Optimizing the Omega Ratio using Linear Programming

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Abstract

The Omega Ratio is a recent performance measure. It captures both, the downside and upside potential of the constructed portfolio, while remaining consistent with utility maximization. In this paper, a new approach to compute the maximum Omega Ratio as a linear program is derived. While the Omega ratio is considered to be a non-convex function, we show an exact formulation in terms of a convex optimization problem, and transform it as a linear program. The convex reformulation for the Omega Ratio maximization is a direct analogue to mean-variance framework and the Sharpe Ratio maximization.

Keywords: Omega Ratio, Performance Measure, Downside Risk, Risk, Return

1 Introduction

Investors face the challenging problem of how to distribute their current wealth over a set of available assets with the objective to earn the highest possible future wealth. One of the first mathematical models for this problem was formulated by Markowitz (1952), who observed that a prudent investor does not aim solely at maximizing the expected return of an investment, but also at minimizing its risk. In the Markowitz model the risk of a portfolio is measured by the variance of the portfolio return. For this reason the Markowitz model is often referred to as the mean-variance framework.

Aside from measuring risk, investors often want to evaluate the performance of different portfolio strategies in order to compare or rank them. One way to do so is by using performance measures. Intuitively speaking, a performance measure should take into account the "reward" or upside potential, as well as the risk of the strategy that has to be evaluated. Often, performance measures are a ratio of reward over risk. The most popular performance measure is the Sharpe Ratio, which was proposed by Sharpe (1966). The Sharpe Ratio is inspired by the Markowitz model and is defined as the ratio of the portfolio’s expected excess relative to the risk free rate return over the standard deviation of the portfolio return. A criticism that is often raised against the Sharpe Ratio is that it is only appropriate when the portfolio return is elliptically distributed. Indeed, the main problem with the Sharpe
The Omega Ratio is defined as the ratio of the light gray area (BCD) over the dark gray area (ABE). The light gray area on the right of the threshold $\tau$ (in this example 0.005) and above the cumulative distribution represents the upside potential. The dark gray area on the left of the threshold and below the cumulative distribution represents the downside potential (risk).

The Omega Ratio is that it only take into account the first- and second-order moments of the portfolio return, but neglects any other higher order moments. When the portfolio return is skewed or exhibits fat tails, then the Sharpe Ratio might result in counterintuitive performance evaluations and rankings.

In order to alleviate the problems associated with the Sharpe Ratio, a number of alternative performance measures have been suggested (see, e.g., Sortino and Price (1994), Keating and Shadwick (2002), Bacmann and Scholz (2003)). Recently, Keating and Shadwick (2002) suggested the use of a new performance measure, which they termed the \textit{Omega Ratio}. A particular feature of the Omega Ratio is that it uses the entire probability distribution of the portfolio return to compute its performance and, therefore, does not suffer from the drawbacks of the mean-variance framework.

The Omega Ratio makes use of a threshold value $\tau$ to distinguish the upside from the downside. This means that portfolio returns above $\tau$ are considered profits, whereas returns below $\tau$ are considered losses. The choice of the value for $\tau$ is left to the investor and is typically taken to be equal to the risk-free rate. The Omega Ratio is defined as the ratio of the area on the right of the threshold and above the cumulative distribution, over the area on the left of the threshold and below the cumulative distribution. We refer to Figure 1 to illustrate the intuition behind the Omega Ratio. Figure 1 also makes it clear that the Omega Ratio is computed by taking the entire probability distribution of the portfolio return into account.

In this paper we address portfolio optimization models that aim to maximize the Omega Ratio subject to additional constraints on the portfolio weights. Traditionally, the Omega Ratio is only used to evaluate and compare fixed portfolio strategies. In contrast, fairly little work has been done on the Omega Ratio maximization. The main reason for this is that the Omega Ratio is non-convex, which renders the Omega Ratio maximization problems difficult to solve.
A number of heuristic optimization methods have been proposed to find good solutions for Omega Ratio maximization problems. The most common approaches are based on the threshold accepting heuristic and other simulating annealing variants (Gilli et al., 2006, 2008, Passow, 2004). These kind of heuristics generate solutions by carrying out local searches and by moving through neighboring solutions in an attempt to improve the objective value. None of these heuristics guarantee that the global optimum will be found. Furthermore, threshold accepting methods can become numerically unstable, and need considerable fine tuning of the parameters (Gilli et al., 2006). Mausser et al. (2006) proposed a method to solve this problem as a linear program under certain conditions, but as Kane et al. (2005) state, this methodology cannot cope with the general case.

In this paper, we show that the Omega Ratio maximization problem can be reformulated equivalently as a quasi-convex optimization problem. Quasi-convex optimization problems can be solved to global optimality in polynomial time, and, we show that the problem reduces to a Linear Program when the portfolio return distribution can be approximated by discrete samples. This enables the Omega Ratio maximization for large-scale portfolios using the Omega Ratio.

2 Exact solution of the Omega Ratio maximization

Consider a market with \( n \) stocks. We denote the current time as \( t = 0 \) and the end of the investment horizon as \( t = T \). A portfolio is completely characterized by a vector of weights \( \mathbf{w} \in \mathbb{R}^n \), whose components add up to 1. The element \( w_i \) denotes the percentage of total wealth invested in the \( i \)th stock at time \( t = 0 \). Furthermore, let \( \tilde{r}_i \) indicate the random return of asset \( i \) and with boldface the vector of variables \( \tilde{\mathbf{r}} \in \mathbb{R}^n \). The random return of a portfolio of assets is defined as \( \tilde{r}_p = \mathbf{w}^\top \tilde{\mathbf{r}} \). Let \( F(r_i) \) and \( f(r_i) \) denote the cumulative density function and the probability density function, respectively. For an asset \( i \), Keating and Shadwick (2002) define the Omega Ratio as:

\[
\Omega(\tilde{r}_i) = \int_{\tau}^{+\infty} \left[ 1 - F(r_i) \right] dr_i \int_{-\infty}^{\tau} F(r_i) dr_i.
\]  

(1)

By integration by parts and some algebraic transformation, the Omega Ratio can be written as

\[
\Omega(\tilde{r}_i) = \int_{\tau}^{+\infty} (\tilde{r}_i - \tau) f(r_i) dr_i \int_{-\infty}^{\tau} (\tau - \tilde{r}_i) f(r_i) dr_i = \frac{\mathbb{E}[(\tilde{r}_i - \tau)^+] \mathbb{E}[\tau - \tilde{r}_i]}{\mathbb{E}[(\tau - \tilde{r}_i)^+] + 1}.
\]  

(2)

Therefore, the Omega Ratio of a portfolio is given by:

\[
\Omega(\tilde{r}_p) = \mathbf{w}^\top \mathbb{E}[\tilde{\mathbf{r}}] - \tau \mathbb{E}[\tau - \mathbf{w}^\top \tilde{\mathbf{r}}] + 1.
\]  

(3)
In this paper, we investigate portfolio optimization problems that aim to maximize the Omega Ratio subjected to additional constraints on portfolio weights. The Omega maximization problem can be written as

$$\max_{w \in \mathbb{R}^n} \frac{w^\top \mathbb{E}[\tilde{r}] - \tau}{\mathbb{E}[(\tau - w^\top \tilde{r})^+]}$$

subject to

$$w^\top 1 = 1$$

$$w \preceq w \preceq w.$$  

The objective is to determine the allocation that gives the optimal weights ($w \in \mathbb{R}^n$) that result in the portfolio with the maximum Omega Ratio. The constraints above relate to the budget constraint and the upper and lower bound on any individual investment. Any additional convex constraints can be also taken into consideration.

The rest of this chapter is organized as follows. First, we illustrate how the exact solution for the Omega Ratio maximization problem can be obtained by solving a sequence of optimization problems. The methodology is a direct analog to the mean-variance and Sharpe maximization frameworks. Second, we show that the exact solution can be also obtained by solving a single optimization problem.

### 2.1 Risk versus return optimization

The objective function (4) is not in general convex in $w$. Consequently, local optima may exist. However, the objective function is a ratio of concave (linear) and a convex function of $w$, and therefore the function is quasi-concave (Boyd and Vandenberghe, 2004). Quasi-concave functions can exhibit a number of local optima, which are not necessarily global. This explains the empirical evidence described by the Kane et al. (2005) that Omega maximization problem result in a number of local optima. However, sub-level sets of a quasi-concave function can be represented via a family of concave inequalities. Moreover, quasi-concave problems can be solved to global optimality by solving a number of concave problems (Boyd and Vandenberghe, 2004).

An exact solution for the Omega Ratio maximization problem can be obtained solving a sequence of optimization problems by changing $c \in [l, u]$. This approach is a direct analogue to mean-variance framework. The objective is to minimize the risk of a portfolio given the required expected return over the threshold $\tau$.

$$\min_{w \in \mathbb{R}^n} \left\{ \mathbb{E}[(\tau - w^\top \tilde{r})^+] : w^\top \mathbb{F} - \tau \geq c, \underline{w} \leq w \leq \overline{w}, 1^\top w = 1 \right\}$$

Note that the optimization problem (7) for a given value of $c$ belongs in the class of convex problems. The highest value of $c (u)$ is equal to $w^\star^\top \mathbb{F} - \tau$, where $w^\star$ is defined by the following problem:

$$\max_{w \in \mathbb{R}^n} \left\{ w^\top \mathbb{F} : \underline{w} \leq w \leq \overline{w}, 1^\top w = 1 \right\}$$
By setting \( c = u \), the solution of the optimization problem (7) is the portfolio with the maximum expected return and the lower associated risk; as measured by the denominator of the Omega Ratio. The lowest value of \( c \) (\( l \)) is equal to \( w^+ \cdot \mathbf{r} - \tau \), where \( w^+ \) is determined by solving the problem (7) without the constraint for the numerator:

\[
\min_{w \in \mathbb{R}^n} \left\{ \mathbb{E} \left[ (\tau - w^+ \cdot \mathbf{r})^+ \right] : \underline{w} \leq w \leq \overline{w}, \mathbf{1}^T w = 1 \right\}
\] (9)

The global minimum risk portfolio (as measured by the 1st order lower partial moment) is given by setting \( c = l \). The portfolio with the maximum Omega can be obtained by calculating the Omega for each solution given by the sequence of the optimization problems (7).

The graphical illustration of this problem displays interesting similarities with the mean-variance framework and Sharpe Ratio maximization. The output extracted by changing \( c \in [l, u] \) in (7) is a Pareto frontier in the Omega numerator and denominator. This frontier exhibits similarities with the efficient frontier of mean variance optimization. No point above the frontier is attainable and for any point below, there exists a solution that an investor can chose to be better off. We call this the Omega frontier.

The Omega frontier is concave and non-decreasing. This arises from the convexity property of the optimization problem (7). The affiliated Omega Ratio for each point on the frontier is given by the slope of the line passing through it and the origin. The goal is to find the line with the maximum slope that passes through the origin and a
point on the frontier. Since the frontier is non-decreasing and concave, the tangent from the origin to the frontier yields the portfolio with the maximum Omega.

To sum up, the exact solution of the Omega maximization problem can be found either by taking the combination that yields the maximum ratio, or, by finding the tangent from the origin to the frontier. Faster algorithms - such as bi-section - could be used to solve the above problem. The choice of a much simpler and slower algorithm is to illustrate the analogy between the mean-variance framework combined with Sharpe Ratio maximization and Omega Ratio maximization. In the next subsection, we consider a more direct method for computing the maximum Omega Ratio.

2.2 Exact reformulation as a linear program

In the previous section, we have shown that the exact solution can be found by solving a family of convex problems in an analogy to the mean-variance framework and Sharpe maximization. In this subsection we present a direct method to solve the problem as a linear program.

The discrete analogue for Omega following from equation (3) is

\[
\Omega = \frac{w^T \bar{r} - \tau}{\sum_j [\tau - w^T r_j]^+ p_j} \tag{10}
\]

Then, the optimization problem is

\[
\max_{w} \frac{w^T \bar{r} - \tau}{\sum_j [\tau - w^T r_j]^+ p_j} \tag{11}
\]

s.t. \( \sum w_i = 1 \) \tag{12}

\[
w \leq w \leq \bar{w} \tag{13}
\]

Let \( r_1, r_2, \ldots, r_m \) denote \( m \) samples of the random asset returns \( \bar{r} \), \( u_j = [\tau - w^T r_j]^+ \) and \( p_j = p = 1/m \), then problem can be written as:

\[
\max_{w \in \mathbb{R}^n, u \in \mathbb{R}^m} \frac{w^T \bar{r} - \tau}{(1/m)1^T u} \tag{14}
\]

s.t. \( u_i \geq \tau - w^T r_i \quad \forall i = 1, \ldots, m \)

\[
u_i \geq 0 \quad \forall i = 1, \ldots, m
\]

\[
1^T w = 1
\]

\[
w \leq w \leq \bar{w}
\]

\[
w^T \bar{r} \geq \tau.
\]

The above minimization problem it is a linear-fractional program, since it is a minimization of a ratio of affine
functions over a polyhedron (Boyd and Vandenberghe, 2004).

Let the feasible set \(\{w,u | 1^Tw = 1, u \geq \tau - w^Tr, u \geq 0, w^T\bar{r} \geq \tau, w \leq \mathbf{w} \leq \bar{w}\} \neq \emptyset\), the optimization problem (14) can be written as an equivalent linear program as follows (see Boyd and Vandenberghe (2004) p. 39-42,151-152)

\[
\begin{align*}
\max_{s \in \mathbb{R}^n, q \in \mathbb{R}^m, z \in \mathbb{R}} & \quad s^T\bar{r} - \tau z \\
\text{s.t.} & \quad q_i \geq \tau z - s^Tr_i \quad \forall i = 1, \ldots, m \\
& \quad q_i \geq 0 \quad \forall i = 1, \ldots, m \\
& \quad 1^Tq = 1 \\
& \quad 1^Ts = z \\
& \quad z\mathbf{w} \leq s \leq z\bar{w} \\
& \quad s^T\bar{r} \geq \tau z \\
& \quad z \geq 0
\end{align*}
\]

(15)

It can be shown that the Omega maximization problem is equivalent to linear program (15), by introducing the scalar variable \(z\) to homogenize the problem. A change in the initial variables is needed, in order to formulate the problem (14) as a linear program. The family of variables \(u\) changes to the new variables \(q\) and its dimension is equal to the number of asset returns. The family of variables \(w\) transforms to the new family of variables \(s\), where \(s\) is a vector of same dimension as assets. At the end of the procedure \(s\) needs to be normalized. The rescaled \(s\) maximizes (14).

While Omega maximization has been considered as a non-convex problem, we shown that the global maximum can be obtained solving a single linear program. This is due the fact that the Omega ratio is quasi-concave in portfolio weights. Heuristic, meta-heuristic, and threshold acceptance methods cannot ensure global optimality, are complicated in implementation and are time-consuming. In addition, heuristic optimization techniques need experience in order to be tuned (Gilli et al., 2006). The above linear program not only ensures global optimality, but is also simple and fast to solve. This allows us to consider a larger number of assets, add additional constraints, and examine more complex formulations. Broadly, this result will help research on this performance measure, and make its implementation by practitioners effortless and more attractive.

3 Conclusions

Despite the fact that the Omega Ratio maximization has been considered to be a non-convex problem, the exact solution can be found by solving either a family of linear programs or a single fractional linear program. Maximizing the Omega Ratio by frontier approach is a direct analogue to the mean-variance framework and Sharpe maximization. However, the proposed methodology in subsection 2.2 (i.e. the fractional linear program) is computationally more efficient. The proposed methodologies are based on the fact that the Omega Ratio is a quasi-concave func-
tion. The existence of local optima has led the researchers into employing global optimization methods to solve this problem. The methodology in this paper also permits the consideration of further extensions to robust optimization under uncertainty, analogous to the approach considered by Pinar and Tutuncu (2005) for the Sharpe ratio.

References


