the maximum expected return achievable becomes higher.

CVaR robust actual frontiers generated using the RS and the CHI sampling techniques also have different "average" performance. The "average" CVaR-based actual frontiers in Figure 4(d)–(f) achieve lower maximum expected returns than the corresponding ones in Figure 4(a)–(c). This happens because the μ-samples generated using the CHI technique have larger deviations, which leads to worse mean loss scenarios.

It is also important to note that, although changing the confidence level affects the highest expected return achievable, the deviation of the CVaR robust actual frontiers from the true efficient frontier does not seem to be affected. In addition, on "average", the deviation seems to be relatively insensitive for different sampling methods. On the other hand, the deviation from the true efficient frontier for the min-max actual frontiers varies significantly with the return percentile, which specifies μ^L. This can be observed from Figure 5(a)-(c) where 100 min-max actual frontiers in each plot are computed based on different percentiles corresponding to μ^L. The μ samples, based on which the percentiles are calculated, are generated using the CHI sampling technique. Note



(a) RS: 90% confidence level (b) RS: 60% confidence level (c) RS: 30% confidence level

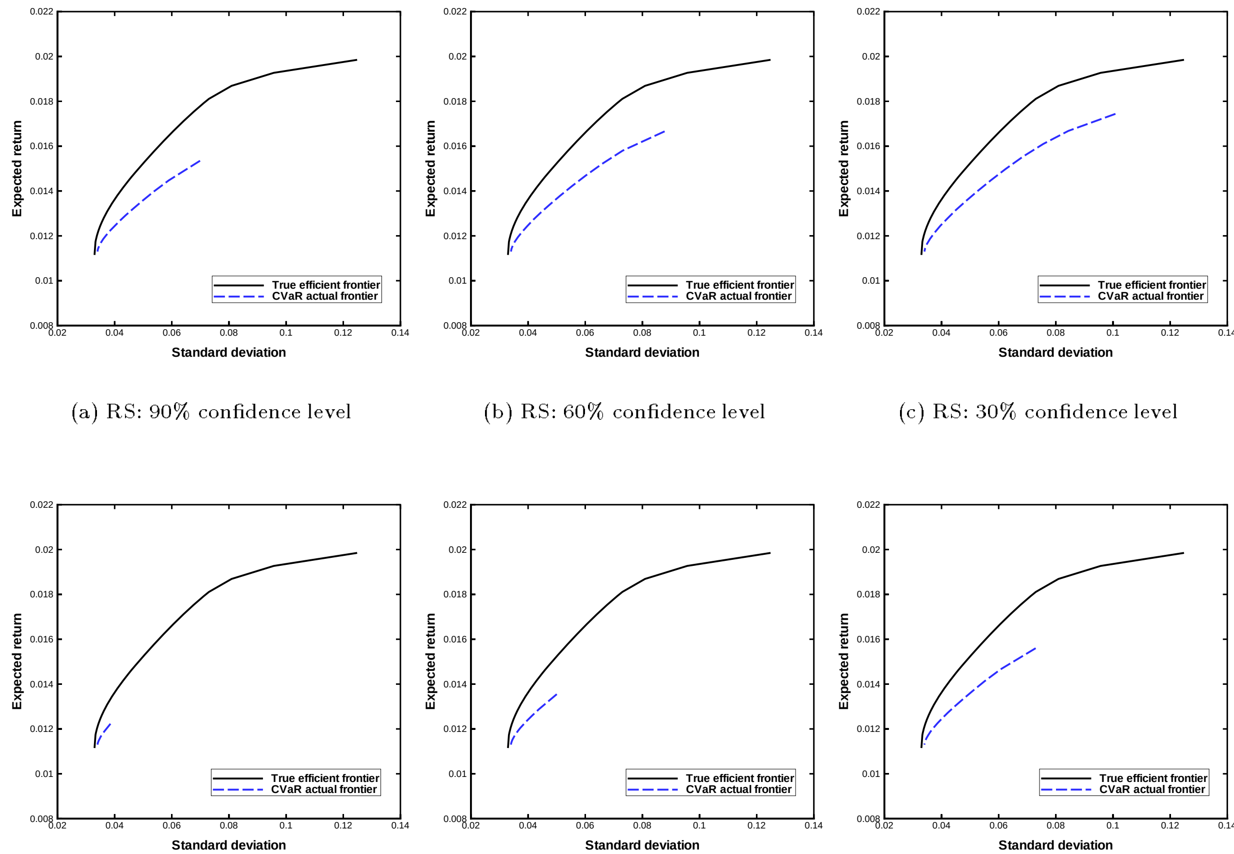
(d) CHI: 90% confidence level (e) CHI: 60% confidence level (f) CHI: 30% confidence level

Figure 3: 100 CVaR robust actual frontiers calculated based on 10,000 μ-samples. The 10-asset example (with data in Table 6).

that the same μ samples, used for generating the CVaR actual frontiers in Figure 3(d)-(f), are also used here. Note also that zero percentile corresponds to the case when μ^L equals to the worst return scenario, and the resulting min-max actual frontiers in Figure 5(a) consist of the portfolios which have the best performance for the worst sample scenario. When choosing 50 percentile for μ^L, half of the μ samples are excluded from the uncertainty set. As can be seen clearly, when the percentile value changes from 0 to 50, not only the variation but also the overall appearance of the min-max actual frontiers change significantly. This causes their actual "average" frontiers, which are plotted in Figure 5(d)-(f), have different deviations from the true efficient frontier. In addition, for this 10-asset example, the min-max actual frontiers in Figure 5(a)-(c) exhibit more variations in comparison with the CVaR actual frontiers in Figure 3(d)-(f).

CVaR robust portfolios are more diversified.

It is common sense that portfolio diversification reduces risk. Portfolio diversification means spreading the total investment across a wide variety of assets and the exposure to individual asset risk is reduced.



(a) RS: 90% confidence level

(b) RS: 60% confidence level

(c) RS: 30% confidence level

(d) CHI: 90% confidence level

(e) CHI: 60% confidence level

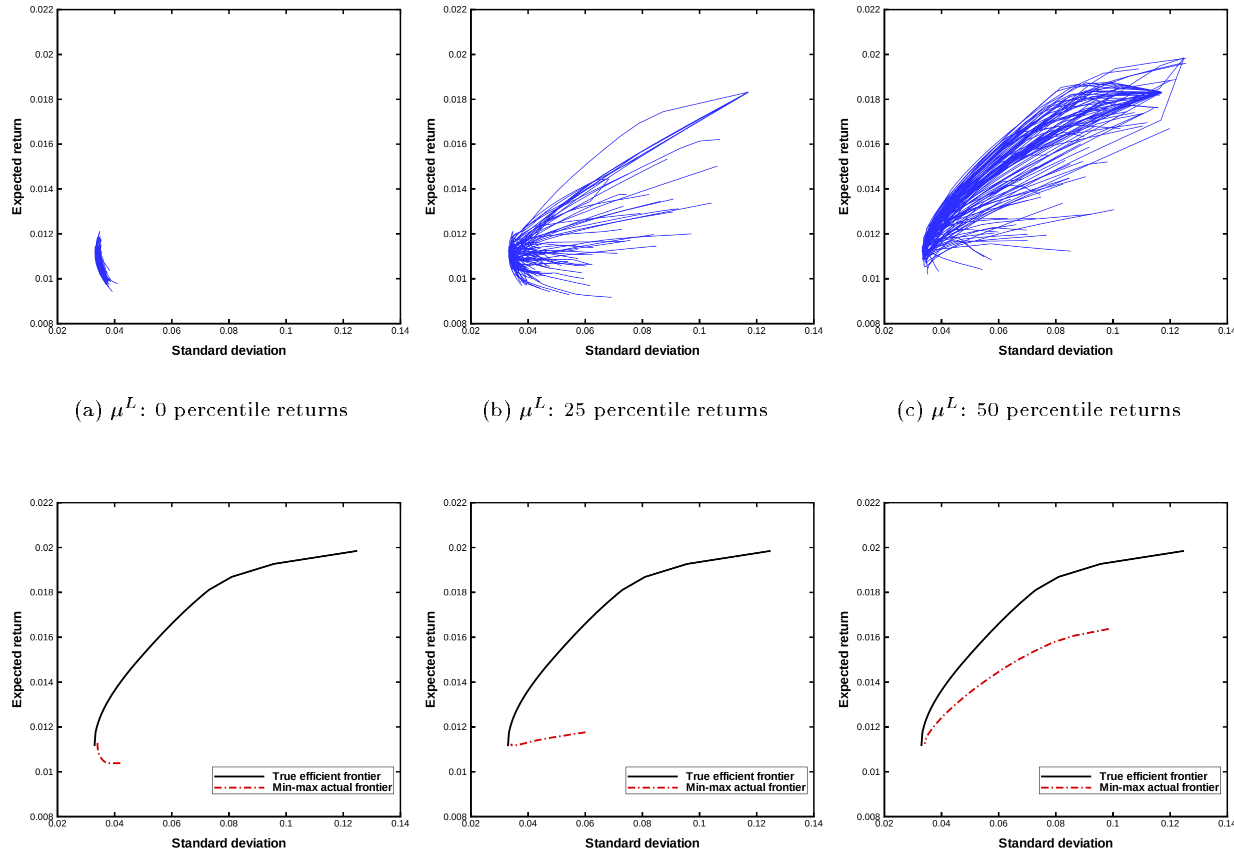
(f) CHI: 30% confidence level

Figure 4: Average CVaR robust actual frontiers calculated based on 10,000 μ-samples. The 10-asset example.

The traditional MV model (1) has the following diversification characteristics. As the risk-aversion parameter λ decreases, the level of diversification decreases. This will increase both the portfolio expected return and its associated return risk. When $\lambda = 0$, the portfolio typically achieves the highest expected return by allocating all investment in the highest-return asset without considering the associated return risk. The portfolio with $\lambda = 0$ is referred to as the maximum-return portfolio. In fact, even with $\lambda \neq 0$ but sufficiently small, the optimal MV portfolio tends to concentrate on a single asset. Given that the exact mean return is unknown, this means that the optimal MV portfolios can concentrate on a wrong asset due to estimation error. This can result in potentially disastrous performance in practice.

For the min-max robust MV model (2) with an interval uncertainty set for μ, the min-max robust portfolio is determined by the lower bound of the interval, μ^L. Thus, for the maximum-return portfolio computed from the min-max robust model, the allocation is still typically concentrated in a single asset. *Note that this is independent of the values of* μ^L. Moreover, due to estimation error, this allocation concentration may not necessarily result in a higher actual portfolio expected return. As an example, Figure 5 depicts that, on "average", the maximum expected return of the min-max actual frontier is significantly lower than the one of the true efficient frontier.

Instead of focusing on the single worst-case scenario μ^L of μ, the CVaR robust formulation yields an optimal portfolio by considering the $(1-\beta)$-tail of the mean loss distribution. This forces the resulting portfolio to be



(a) μ^L: 0 percentile returns

(b) μ^L: 25 percentile returns

(c) μ^L: 50 percentile returns

(d) Average. μ^L: 0 percentile returns

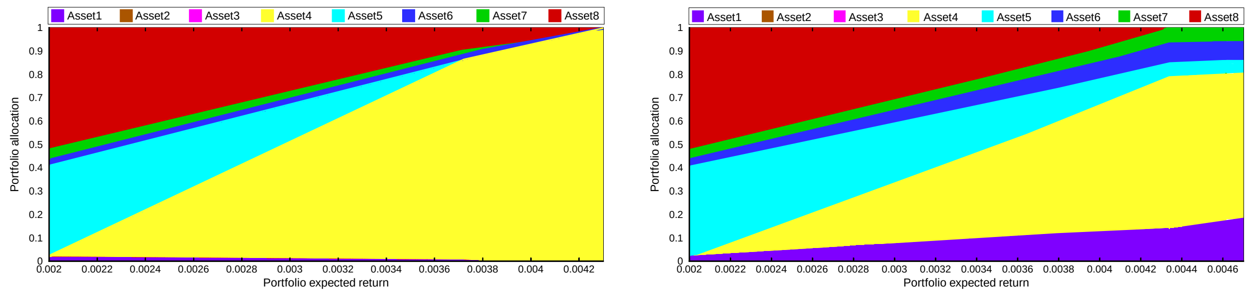
(e) Average. μ^L: 25 percentile returns

(f) Average. μ^L: 50 percentile returns

Figure 5: 100 min-max actual frontiers based on different percentiles for μ^L for the 10-asset example. Samples of μ are generated using the CHI technique.

more diversified. Therefore, even when ignoring return risk (i.e., $\lambda = 0$), the allocation of the CVaR robust portfolio (which typically achieves the maximum-return for the given β) is usually distributed among more than one asset, if β is not too small. We illustrate this next with examples.

Our first example illustrates the diversification property of the maximum-return portfolio computed from the CVaR robust model. We compute both the min-max robust and the CVaR robust($\beta = 90\%$) actual frontiers for the 8-asset example with data given in Table 5 in Appendix B. The computations are based on 10,000 mean return samples generated from the CHI-sampling technique. Each frontier consists of the portfolios computed using a sequence of λ ranging from 0 to 1000. We compare the composition graphs of the portfolios on the two actual frontiers. They are presented in Figure 6(a) and 6(b) respectively. For the minimum-return portfolio at the left-most end of each composition graph, most of the investment is allocated in Asset 5 and Asset 8. As the expected return value increases from left to right, both assets are gradually replaced by a mixture of other assets. However, close to the maximum-return end of the graphs, the compositions in Figure 6(b) are more diversified than that in Figure 6(a). In Appendix D, Table 7(a) and 7(b) list the portfolio weights of the two actual frontiers for each λ value. When $\lambda = 0$, the min-max robust maximum-return portfolio in Table 7(a) focuses all holdings in Asset 4, whereas the CVaR robust portfolio are diversified into five different assets, see also Table 7(b) in Appendix D.

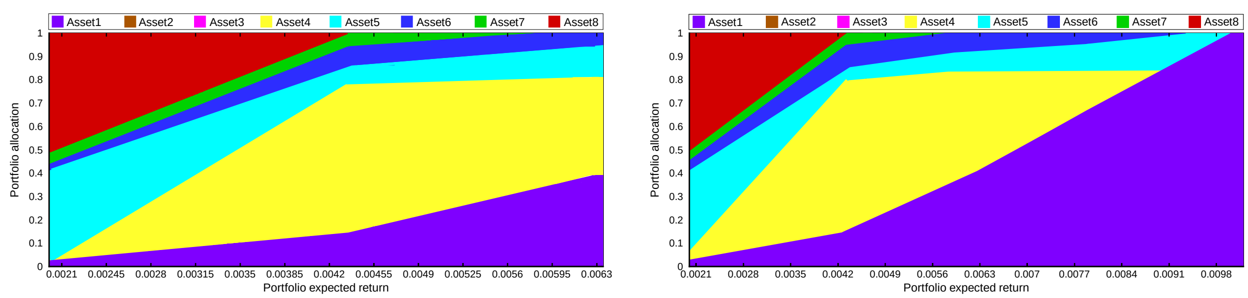


(a) Min-Max robust portfolios

(b) CVaR robust (90%) portfolios

Figure 6: Compositions of min-max robust and CVaR robust (90%) portfolio weights.

Next, we illustrate the impact of the choice of the confidence level β on diversification. Using the same data as in the first example, we compute the CVaR robust actual frontiers for $\beta = 60\%$ and $\beta = 30\%$. The portfolios' composition graphs are presented in Figure 7(a) and 7(b), respectively. The portfolio weights corresponding to the frontiers are listed in Table 8(a) and 8(b) in Appendix D, respectively. Comparing the compositions in Figure 6(b), 7(a) and 7(b), it can be observed that the weights become less diversified as the value of β decreases. In particular, when $\lambda = 0$, the CVaR robust portfolio for $\beta = 30\%$ in Table 8(b) allocate all investment in a single asset. Unlike the min-max robust portfolio in Table 7(a), which is concentrated on Asset 4, this portfolio is concentrated in Asset 1.



(a) CVaR robust (60%) portfolios

(b) CVaR robust (30%) portfolios

Figure 7: Compositions of CVaR robust (60%) and (30%) portfolio weights.

For the CVaR robust model, the relationship between decrease in diversification and decrease in β further confirm that it is reasonable to regard β as a risk aversion parameter for estimation risk. An investor who is risk averse to the estimation risk can naturally choose a large β value and obtain a more diversified portfolio. As discussed before, this portfolio typically achieves less expected return. The risk averse investor can also expect less variations, with respect to the initial data, in the portfolios generated from the CVaR robust model with a large β.

5 An Efficient computational Technique for Computing CVaR Robust Portfolios

One potential disadvantage of the CVaR robust formulation (14), in comparison to the min-max robust formulation (2), is that it may require more time to compute a CVaR robust portfolio than a min-max robust portfolio.

In Section 3, we have shown that the CVaR robust portfolio optimization problem (14) can be approximated by a QP problem (18). Given a finite number of mean return samples, the LP approach uses a piecewise linear function to approximate the continuous differentiable CVaR function. When more samples are used, the approximation becomes more accurate. However, we illustrate that this QP approach can become inefficient for large scale CVaR optimization problems. These computational efficiency issues have been investigated in Alexander et al. (2006) for CVaR minimization problems. The main difference is that the CVaR robust MV portfolio problem (14) in this paper has an additional quadratic term $x^T Qx$ because variance is used as the return risk measure. We now compare the QP approach (18) and the smooth technique proposed in Alexander et al. (2006) in term of the efficiency for computing CVaR robust MV portfolios. We note that the machine used in this study is different from the one used in Alexander et al. (2006), and the computing platform and softwares are also different versions. The computation in this paper is done in MATLAB version 7.3 for Windows XP, and run on a Pentium 4 CPU 3.00GHz machine with 1 GB RAM. QP problems are solved using the MOSEK Optimization Toolbox for MATLAB version 7.

In Section 3, we have stated that a CVaR robust MV portfolio can be computed approximately by solving a QP (18):

$$\begin{array}{ll} \min\limits_{x,z,\alpha} & \alpha + \dfrac{1}{m(1-\beta)} \displaystyle\sum_{i=1}^{m} z_i + \lambda x^T \bar{Q} x \\ \text{s.t.} & x \in \Omega \,, \\ & z_i \geq 0 \,, \\ & z_i + \mu_i^T x + \alpha \geq 0, \;\; i = 1, \ldots, m \,. \end{array}$$

A convex QP is one of the simplest constrained optimization problems, and can be solved quickly using software such as MOSEK. However, this QP approach can become inefficient when the number of simulations and the number of assets become large. In this formulation, generating a new sample will add an additional variable and constraint. For n risky assets and m mean return samples, the problem has a total of $O(n+m)$ variables and $O(n+m)$ constraints. Alexander et al. (2006) analyze the computation cost of both the simplex method and the interior-point method when they are used in the LP approach for CVaR optimization. They show that computational costs using both methods can quickly become quite large as the number of samples and/or assets becomes large. The efficiency of a QP solver such as MOSEK depends heavily on the sparsity structures of the QP problem. The QP problem (18) has an m-by-$(n+1)$ dense block in the constraint matrix.

In Table 1 we report the CPU time required to solve the simulation CVaR optimization problem (18) for different asset examples with different number of simulations. In this computation, we set the risk aversion parameter $\lambda = 0$; thus (18) is a LP. Both the RS technique and the CHI technique are considered to generate the mean return samples.

From Table 1, it is clear that, when using MOSEK, the computational cost increases quickly as the sample size and the number of assets increase. For examples, for each size of the RS samples, the CPU time required by the 50-asset example is at least twice as that required by the 8-asset one. When the size of the CHI samples is increased from 10,000 to 25,000, the CPU time is increased by at least 150% for each asset example. Note that the CPU time reported here is for solving a single QP for a given risk aversion parameter λ. To generate an efficient frontier, many QP problems need to be solved for different risk aversion parameter values. This results in very large CPU time difference for generating an efficient frontier.

	RS Tech (CPU sec)			CHI Tech (CPU sec)		
# samples	8 assets	50 assets	148 assets	8 assets	50 assets	148 assets
5000	0.41	1.84	9.77	0.39	1.75	7.06
10,000	0.88	3.56	20.41	0.77	4.25	10.38
25,000	2.78	9.17	32.69	2.56	10.83	34.97

Table 1: CPU time for the QP approach when $\lambda = 0$: $\beta = 0.90$

A Smoothing Approach for CVaR Robust MV Portfolios

As an alternative to the QP approach, we can solve the CVaR minimization problem more efficiently via a smoothing technique proposed by Alexander et al. (2006). The smoothing technique directly exploits the structure of the CVaR minimization problem. It has been shown in Alexander et al. (2006) that the smoothing approach is computationally significantly more efficient than the LP method for CVaR optimization problem. We investigate the computational performance comparison between the QP approach and the smoothing approach for CVaR robust MV portfolios.

As mentioned in Section 3,

$$\min_{x} \left(\mathrm{CVaR}^{\mu}_{\beta}(x) + \lambda x^T \bar{Q} x \right) \equiv \min_{x,\alpha} \left(F_\beta(x,\alpha) + \lambda x^T \bar{Q} x \right) \ ,$$

where

$$F_\beta(x,\alpha) = \alpha + \frac{1}{1-\beta} \int_{\mu \in R^n} [f(x,\mu) - \alpha]^+ p(\mu)\, d\mu \ . \tag{19}$$

Note that the function $F_\beta(x,\alpha)$ is both convex and continuously differentiable when the assumed distribution for μ is continuous.

The QP approach (18) approximates the function $F_\beta(x,\alpha)$ by the following piecewise linear objective function:

$$\bar{F}_\beta(x,\alpha) = \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} [-\mu_i^T x - \alpha]^+ \ , \tag{20}$$

where each μ_i is a mean vector sample. When the number of mean return samples increases to infinity, the approximation approaches to the exact function.

Instead of using $\bar{F}_\beta(x,\alpha)$, Alexander et al. (2006) suggest a piecewise quadratic function $\tilde{F}_\beta(x,\alpha)$ to approximate $F_\beta(x,\alpha)$. Let

$$\tilde{F}_\beta(x,\alpha) = \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} \rho_\epsilon(-\mu_i^T x - \alpha), \tag{21}$$

where $\rho_\epsilon(z)$ is defined as:

$$\begin{cases} z & \text{if } z \geq \epsilon \\ \frac{z^2}{4\epsilon} + \frac{1}{2} z + \frac{1}{4}\epsilon & \text{if} -\epsilon \leq z \leq \epsilon \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

where $\epsilon > 0$ is a given resolution parameter. Note that $\rho_\epsilon(z)$ is continuous differentiable and approximates the piecewise linear function $\max(z,0)$. Figure 8 illustrates smoothness of $\frac{1}{m}\sum_{i=1}^{m} \max(z_i - \alpha, 0)$ and $\frac{1}{m}\sum_{i=1}^{m} \rho_\epsilon(z_i - \alpha)$ for $m = 3$ and $m = 10,000$ respectively.

Applying the smoothing formulation (21), CVaR robust model (14) can be formulated as the following problem:

$$\begin{array}{ll} \min\limits_{x,\alpha} & \alpha + \dfrac{1}{m(1-\beta)} \displaystyle\sum_{i=1}^{m} \rho_\epsilon(-\mu_i^T x - \alpha) + \lambda x^T \bar{Q} x \\ \text{s.t.} & x \in \Omega. \end{array} \tag{23}$$

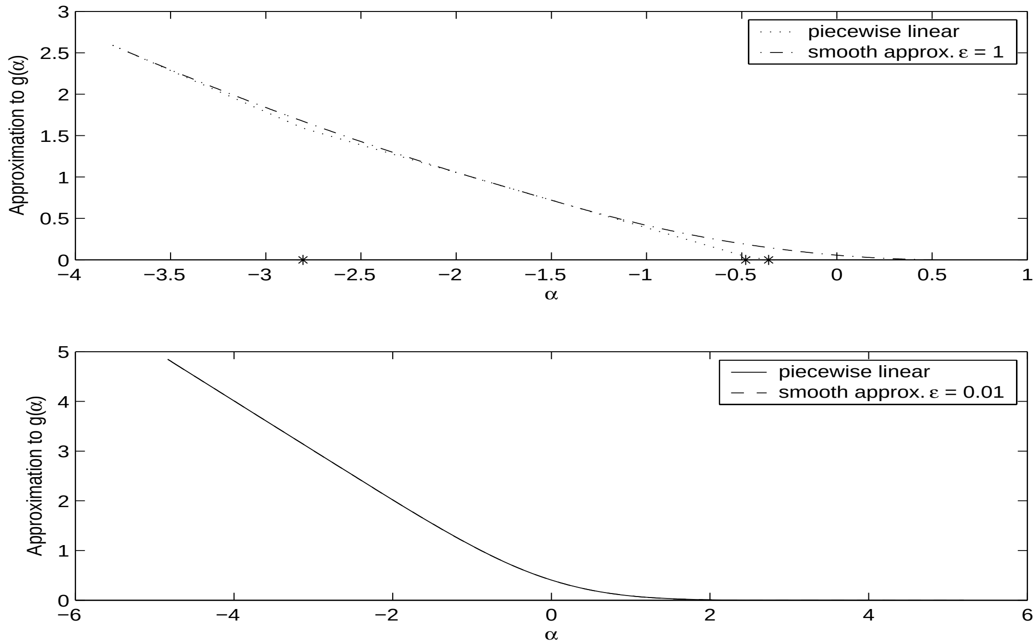


Figure 8: Smooth approximation and piecewise linear approximation for $g(\alpha) = \mathbf{E}(\max(\mu - \alpha, 0))$. For the top plot $m = 3$. For the bottom plot, $m = 10,000$

While QP (18) has a total of $O(n+m)$ variables and $O(n+m)$ constraints, the smoothing formulation (23) only has $O(n)$ variables and $O(n)$ constraints. *Therefore, increasing the sample size m does not change the number of variables and constraints.*

In Table 2, we report the CPU time for the smoothing method (23) for the same examples in Table 1, which is included again for comparison. The smoothed minimization problem (23) is solved using the interior point method by Coleman and Li (1996) for nonlinear minimization with bound constraints. The computation is done for both the RS and CHI sampling techniques, for which the CPU time is reported in Table 2(a) and 2(b) respectively. Comparing the CPU time between the two approaches, we observe that the smoothing approach is much more efficient than the QP approach for both sampling techniques. The problem of 148 assets and 25,000 samples can now be solved in less than eleven CPU seconds using the smoothing approach; while the same problems are solved in more than thirty CPU seconds via the QP approach. The CPU efficiency gap increases as the scale of the problem (including sample size and the number of assets) becomes larger. For 8 assets and 5000 samples, there is a small difference between the CPU time used by the two approaches. However, when the number of assets exceeds 50 and the sample size exceeds 5000, the difference becomes significant. These comparisons show that the smoothing approach achieves significantly better computational efficiency.

Using four different λ values, Table 3 illustrates that, while the CPU time required by QP increases significantly with the risk aversion parameter, the time required by the smoothing method is relatively insensitive to the value of λ.

To analyze the accuracy of the smoothing approach (23), we determine the following relative difference in the CVaR^{μ} value computed via the smoothing approach,

$$Q_{\text{CVaR}^{\mu}} = \frac{\text{CVaR}_s^{\mu} - \text{CVaR}_m^{\mu}}{|\ \text{CVaR}_m^{\mu}\ |} \tag{24}$$

where CVaR_m^{μ} and CVaR_s^{μ} are the CVaR^{μ} values obtained by using the QP approach (18) and the smoothing approach (23), respectively. Table 4 compares the $\text{Q}_{\text{CVaR}^{\mu}}$ in percentage for different sample sizes and ϵ values.

	MOSEK (CPU sec)			Smoothing (CPU sec)		
# samples	8 assets	50 assets	148 assets	8 assets	50 assets	148 assets
5000	0.41	1.84	9.77	0.34	0.50	2.55
10,000	0.88	3.56	20.41	0.56	1.34	4.08
25,000	2.78	9.17	32.69	1.22	3.28	8.11

(a) RS Technique

	MOSEK (CPU sec)			Smoothing (CPU sec)		
# samples	8 assets	50 assets	148 assets	8 assets	50 assets	148 assets
5000	0.39	1.75	7.06	0.42	0.34	1.98
10,000	0.77	4.25	10.38	0.75	0.50	4.13
25,000	2.56	10.83	34.97	1.77	1.36	10.25

(b) CHI Technique

Table 2: CPU time for computing maximum-return portfolios ($\lambda = 0$) MOSEK vs. Smoothing ($\epsilon = 0.005$): $\beta = 90\%$

	MOSEK (CPU sec)				Smoothing (CPU sec)			
# samples	$\lambda = 0$	0.1	10	1000	0	0.1	10	1000
5000	10.42	11.13	14.75	15.19	2.31	2.16	2.14	2.58
10,000	18.33	42.77	29.41	36.66	3.70	3.55	4.00	3.36
25,000	29.59	89.06	95.31	122.72	7.66	7.95	7.16	7.58

Table 3: CPU time for different λ values for the 148-asset example: $\beta = 90\%$ ($\epsilon = 0.005$)

As can be seen, given the same ϵ, the absolute value of $Q_{\mathrm{CVaR}^{\mu}}$ decreases when the sample size increases, which indicates the difference between the CVaR^{μ} values approximated by the two approaches become smaller. In addition, decreasing the value of ϵ also decreases the difference.

# samples	50 assets	148 assets	200 assets
10000	-1.1225	-0.2253	-0.2260
25000	-0.0939	-0.0889	-0.0883
50000	-0.0513	-0.0459	-0.0472

(a) $\epsilon = 0.005$

# samples	50 assets	148 assets	200 assets
10000	-0.2974	-0.2236	-0.2234
25000	-0.0934	-0.0882	-0.0880
50000	-0.0504	-0.0454	-0.0466

(b) $\epsilon = 0.001$

Table 4: Relative difference $Q_{\mathrm{CVaR}^{\mu}}$ (in percentage) for different sample sizes and ϵ values, $\beta = 95\%$ and $\lambda = 0$

6 Concluding Remarks

The classical MV portfolio optimization is typically based on the nominal estimates of mean returns and a covariance matrix from a set of return samples. Given that the number of return samples is limited in practice, MV frontiers can vary significantly with the set of initial return samples, potentially resulting in extremely poor actual performance.

In this paper, we investigate the impact of estimation risk and how it is addressed in a robust MV portfolio optimization formulation. We consider estimation risk only in mean returns and assume that the covariance matrix is known.

Recently, min-max robust portfolio optimization has been proposed to address the estimation risk. We show that, with an ellipsoidal uncertainty set based on the statistics of the sample mean estimates, the robust portfolio from the min-max robust mean-variance model equals to the optimal portfolio from the standard mean-variance model based on the nominal mean estimate but with a larger risk aversion parameter. Assuming that the uncertainty set is an interval $[\mu^L, \mu^U]$, the min-max robust portfolio is essentially the MV optimal portfolio generated based on the lower bound μ^L, which can be difficult to select in general. The min-max robust MV portfolio can also be very sensitive to the initial data used to generate an uncertainty set. The min-max robust optimization problem becomes more complex when other types of uncertainty sets are used. By nature, the min-max robust model emphasizes the best performance under the worst-case scenario. Adjustment of the level of conservatism in the min-max robust model can be achieved by excluding bad scenarios from the uncertainty sets, which is unappealing. The min-max robust portfolio also ignores any probability information in the uncertain data.

We propose a CVaR robust MV portfolio formulation to address estimation risk. In this model, a robust portfolio is determined based on a set of worst-case mean returns, rather than nominal estimates (classical MV) or a single worst-case scenario (min-max robust). When the confidence level β is high, CVaR robust optimization focuses on a small set of extreme mean loss scenarios. The resulting portfolios are optimal against the average of these extreme mean loss scenarios and tend to be more robust. In addition, actual frontiers with a larger confidence level β tend to be shorter, with more difficulty in achieving higher expected returns.

More aggressive MV portfolios can be generated with a smaller confidence level β in the CVaR robust framework. In contrast to the min-max robust model, the decrease in the level of the conservatism is achieved by including a larger set of poor mean returns; this results in less focus on the extreme poor scenarios. Decreasing the confidence level β corresponds to more acceptance of estimation risk. Indeed, it seems reasonable to regard β as a risk aversion parameter for estimation risk. Our computational results also suggest that there is little variation in the efficiency of the actual frontiers from the CVaR robust formulation.

In some sense, min-max robust model is essentially quantile-based, assuming that the uncertainty set is determined based on quantile of the uncertain parameters. CVaR robust model, on the other hand, is tail-based. Because of this, there is a crucial difference in the diversification of the robust portfolios generated from the two approaches. In spite of the robust objective, the investment allocation from the min-max robust portfolio with $\lambda = 0$ (which achieves the maximum return), typically concentrates on a single asset, no matter what confidence level is used to determine μ^L. The corresponding CVaR robust portfolio, on the other hand, typically consists of multiple assets even for a high confidence level, e.g., $\beta = 90\%$. The level of diversification decreases as the confidence level decreases.

In addition, we investigate the computational issues in the CVaR robust model, and implement a smoothing technique for computing CVaR robust portfolios. Unlike the QP approach, which uses a piecewise linear function to approximate the CVaR function, the smoothing approach uses a continuously differentiable piecewise quadratic function. We show that the smoothing approach is computational more efficient for computing CVaR robust portfolios. In addition, as the number of mean return samples increases, the difference between the CVaR values approximated by the two approaches become smaller.

In Schöttle and Werner (2008), it has been shown that, among 14 strategies considered (including the min-max robust strategy), no strategy can consistently outperform the naive strategy, based on out-of-sample performance. It will be interesting to investigate the degree of improvement of the proposed CVaR robust strategy in economic terms.

APPENDIX

A Proofs of Theorems

We first prove Theorem 2.1, which is stated here again for convenience.

Theorem A.1. *Assume that Q is symmetric positive definite and $\chi \geq 0$. The min-max robust portfolio for (6) is an optimal portfolio of the mean-standard deviation problem (5) with nominal estimates $\bar{\mu}$ and Q for a larger risk aversion parameter $\lambda + \sqrt{\chi}$.*

Proof. For any feasible x, let μ^* be the minimizer of the inner optimization problem in (6) with respect to μ, i.e., μ^* solves

$$\begin{array}{ll} & \min_{\mu} \mu^T x \\ \text{s.t.} & (\bar{\mu} - \mu)^T Q^{-1} (\bar{\mu} - \mu) \leq \chi. \end{array}$$

Then there exists some $\rho < 0$ such that

$$x - \rho Q^{-1}(\mu^* - \bar{\mu}) = 0.$$

Note that $\rho \neq 0$ since $x = 0$ is not a feasible point for (6). Thus

$$\mu^* = \bar{\rho} Q x + \bar{\mu}, \quad \text{where } \bar{\rho} = \frac{1}{\rho} < 0.$$

From

$$Q^{-\frac{1}{2}}(\mu^* - \bar{\mu}) = \bar{\rho} Q^{\frac{1}{2}} x$$

and

$$(\bar{\mu} - \mu^*)^T Q^{-1} (\bar{\mu} - \mu^*) = \chi,$$

we have

$$\bar{\rho}^2 = \frac{\chi}{x^T Q x} \quad \text{and} \quad \bar{\rho} = -\frac{\sqrt{\chi}}{\sqrt{x^T Q x}}.$$

Thus the min-max robust mean-standard deviation portfolio can be obtained from

$$\begin{array}{ll} & \min_{x} -\bar{\mu}^T x + (\lambda + \sqrt{\chi})\sqrt{x^T Q x} \\ \text{s.t.} & e^T x = 1, \quad x \geq 0. \end{array}$$

This completes the proof. $\square$

We now prove Theorem 2.2 which is stated again here for convenience.

Theorem A.2. *Assume that Q is symmetric positive definite and $\chi \geq 0$. Any robust portfolio from the min-max robust mean-variance model (7) is an optimal portfolio from the standard mean-variance model based on the nominal estimates $\bar{\mu}$ and Q with a risk aversion parameter $\hat{\lambda} \geq \lambda$.*

Proof. From the proof of Theorem 2.1, the min-max robust mean-variance problem (7) is equivalent to

$$\begin{array}{ll} & \min_{x} -\bar{\mu}^T x + \lambda x^T Q x + \sqrt{\chi}\sqrt{x^T Q x} \\ \text{s.t.} & e^T x = 1, \quad x \geq 0\ . \end{array}$$

Since this is a convex programming problem, it is easy to show that there exists $\tilde{\chi} \geq 0$ such that the above problem is equivalent to

$$\begin{aligned} & \min_x -\bar{\mu}^T x + \lambda x^T Q x \\ \text{s.t.} \quad & \sqrt{x^T Q x} \leq \tilde{\chi} \\ & e^T x = 1, \quad x \geq 0 \ . \end{aligned}$$

In addition, the above problem is equivalent to

$$\begin{aligned} & \min_x -\bar{\mu}^T x + \lambda x^T Q x \\ \text{s.t.} \quad & x^T Q x \leq \tilde{\chi}^2 \\ & e^T x = 1, \quad x \geq 0 \ . \end{aligned}$$

From the convexity of the problem and the Kuhn-Tucker conditions, there exists $\tilde{\lambda} \geq 0$ such that the above problem is equivalent to

$$\begin{aligned} & \min_x -\bar{\mu}^T x + \lambda x^T Q x + \tilde{\lambda} x^T Q x \\ \text{s.t.} \quad & e^T x = 1, \quad x \geq 0 \ . \end{aligned}$$

This completes the proof. $\square$

B Tables of Mean Returns and Covariance Matrix

Table 5: Mean vector and covariance matrix for a 8-asset portfolio problem

$10^{-2}\times$	Asset1	Asset2	Asset3	Asset4	Asset5	Asset6	Asset7	Asset8
	1.0160	0.47460	0.47560	0.47340	0.47420	-0.0500	-0.1120	0.0360

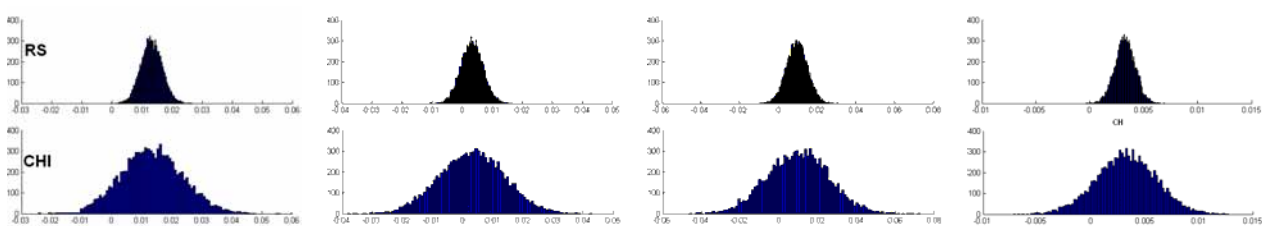
$10^{-2}\times$	Asset1	Asset2	Asset3	Asset4	Asset5	Asset6	Asset7	Asset8
Asset1	0.0980							
Asset2	0.0659	0.1549						
Asset3	0.0714	0.0911	0.2738					
Asset4	0.0105	0.0058	-0.0062	0.0097				
Asset5	0.0058	0.0379	-0.0116	0.0082	0.0461			
Asset6	-0.0236	-0.0260	0.0083	-0.0215	-0.0315	0.2691		
Asset7	-0.0164	0.0079	0.0059	-0.0003	0.0076	-0.0080	0.0925	
Asset8	0.0004	-0.0248	0.0077	-0.0026	-0.0304	0.0159	-0.0095	0.0245

Table 6: Mean vector and covariance matrix for a 10-asset portfolio problem

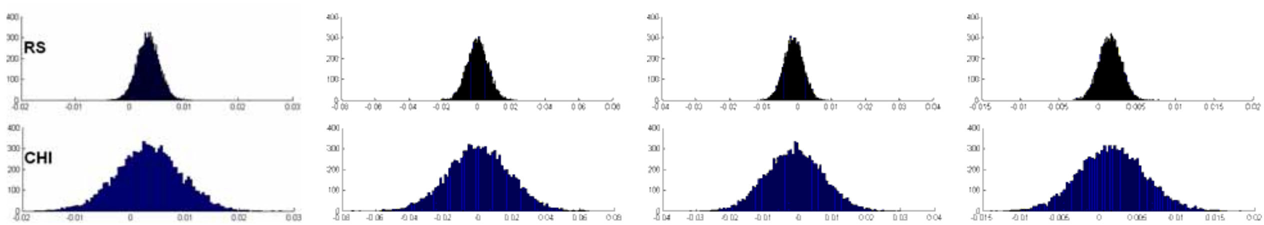
$10^{-2}\times$	Asset1	Asset2	Asset3	Asset4	Asset5	Asset6	Asset7	Asset8	Asset9	Asset10
	1.0720	1.7618	1.8270	1.0761	1.9845	1.4452	0.9910	1.6353	1.3755	1.8315

$10^{-2}\times$	Asset1	Asset2	Asset3	Asset4	Asset5	Asset6	Asset7	Asset8	Asset9	Asset10
Asset1	0.2516									
Asset2	0.0766	1.3743								
Asset3	0.1104	0.2847	1.3996							
Asset4	0.1314	0.0930	0.1027	0.1928						
Asset5	0.0157	0.5610	0.4725	0.0451	1.5981					
Asset6	0.0554	0.3457	0.2769	0.0898	0.3490	0.4787				
Asset7	0.0937	0.0253	0.0759	0.1010	0.0714	0.0643	0.1664			
Asset8	0.1646	0.1757	0.3200	0.1641	0.4721	0.2669	0.1020	0.9013		
Asset9	0.0509	0.1810	0.3275	0.0993	0.2978	0.1783	0.0635	0.1534	0.5731	
Asset10	0.1515	0.3445	0.3627	0.0966	0.4740	0.2651	0.0611	0.3596	0.2154	1.4041

C Distributions from Resampling and CHI Sampling Technique



(a) Asset 1 (b) Asset 2 (c) Asset 3 (d) Asset 4



(e) Asset 5 (f) Asset 6 (g) Asset 7 (h) Asset 8

Figure 9: Distribution of mean return samples generated by sampling techniques RS(top) and CHI(bottom) for each asset in Table 5.

D Tables of Portfolio Weights

Table 7: Portfolio Weights for Min-Max Robust and CVaR robust (90%) Actual Frontiers

λ	Asset1	Asset2	Asset3	Asset4	Asset5	Asset6	Asset7	Asset8
0	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
100	0.00	0.00	0.00	0.73	0.07	0.00	0.00	0.20
200	0.00	0.00	0.00	0.35	0.30	0.01	0.04	0.43
300	0.00	0.00	0.00	0.23	0.30	0.01	0.04	0.43
400	0.00	0.00	0.00	0.17	0.32	0.02	0.04	0.46
500	0.01	0.00	0.00	0.13	0.34	0.02	0.04	0.47
600	0.01	0.00	0.00	0.10	0.35	0.02	0.04	0.48
700	0.01	0.00	0.00	0.09	0.35	0.02	0.04	0.49
800	0.01	0.00	0.00	0.07	0.36	0.02	0.05	0.49
900	0.01	0.00	0.00	0.06	0.36	0.02	0.05	0.50
1000	0.01	0.00	0.00	0.05	0.37	0.02	0.05	0.50

(a) Min-Max Robust Portfolios Weights

λ	Asset1	Asset2	Asset3	Asset4	Asset5	Asset6	Asset7	Asset8
0	0.18	0.00	0.00	0.63	0.05	0.08	0.06	0.00
100	0.05	0.00	0.00	0.16	0.30	0.04	0.06	0.39
200	0.04	0.00	0.00	0.09	0.34	0.04	0.06	0.44
300	0.04	0.00	0.00	0.06	0.36	0.03	0.05	0.47
400	0.03	0.00	0.00	0.04	0.37	0.03	0.05	0.48
500	0.03	0.00	0.00	0.03	0.37	0.03	0.05	0.49
600	0.03	0.00	0.00	0.02	0.37	0.03	0.05	0.49
700	0.03	0.00	0.00	0.02	0.38	0.03	0.05	0.50
800	0.03	0.00	0.00	0.01	0.38	0.03	0.05	0.50
900	0.03	0.00	0.00	0.01	0.38	0.03	0.05	0.50
1000	0.03	0.00	0.00	0.01	0.38	0.03	0.05	0.51

(b) CVaR robust (90%) Portfolios Weights

Table 8: Portfolio weights for CVaR robust (60%) and (30%) Actual Frontiers

λ	Asset1	Asset2	Asset3	Asset4	Asset5	Asset6	Asset7	Asset8
0	0.39	0.00	0.00	0.42	0.13	0.06	0.00	0.00
100	0.06	0.00	0.00	0.21	0.28	0.05	0.06	0.35
200	0.04	0.00	0.00	0.11	0.33	0.04	0.06	0.43
300	0.04	0.00	0.00	0.07	0.35	0.03	0.05	0.46
400	0.03	0.00	0.00	0.05	0.36	0.03	0.05	0.47
500	0.03	0.00	0.00	0.04	0.37	0.03	0.05	0.48
600	0.03	0.00	0.00	0.03	0.37	0.03	0.05	0.49
700	0.03	0.00	0.00	0.02	0.38	0.03	0.05	0.49
800	0.03	0.00	0.00	0.02	0.38	0.03	0.05	0.50
900	0.03	0.00	0.00	0.01	0.38	0.03	0.05	0.50
1000	0.02	0.00	0.00	0.01	0.38	0.03	0.05	0.50

(a) CVaR robust (60%) Portfolios Weights

λ	Asset1	Asset2	Asset3	Asset4	Asset5	Asset6	Asset7	Asset8
0	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
100	0.07	0.00	0.00	0.25	0.26	0.05	0.06	0.31
200	0.05	0.00	0.00	0.12	0.32	0.04	0.05	0.41
300	0.04	0.00	0.00	0.08	0.35	0.03	0.05	0.45
400	0.03	0.00	0.00	0.05	0.36	0.03	0.05	0.47
500	0.03	0.00	0.00	0.04	0.37	0.03	0.05	0.48
600	0.03	0.00	0.00	0.03	0.37	0.03	0.05	0.49
700	0.03	0.00	0.00	0.02	0.38	0.03	0.05	0.49
800	0.03	0.00	0.00	0.02	0.38	0.03	0.05	0.50
900	0.03	0.00	0.00	0.01	0.38	0.03	0.05	0.50
1000	0.02	0.00	0.00	0.01	0.38	0.03	0.05	0.50

(b) CVaR robust (30%) Portfolios Weights

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