15 Newton's Algorithm

Goal

Newton's algorithm, conjugate gradient, auto-differentiation, local quadratic rate of convergence, affine invariance, cubic regularization

Alert 15.1: Convention

Nice surveys for Newton's algorithm include Polyak (2006) and Ypma (1995).

Gray boxes are not required hence can be omitted for unenthusiastic readers.

This note is likely to be updated again soon.

Polyak, R. A. (2006). "Nonlinear Rescaling as Interior Quadratic Prox Method in Convex Optimization". *Computational Optimization and Applications*, vol. 35, pp. 347–373.

Ypma, T. J. (1995). "Historical Development of the Newton-Raphson Method". *SIAM Review*, vol. 37, no. 4, pp. 531–551.

Definition 15.2: Problem

In this lecture we consider solving the smooth minimization problem:

$$\min_{\mathbf{w}} \ f(\mathbf{w}),$$

where f is sufficiently smooth (twice or thrice continuously differentiable).

Algorithm 15.3: Newton

Newton's algorithm iteratively minimizes the second order Taylor series of the smooth objective f:

$$\begin{aligned}\mathbf{w}_{t+1} &= \operatorname*{argmin}_{\mathbf{w}} \ f(\mathbf{w}_t) + \langle f'(\mathbf{w}_t), \mathbf{w} - \mathbf{w}_t\rangle + \tfrac{1}{2\eta_t}\langle f_t''(\mathbf{w}_t)(\mathbf{w} - \mathbf{w}_t), \mathbf{w} - \mathbf{w}_t\rangle \\ &= \mathbf{w}_t - \eta_t[f_t''(\mathbf{w}_t)]^{-1} \cdot f_t'(\mathbf{w}_t), \end{aligned} \tag{15.1}$$

where the step size η_t is often set to the constant 1 (at least in later stages when we are sufficiently close to the local optimum).

Often, we also add some regularization (e.g. Levenberg-Marquardt) to the Hessian so that its inverse is well-behaved.

History 15.4: Early studies of Newton's algorithm

To mention just a few: Fine (1916), Bennett (1916), Kantorovich (1948, 1949, 1957), Cheney and Goldstein (1959), Goldstein (1965), and Goldfeld et al. (1966).

Fine, H. B. (1916). "On Newton's method of approximation". *Proceedings of National Academy of Sciences*, vol. 2, no. 9, pp. 546–552.

Bennett, A. A. (1916). "Newton's Method in General Analysis". *Proceedings of National Academy of Sciences*, vol. 2, no. 10, pp. 592–598.

Kantorovich, L. V. (1948). "On Newton's method for functional equations". *Dokl. Akad. Nauk SSSR*, vol. 59, pp. 1237–1240.

— (1949). "On the Newton method". *Proceedings of the Steklov Institute of Mathematics*, vol. 28, pp. 104–144.

— (1957). "Some further applications of the Newton method for functional equations". *Vestn. LGU, Ser. Math. Mech.*, vol. 7, pp. 68–103.

Cheney, E. W. and A. A. Goldstein (1959). "Newton's Method for Convex Programming and Tchebycheff Approximation". *Numerische Mathematik*, vol. 1, pp. 253–268.

Goldstein, A. A. (1965). "On Newton's method". *Numerische Mathematik*, vol. 7, pp. 391–393.
Goldfeld, S. M., R. E. Quandt, and H. F. Trotter (1966). "Maximization by Quadratic Hill-Climbing". *Econometrica*, vol. 34, no. 3, pp. 541–551.

Alert 15.5: Affine equivariance and invariance

Newton's Algorithm 15.3 is affine equivariant. Indeed, consider the change-of-variable $\mathbf{w} = A\mathbf{z}$ for any invertible linear map A, we have

$$(f \circ A)'(\mathbf{z}) = A^\top f'(A\mathbf{z}), \quad (f \circ A)''(\mathbf{z}) = A^\top f''(A\mathbf{z})A, \implies \mathbf{z}_{t+1} = \mathbf{z}_t - \eta_t A^{-1}[f''(A\mathbf{z}_t)]^{-1} f'(A\mathbf{z}_t),$$

and hence $\mathbf{w}_{t+1} = A\mathbf{z}_{t+1}$ if we start with $\mathbf{w}_0 = A\mathbf{z}_0$.

Newton's Algorithm 15.3 is also affine invariant w.r.t. the Euclidean metric. Indeed, if we change the inner product to $\langle \mathbf{w}, \mathbf{z} \rangle_Q := \langle Q\mathbf{w}, \mathbf{z} \rangle$ for some (symmetric) positive definite Q, we have

$$f' \to Q^{-1} f' \quad \text{and} \quad f'' \to Q^{-1} f''$$

so that the Newton update remains the same.

Alert 15.6: Scaling invariance

Perhaps more surprisingly, Newton's Algorithm 15.3 is also scaling invariant: if we change f to αf for any $\alpha \in \mathbb{R}$, Newton's update still remains the same.

This simple observation actually reveals the true nature of Newton's algorithm: it merely aims to solve the nonlinear equation

$$f'(\mathbf{w}) = \mathbf{0},$$

but does not care if $\mathbf{w}$ is a (local) minimizer or maximizer.

Theorem 15.7: Local quadratic rate under strong convexity

Suppose f is σ-strongly convex and f'' is L-Lipschitz continuous (w.r.t. the ℓ_2 norm), and $q = \frac{\mathsf{L}}{2\sigma^2}\|f'(\mathbf{w}_0)\|_2 < 1$, then for all t:

$$\|\mathbf{w}_t - \mathbf{w}_*\|_2 \le \tfrac{1}{\sigma}\|f'(\mathbf{w}_t)\|_2 \le \tfrac{2\sigma}{\mathsf{L}} q^{2^t}, \tag{15.2}$$

where $\mathbf{w}_*$ is the unique minimizer of f.

Proof: According to Proposition 1.12, the L-Lipschitz continuity of f'' implies that

$$\|f'(\mathbf{w}_t + \mathbf{z}) - f'(\mathbf{w}_t) - f''(\mathbf{w}_t)\mathbf{z}\|_2 \le \tfrac{\mathsf{L}}{2}\|\mathbf{z}\|_2^2.$$

Taking $\mathbf{z} = -[f''(\mathbf{w}_t)]^{-1} f'(\mathbf{w}_t) =: \mathbf{w}_{t+1} - \mathbf{w}_t$ we obtain

$$\|f'(\mathbf{w}_{t+1})\|_2 \le \tfrac{\mathsf{L}}{2}\|[f''(\mathbf{w}_t)]^{-1} f'(\mathbf{w}_t)\|_2^2 \le \tfrac{\mathsf{L}}{2}\|[f''(\mathbf{w}_t)]^{-1}\|_{\mathrm{sp}}^2 \cdot \|f'(\mathbf{w}_t)\|_2^2 \le \tfrac{\mathsf{L}}{2\sigma^2}\|f'(\mathbf{w}_t)\|_2^2.$$

Therefore, telescoping yields for $t \ge 0$:

$$\tfrac{\mathsf{L}}{2\sigma^2}\|f'(\mathbf{w}_{t+1})\|_2 \le \left(\tfrac{\mathsf{L}}{2\sigma^2}\|f'(\mathbf{w}_t)\|_2\right)^2 \le \cdots \le \left(\tfrac{\mathsf{L}}{2\sigma^2}\|f'(\mathbf{w}_0)\|_2\right)^{2^{t+1}}.$$

Lastly, it follows from the strong convexity of f that (see Proposition 3.22)

$$\|f'(\mathbf{w}_t)\|_2 = \|f'(\mathbf{w}_t) - f'(\mathbf{w}_*)\|_2 \ge \sigma\|\mathbf{w}_t - \mathbf{w}_*\|_2.$$

■

The condition $q = \frac{\mathsf{L}}{2\sigma^2}\|f'(\mathbf{w}_0)\|_2 < 1$ implies that

$$\|\mathbf{w}_0 - \mathbf{w}_*\| < \tfrac{2\sigma}{\mathsf{L}},$$

i.e., the starting point $\mathbf{w}_0$ is sufficiently close to the minimizer $\mathbf{w}_*$. Inspecting (15.2) we observe that once the iterate $\mathbf{w}_t$ enters a small ball around $\mathbf{w}_*$ (such that $f'(\mathbf{w}_t) < \frac{2\sigma^2}{\mathsf{L}}$, implying the radius is less than $\frac{2\sigma}{\mathsf{L}}$), it will remain there and converge to $\mathbf{w}_*$ at a qudratic rate. Thus, the constants L and σ can be relativized, as long as we initialize carefully.

Example 15.8: Newton may not converge faster than linearly

Let us consider the simple univariate function

$$f(w) := |w|^{5/2}.$$

Clearly, we have $f'(w) = \frac{5}{2}\operatorname{sign}(w)|w|^{3/2}$ and $f''(w) = \frac{15}{4}|w|^{1/2}$. Note that f'' is not Lipschitz continuous and f is not strongly convex. The Newton update is:

$$w_{t+1} = w_t - \tfrac{4}{15}|w_t|^{-1/2} \cdot \tfrac{5}{2}\operatorname{sign}(w_t)|w_t|^{3/2} = w_t - \tfrac{2}{3}w_t = \tfrac{1}{3}w_t,$$

which converges to 0, the unique minimizer, at a linear rate.

Example 15.9: Newton may cycle

Consider the simple univariate function

$$f(w) = -\tfrac{1}{4}w^4 + \tfrac{5}{2}w^2, \qquad f'(w) = -w^3 + 5w, \quad f''(w) = -3w^2 + 5,$$

which, around 0, is locally (strongly) convex and f'' is locally Lipschitz continuous. The Newton update is:

$$w_{t+1} = w_t - \frac{-w_t^3 + 5w_t}{-3w_t^2 + 5} = \frac{2w_t^3}{3w_t^2 - 5}.$$

Thus, with $w_0 = 1$ we enter a cycle $1 \underset{t \leftarrow t+1}{\rightleftarrows} -1$. We verify that restricted to the unit ball around the origin, $\mathsf{L} = 6$ and $\sigma = 2$, so that $q = \frac{\mathsf{L}}{2\sigma^2}\|f'(w_0)\|_2 = 6 \times 4/2^3 = 3 \not< 1$.

Example 15.10: Newton can be chaotic

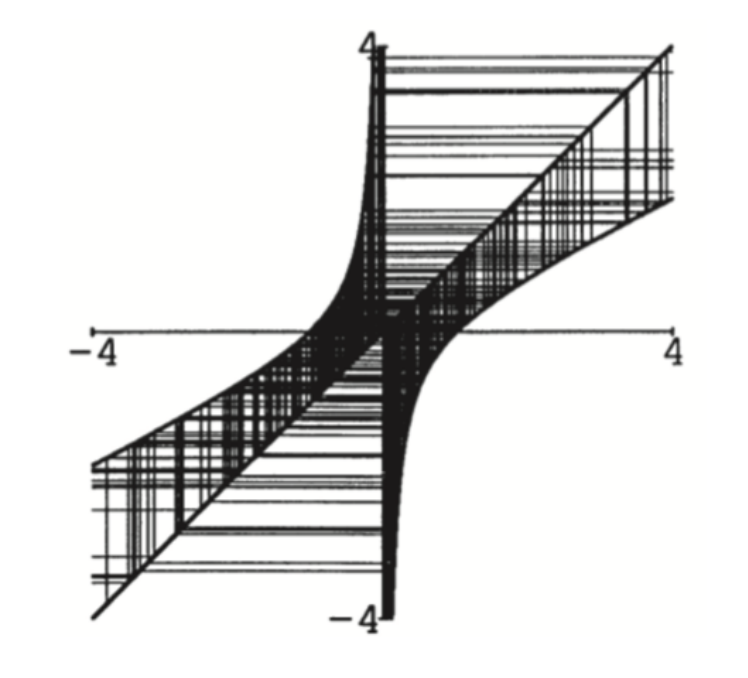
Consider the simple univariate function

$$f(w) = \tfrac{1}{3}w^3 + w, \qquad f'(w) = w^2 + 1, \qquad f''(w) = 2w.$$

Note that f, being nonconvex, tends to $-\infty$ as $w \to -\infty$ while f'' is 2-Lipschitz continuous and vanishes at $w = 0$. The Newton update is:

$$w_{t+1} = w_t - \frac{w_t^2 + 1}{2w_t} = \tfrac{1}{2}(w_t - \tfrac{1}{w_t}),$$

which behaves chaotically:



Jarre and Toint, 2016; Mascarenhas, 2007, 2008

Jarre, F. and P. L. Toint (2016). "Simple examples for the failure of Newton's method with line search for strictly convex minimization". *Mathematical Programming*, vol. 158, pp. 23–34.

Mascarenhas, W. F. (2007). "On the divergence of line search methods". *Computational and Applied Mathematics*, vol. 26, no. 1, pp. 129–169.

— (2008). "Newton's iterates can converge to non-stationary points". *Mathematical Programming*, vol. 112, pp. 327–334.

Algorithm 15.11: Cubic regularization (Griewank, 1981; Nesterov and Polyak, 2006)

Since gradient descent minimizes a quadratic upper bound of our function, it is natural to consider minimizing the following cubic upper bound as an alternative to Newton's update (15.1):

$$\mathbf{w}_{t+1} = \operatorname*{argmin}_{\mathbf{w}} \underbrace{f(\mathbf{w}_t) + \langle f'(\mathbf{w}_t), \mathbf{w} - \mathbf{w}_t\rangle + \tfrac{1}{2}\langle f''(\mathbf{w}_t)(\mathbf{w} - \mathbf{w}_t), \mathbf{w} - \mathbf{w}_t\rangle + \tfrac{1}{6\eta_t}\|\mathbf{w} - \mathbf{w}_t\|_2^3}_{\bar{f}_t(\mathbf{w}) = \bar{f}_{\eta_t}(\mathbf{w};\mathbf{w}_t)}. \tag{15.3}$$

Setting the derivative at $\mathbf{w}_{t+1}$ to zero we obtain:

$$f'(\mathbf{w}_t) + f''(\mathbf{w}_t)(\mathbf{w}_{t+1} - \mathbf{w}_t) + \tfrac{1}{2\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2 \cdot (\mathbf{w}_{t+1} - \mathbf{w}_t) = \mathbf{0}, \tag{15.4}$$

$$\text{also } \langle f'(\mathbf{w}_t), \mathbf{w}_{t+1} - \mathbf{w}_t\rangle + \langle f''(\mathbf{w}_t)(\mathbf{w}_{t+1} - \mathbf{w}_t), \mathbf{w}_{t+1} - \mathbf{w}_t\rangle + \tfrac{1}{2\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3 = 0.$$

In other words,

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \left[f''(\mathbf{w}_t) + \tfrac{1}{2\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2 \cdot \mathrm{Id}\right]^{-1} f'(\mathbf{w}_t),$$

which is essentially Newton's update with an adaptive Levenberg-Marquardt regularization. In particular, noting that usually $\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2 \to 0$, the regularization automatically dies down as we progress, so cubic regularization eventually behaves similarly to Newton's update.

Griewank, A. O. K. (1981). "The modification of Newton's method for unconstrained optimization by bounding cubic terms".

Nesterov, Y. and B. T. Polyak (2006). "Cubic regularization of Newton method and its global performance". *Mathematical Programming*, vol. 108, pp. 177–205.

Exercise 15.12: Nitpicking the proof

Before we continue, let us revisit our proof in Theorem 2.21 for gradient descent (set $r \equiv 0$ there). Can you recycle and adapt the proof for cubic regularization (15.3)?

[You may assume f is convex if it helps, although this is not really needed below.]

Alert 15.13: Strong duality

We point out that the subproblem in (15.3) does not appear to be convex, since we do not assume f to be convex. However, we may consider the standard semidefinite program (SDP) relaxation:

$$\min_{Z \succeq \mathbf{0}, Z_{11}=1, \langle I, Z\rangle = \zeta+1} f(\mathbf{w}_t) + \left\langle \frac{1}{2}\begin{bmatrix} 0 & f'(\mathbf{w}_t)^\top \\ f'(\mathbf{w}_t) & f''(\mathbf{w}_t) \end{bmatrix}, Z \right\rangle + \tfrac{1}{6\eta_t}\zeta^{3/2}, \tag{15.5}$$

where we think of $Z = \begin{bmatrix} 1 \\ \mathbf{w}-\mathbf{w}_t \end{bmatrix}\begin{bmatrix} 1 \\ \mathbf{w}-\mathbf{w}_t \end{bmatrix}^\top$. Since we only have two linear constraints on Z, and the objective is linear (and hence concave) in Z, it follows that an optimal Z can be chosen as an extreme point of the constraint set, which is known to have rank exactly 1 (Pataki, 1998, see also Barvinok, 1995; Polyak, 1998). In other words, the SDP relaxation is equivalent to the original, seemingly nonconvex, problem (15.3)!

We now make two important observations from the convex equivalent (15.5):

- From the KKT conditions we have

$$f''(\mathbf{w}_t) + \tfrac{1}{2\eta_t}\|\mathbf{w}_t - \mathbf{w}_{t+1}\|_2 \cdot \mathrm{Id} \succeq \mathbf{0}, \tag{15.6}$$

  whereas the second-order necessary condition for (15.3) would lose the factor $\frac{1}{2}$ in the second term.

- Setting $\mathbf{w} = \mathbf{w}_t$ and $\mathbf{w} = \mathbf{w}_{t+1}$ in Z respectively we conclude

$$0 \geq \langle f'(\mathbf{w}_t), \mathbf{w}_{t+1} - \mathbf{w}_t\rangle + \tfrac{1}{2}\langle f''(\mathbf{w}_t)(\mathbf{w}_{t+1} - \mathbf{w}_t), \mathbf{w}_{t+1} - \mathbf{w}_t\rangle + \tfrac{1}{4\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3,$$

  or put it succinctly,

$$f(\mathbf{w}_t) \geq \bar{f}_t(\mathbf{w}_{t+1}) + \tfrac{1}{12\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3. \tag{15.7}$$

  Note that we do not (need to) know if the above inequality holds for all $\mathbf{w}$ in place of $\mathbf{w}_t$.

Pataki, G. (1998). "On the Rank of Extreme Matrices in Semidefinite Programs and the Multiplicity of Optimal Eigenvalues". *Mathematics of Operations Research*, vol. 23, no. 2, pp. 339–358.

Barvinok, A. I. (1995). "Problems of distance geometry and convex properties of quadratic maps". *Discrete & Computational Geometry*, vol. 13, pp. 189–202.

Polyak, B. T. (1998). "Convexity of Quadratic Transformations and Its Use in Control and Optimization". *Journal of Optimization Theory and Applications*, vol. 99, pp. 553–583.

Exercise 15.14: Filling in the details

Prove (15.6) and (15.7).

[Hint: for the latter, apply Proposition 2.20 with $\mathbf{w} = \mathbf{w}_t$ and $\mathbf{w}_\star = \mathbf{w}_{t+1}$ in Z, noting that the Bregman divergence

$$\mathsf{D}_{(\cdot)^{3/2}}(\alpha, \beta) = \alpha^{3/2} + \tfrac{1}{2}\beta^{3/2} - \tfrac{3}{2}\alpha\beta^{1/2},$$

and at optimality $\zeta = \|\mathbf{w}_t - \mathbf{w}_{t+1}\|_2^2$.]

Exercise 15.15: Tighter bound under convexity

Suppose f is convex. Prove that

$$f(\mathbf{w}_t) \geq \bar{f}_t(\mathbf{w}_{t+1}) + \tfrac{1}{3\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3,$$

which is a factor of 4 improvement compared to (15.7). Needless to say, the inequality (15.6) is now trivial.

Proposition 15.16: Sandwiching cubic regularization

Suppose f'' is $\mathsf{L} = \mathsf{L}^{[2]}$-Lipschitz continuous (w.r.t. the ℓ_2 norm). Then, the cubic regularization iterates $\{\mathbf{w}_t\}$ in (15.3) satisfy the following sandwich inequality:

$$f(\mathbf{w}_{t+1}) - \tfrac{1}{6}(\mathsf{L} - \tfrac{1}{\eta_t})\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3 \leq \bar{f}_t(\mathbf{w}_{t+1}) \leq f(\mathbf{w}_t) - \tfrac{1}{12\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3.$$

In particular,

- *if $\eta_t \leq \frac{3}{2\mathsf{L}}$, then $f(\mathbf{w}_{t+1}) \leq f(\mathbf{w}_t)$, i.e., cubic regularization is descending;*
- *if $\eta_t \leq \frac{1}{\mathsf{L}}$, then $f(\mathbf{w}_{t+1}) \leq \bar{f}_t(\mathbf{w}_{t+1}) \leq f(\mathbf{w}_t)$.*

Proof: The first inequality follows from Theorem 1.13:

$$f(\mathbf{w}_{t+1}) \leq f(\mathbf{w}_t) + \langle f'(\mathbf{w}_t), \mathbf{w}_{t+1} - \mathbf{w}_t\rangle + \tfrac{1}{2}\langle f''(\mathbf{w}_t)(\mathbf{w}_{t+1} - \mathbf{w}_t), \mathbf{w}_{t+1} - \mathbf{w}_t\rangle + \tfrac{\mathsf{L}}{6}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3,$$

while the second inequality was already established in (15.7). ■

For comparison, gradient descent requires $\eta_t \leq \frac{2}{\mathsf{L}^{[1]}}$ to guarantee descending (see Theorem 1.17).

Exercise 15.17: Bigger step size under convexity

If f is additionally convex in Proposition 15.16, then

$$f(\mathbf{w}_{t+1}) - \tfrac{1}{6}(\mathsf{L} - \tfrac{1}{\eta_t})\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3 \leq \bar{f}_t(\mathbf{w}_{t+1}) \leq f(\mathbf{w}_t) - \tfrac{1}{3\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3.$$

In particular, if $\eta_t \leq \frac{3}{\mathsf{L}}$, then $f(\mathbf{w}_{t+1}) \leq f(\mathbf{w}_t)$, i.e., cubic regularization is descending.

Proposition 15.18: Relating progress

Suppose f'' is $\mathsf{L} = \mathsf{L}^{[2]}$-Lipschitz continuous (w.r.t. the ℓ_2 norm). Then, the iterates $\{\mathbf{w}_t\}$ in (15.3) satisfy:

$$\|f'(\mathbf{w}_{t+1})\|_2 \leq \tfrac{1}{2}(\mathsf{L} + \tfrac{1}{\eta_t})\|\mathbf{w}_t - \mathbf{w}_{t+1}\|_2^2 \tag{15.8}$$

$$f''(\mathbf{w}_{t+1}) \succeq -(\mathsf{L} + \tfrac{1}{2\eta_t})\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2 \cdot \mathrm{Id}$$

$$\bar{f}_t(\mathbf{w}_{t+1}) \leq \min_{\mathbf{w}} \; f(\mathbf{w}) + \tfrac{1}{6}(\mathsf{L} + \tfrac{1}{\eta_t})\|\mathbf{w} - \mathbf{w}_t\|_2^3, \tag{15.9}$$

where the right-hand side is the cubic proximal point.

Proof: For the first inequality, apply (15.4) and Theorem 1.13. For the second inequality, apply (15.6) and Lipschitz continuity of f''. The last inequality again follows from Theorem 1.13. ■

Theorem 15.19: Sublinear rate of cubic regularization (Nesterov and Polyak, 2006)

Suppose f'' is $\mathsf{L} = \mathsf{L}^{[2]}$-Lipschitz continuous (w.r.t. the ℓ_2 norm) and f is bounded from below by $f_\star$. Then, assuming $\eta_t \in [0, \frac{3}{2\mathsf{L}}]$, the cubic regularization iterates $\{\mathbf{w}_t\}$ in (15.3) satisfy:

$$\sum_{t=0}^{\infty}(\tfrac{1}{4\eta_t} - \tfrac{\mathsf{L}}{6})(\tfrac{2\eta_t}{1+\eta_t\mathsf{L}})^{3/2}\|f'(\mathbf{w}_{t+1})\|_2^{3/2} \le \sum_{t=0}^{\infty}(\tfrac{1}{4\eta_t} - \tfrac{\mathsf{L}}{6})\|\mathbf{w}_t - \mathbf{w}_{t+1}\|_2^3 \le f(\mathbf{w}_0) - f_\star.$$

In particular, if $\eta_t = \frac{1}{\mathsf{L}}$, we have $\sum_t \|\frac{f'(\mathbf{w}_{t+1})}{\mathsf{L}}\|_2^{3/2} \le \sum_t \|\mathbf{w}_t - \mathbf{w}_{t+1}\|_2^3 \le \frac{12(f_0 - f_\star)}{\mathsf{L}}$.

Proof: We simply telescope the sandwich inequality in Proposition 15.18:

$$f(\mathbf{w}_0) - f_\star \ge \sum_{t=0}^{\infty}(f(\mathbf{w}_t) - f(\mathbf{w}_{t+1})) \ge \sum_{t=0}^{\infty}(\tfrac{1}{4\eta_t} - \tfrac{\mathsf{L}}{6})\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3.$$

When $\eta_t \le \frac{3}{2\mathsf{L}}$, we further apply (15.8). ■

Of course, when f is convex, we can replace the factor $\frac{1}{4\eta_t} - \frac{\mathsf{L}}{6}$ with $\frac{1}{2\eta_t} - \frac{\mathsf{L}}{6}$.
It follows from Proposition 15.18 that if $\mathbf{w}_*$ is a limit point of $\mathbf{w}_t$, then we must have

$$f'(\mathbf{w}_*) = 0, \qquad f''(\mathbf{w}_*) \succeq \mathbf{0}.$$

In other words, $\mathbf{w}_*$ cannot be a local maximizer (or a strict saddle, namely the Hessian has both a positive and negative eigenvalue), which is in sharp contrast to Newton's algorithm (see Alert 15.6)!

Nesterov, Y. and B. T. Polyak (2006). "Cubic regularization of Newton method and its global performance". *Mathematical Programming*, vol. 108, pp. 177–205.

Remark 15.20: Making gradient small

Theorem 15.19 implies that the (minimum) gradient of cubic regularization decays at the rate of $O(t^{-2/3})$, which is faster than gradient descent, see Theorem 1.17 and Nesterov (2012). See also (Kim and Fessler, 2021; Allen-Zhu, 2018; Carmon et al., 2018; Foster et al., 2019; Ito and Fukuda, 2021).

Nesterov, Y. (2012). "How to make the gradients small". In: *Optima*. Vol. 88, pp. 10–11.

Kim, D. and J. A. Fessler (2021). "Optimizing the Efficiency of First-Order Methods for Decreasing the Gradient of Smooth Convex Functions". *Journal of Optimization Theory and Applications*, vol. 188, pp. 192–219.

Allen-Zhu, Z. (2018). "How To Make the Gradients Small Stochastically: Even Faster Convex and Nonconvex SGD". In: *Advances in Neural Information Processing Systems*.

Carmon, Y., J. C. Duchi, O. Hinder, and A. Sidford (2018). "Accelerated Methods for NonConvex Optimization". *SIAM Journal on Optimization*, vol. 28, no. 2, pp. 1751–1772.

Foster, D. J., A. Sekhari, O. Shamir, N. Srebro, K. Sridharan, and B. Woodworth (2019). "The Complexity of Making the Gradient Small in Stochastic Convex Optimization". In: *Proceedings of the Thirty-Second Conference on Learning Theory*, pp. 1319–1345.

Ito, M. and M. Fukuda (2021). "Nearly Optimal First-Order Methods for Convex Optimization under Gradient Norm Measure: An Adaptive Regularization Approach". *Journal of Optimization Theory and Applications*, vol. 188, pp. 770–804.

Theorem 15.21: Local quadratic rate of cubic regularization (Nesterov and Polyak, 2006)

Let $q_t := \frac{\mathsf{L}\|f'(\mathbf{w}_t)\|_2}{\sigma_t^2}$, where σ_t is the minimum eigenvalue of $f''(\mathbf{w}_t)$. Suppose f'' is $\mathsf{L} = \mathsf{L}^{[2]}$-Lipschitz

continuous and $\sigma_0 > 0$. *Then, the cubic regularization iterates* (15.3) *satisfy:*

$$q_{t+1} \le \tfrac{1+\alpha}{2}\left(\tfrac{q_t}{1-q_t}\right)^2 \le \tfrac{1+\alpha}{2(1-\beta)^2} q_t^2 \le \tfrac{(1+\alpha)\beta}{2(1-\beta)^2} q_t,$$

as long as $\eta_t \ge \frac{1}{\alpha \mathsf{L}}$, $q_0 \le \beta < 1$ *and* $\frac{(1+\alpha)\beta}{2(1-\beta)^2} < 1$. *Moreover, the gradient norm* $\|f'(\mathbf{w}_t)\|_2$ *converges to 0 and the iterate* $\mathbf{w}_t$ *converges to a local minimizer at a quadratic rate.*

Proof: Using (15.6), we first note that

$$\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2 = \left\| \left[f''(\mathbf{w}_t) + \tfrac{1}{\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2 \cdot \mathrm{Id} \right]^{-1} f'(\mathbf{w}_t) \right\|_2 \le \frac{\|f'(\mathbf{w}_t)\|_2}{\sigma_t + \|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2/\eta_t},$$

and hence assuming $\sigma_t > 0$,

$$\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2 \le \frac{2\|f'(\mathbf{w}_t)\|_2}{\sqrt{\sigma_t^2 + 4\|f'(\mathbf{w}_t)\|_2/\eta_t} + \sigma_t} \le \frac{\|f'(\mathbf{w}_t)\|_2}{\sigma_t}. \tag{15.10}$$

Therefore, applying the L-Lipschitz continuity of f'':

$$\sigma_{t+1} \ge \sigma_t - \mathsf{L}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2 \ge \sigma_t - \frac{\mathsf{L}\|f'(\mathbf{w}_t)\|_2}{\sigma_t} = (1 - q_t)\sigma_t, \tag{15.11}$$

$$\sigma_{t+1} \le \sigma_t + \mathsf{L}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2 \le \sigma_t + \frac{\mathsf{L}\|f'(\mathbf{w}_t)\|_2}{\sigma_t} = (1 + q_t)\sigma_t. \tag{15.12}$$

Moreover, applying (15.8) and assuming $q_t \le 1$ we know

$$q_{t+1} := \frac{\mathsf{L}\|f'(\mathbf{w}_{t+1})\|_2}{\sigma_{t+1}^2} \le \frac{\mathsf{L}(\mathsf{L} + \frac{1}{\eta_t})\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^2}{2\sigma_{t+1}^2} \le \frac{\mathsf{L}(\mathsf{L} + \frac{1}{\eta_t})\|f'(\mathbf{w}_t)\|_2^2}{2(1-q_t)^2\sigma_t^4} = \tfrac{1}{2}\big(1 + \tfrac{1}{\eta_t \mathsf{L}}\big)\big(\tfrac{q_t}{1-q_t}\big)^2.$$

With our choice on η_t and β, $q_{t+1} \le q_t \le \beta \le 1$ and hence $\sigma_{t+1} \ge (1 - q_t)\sigma_t > 0$ are satisfied recursively.

Clearly, $\sum_t q_t < \infty$ and hence it follows from (15.11) and (15.12) that σ_t remains bounded and away from 0. Therefore, the (local) quadratic convergence rate of q_t translates to the same for the gradient $\|f'(\mathbf{w}_t)\|_2$ (by the definition of q_t) and the iterate $\{\mathbf{w}_t\}$ (due to (15.10)). ■

Setting $\eta_t = \infty$, $\alpha = 0$ and $\beta < \frac{1}{2}$ yields a similar result for the Newton's Algorithm 15.3 as Theorem 15.7. This is not surprising, after all as we commented in Algorithm 15.11, cubic regularization gradually reduces to Newton's update.

Nesterov, Y. and B. T. Polyak (2006). "Cubic regularization of Newton method and its global performance". *Mathematical Programming*, vol. 108, pp. 177–205.

Algorithm 15.22: Solving cubic regularization

Let us examine how we can solve the cubic regularization iterate (15.3). Denote $A := \begin{bmatrix} 0 & \mathbf{g}^\top \\ \mathbf{g} & H \end{bmatrix}$, we derive:

$$\begin{aligned}
\min_{\mathbf{z}} \; 2\langle \mathbf{z}, \mathbf{g}\rangle + \langle \mathbf{z}, H\mathbf{z}\rangle + \tfrac{1}{3\eta}\|\mathbf{z}\|_2^3 &= \min_{Z \succeq \mathbf{0}, Z_{11}=1, \langle I, Z\rangle = \zeta + 1} \langle A, Z\rangle + \tfrac{1}{3\eta}\zeta^{3/2} \\
&= \sup_{\lambda,\mu} \min_{\gamma, Z \succeq \mathbf{0}} \mu(\langle \mathbf{e}_1\mathbf{e}_1^\top, Z\rangle - 1) + \lambda(\langle I, Z\rangle - \zeta - 1) + \langle A, Z\rangle + \tfrac{1}{3\eta}\zeta^{3/2} \\
&= \sup_{A + \lambda I + \mu \mathbf{e}_1\mathbf{e}_1^\top \succeq \mathbf{0}} \min_{\zeta} \; -\mu - \lambda(\zeta + 1) + \tfrac{1}{3\eta}\zeta^{3/2}, \\
[\sqrt{\zeta} = 2\eta\lambda_+] &= \sup_{\lambda \ge 0, \mu} \; -(\lambda + \mu) - \tfrac{4}{3}\eta^2\lambda^3 \quad \text{s.t.} \quad \begin{bmatrix} \lambda + \mu & \mathbf{g}^\top \\ \mathbf{g} & H + \lambda I \end{bmatrix} \succeq \mathbf{0}
\end{aligned}$$

$$[\text{Schur's complement}] \;=\; -\inf_{\lambda \geq (-\lambda_{\min}(H))_+} \left\langle (H+\lambda I)^\dagger \mathbf{g}, \mathbf{g} \right\rangle + \tfrac{4}{3}\eta^2\lambda^3, \quad \text{s.t.} \quad \mathbf{g} \in \operatorname{rge}(H+\lambda I),$$

which is a univariate convex minimization problem. Setting derivative to $\mathbf{0}$ we obtain the fixed point equation:

$$2\eta\bar{\lambda} = \|(H+\bar{\lambda}I)^\dagger \mathbf{g}\|_2.$$

The optimal λ_* is given by (see Theorem 19.4)

$$\lambda_\star = \bar{\lambda} \vee (-\lambda_{\min}(H))_+$$

and it follows from (15.4) that (recall $\sqrt{\zeta} = 2\eta\lambda_+ = \|\mathbf{z}\|_2$)

$$(H+\lambda_\star I)\mathbf{z} + \mathbf{g} = \mathbf{0}, \tag{15.13}$$

which has a unique solution if $\lambda_\star = \bar{\lambda} > (-\lambda_{\min}(H))_+$ or $\lambda_{\min}(H) > 0$. When $\lambda_\star = -\lambda_{\min}(H) \geq 0$, we pick any solution of (15.13) such that $\|\mathbf{z}\|_2 = 2\eta\lambda_\star$.

For faster algorithms, see Paternain et al. (2019), Carmon and Duchi (2020), Lieder (2020), and Jiang et al. (2021).

Paternain, S., A. Mokhtari, and A. Ribeiro (2019). "A Newton-Based Method for Nonconvex Optimization with Fast Evasion of Saddle Points". *SIAM Journal on Optimization*, vol. 29, no. 1, pp. 343–368.

Carmon, Y. and J. C. Duchi (2020). "First-Order Methods for Nonconvex Quadratic Minimization". *SIAM Review*, vol. 62, no. 2, pp. 395–436.

Lieder, F. (2020). "Solving Large-Scale Cubic Regularization by a Generalized Eigenvalue Problem". *SIAM Journal on Optimization*, vol. 30, no. 4, pp. 3345–3358.

Jiang, R., M.-C. Yue, and Z. Zhou (2021). "An accelerated first-order method with complexity analysis for solving cubic regularization subproblems". *Computational Optimization and Applications*, vol. 79, pp. 471–506.

Example 15.23: Solving cubic regularization

Consider the following instance:

$$\mathbf{g} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad \eta = 1.$$

We have the primal problem

$$\min_{\mathbf{z}\in\mathbb{R}^2} \; -2z_1 - z_2^2 + \tfrac{1}{3}\|\mathbf{z}\|_2^3,$$

whose first-order necessary condition is:

$$\left.\begin{aligned} -2 \quad + \|\mathbf{z}\|_2 z_1 \quad &= 0 \\ -2z_2 + \|\mathbf{z}\|_2 z_2 \quad &= 0 \end{aligned}\right\} \implies \mathbf{z} = \begin{bmatrix} 1 \\ \pm\sqrt{3} \end{bmatrix}.$$

(The other possibility $\mathbf{z} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$ is a strict saddle after checking the second-order condition (15.6).)

The dual problem is

$$-\inf_{\lambda \geq 1} \; \lambda^{-1} + \tfrac{4}{3}\lambda^3 \implies \lambda_\star = 1.$$

Solving the linear system (15.13) we obtain $z_1 = 1$ and we find $z_2 = \pm\sqrt{3}$ so that $\|\mathbf{z}\|_2 = 2$.

Lemma 15.24: Recursive estimate (e.g., Polyak, 1987, Lemma 6, p. 46)

Consider nonnegative sequences u_t and ξ_t. Let $p > 0$ and $q \in (0, 1]$. Then,

- $u_{t+1} \le u_t(1 - \xi_t u_t^p) \implies u_{t+1} \le u_0 \left(1 + p u_0^p \sum_{\tau=0}^t \xi_\tau\right)^{-1/p}$.
- $u_{t+1} \le u_t(1 - \xi_t u_t^{-q}) \implies u_{t+1}^q \le u_t^q - q\xi_t$.

Proof: For the first claim, we deduce

$$u_{t+1}^{-p} \ge u_t^{-p}(1 - \xi_t u_t^p)^{-p} \ge u_t^{-p}(1 + p\xi_t u_t^p) = u_t^{-p} + p\xi_t,$$

where we applied the convex inequality $(1 - x)^{-p} \ge 1 + px$. Telescoping completes the proof.

For the second claim, we deduce

$$u_{t+1}^q \le u_t^q(1 - \xi_t u_t^{-q})^q \le u_t^q(1 - q\xi_t u_t^{-q}) = u_t^q - q\xi_t,$$

where we applied the concave inequality $(1 - x)^q \le 1 - qx$. ■

Polyak, B. T. (1987). "Introduction to Optimization". Optimization Software.

Theorem 15.25: Global sublinear rate under star convexity (Nesterov and Polyak, 2006)

Suppose f is convex, f'' is $\mathsf{L} = \mathsf{L}_2^{[2]}$-Lipschitz continuous, and the (sub)level set $[\![f \le f(\mathbf{w}_0)]\!]$ is bounded in diameter by ϱ. Then, the cubic regularization iterates (15.3) satisfy for all $t \ge 0$:

$$f(\mathbf{w}_{t+1}) - f_\star \le \frac{f(\mathbf{w}_1) - f_\star}{\left(1 + \sqrt{f(\mathbf{w}_1) - f_\star} \sum_{\tau=1}^t \sqrt{\frac{2}{9(\mathsf{L}+1/\eta_\tau)\varrho^3}}\right)^2} \le \frac{9\varrho^3\mathsf{L}}{2\left(\sum_{\tau=0}^t \sqrt{\frac{\eta_\tau \mathsf{L}}{1+\eta_\tau \mathsf{L}}}\right)^2},$$

provided that for all t, $\frac{1}{\eta_t} \le 2\mathsf{L} + \frac{3}{\eta_{t+1}}$, in particular, if $\eta_{t+1} \le 3\eta_t$, and inequality (15.14) holds, in particular if $\eta_t \le \frac{1}{\mathsf{L}}$.

Proof: Using the condition on the step size η_t and applying (15.9) we have

$$\begin{aligned} f(\mathbf{w}_{t+1}) - f_\star &\le \bar{f}_t(\mathbf{w}_{t+1}) - f_\star \qquad (15.14)\\ &\le \inf_{\mathbf{w}} \; f(\mathbf{w}) - f_\star + \tfrac{1}{6}(\mathsf{L} + \tfrac{1}{\eta_t})\|\mathbf{w} - \mathbf{w}_t\|_2^3 \\ &\le \min_{\beta_t \in [0,1]} f\big((1 - \beta_t)\mathbf{w}_t + \beta_t \mathbf{w}_\star\big) - f_\star + \tfrac{\beta_t^3}{6}(\mathsf{L} + \tfrac{1}{\eta_t})\|\mathbf{w}_\star - \mathbf{w}_t\|_2^3 \\ &\le \min_{\beta_t \in [0,1]} f(\mathbf{w}_t) - f_\star - \beta_t[f(\mathbf{w}_t) - f_\star] + \tfrac{\beta_t^3 \varrho^3}{6}(\mathsf{L} + \tfrac{1}{\eta_t}). \end{aligned}$$

(Note that inequality (15.14) implies that $f(\mathbf{w}_{t+1}) \le f(\mathbf{w}_t) \le \cdots \le f(\mathbf{w}_0)$, i.e. $\{\mathbf{w}_t\}$ remains bounded in diameter ϱ.) Setting the derivative w.r.t. β_t to zero we obtain (see Theorem 19.4):

$$\beta_t = 1 \wedge \sqrt{\frac{2(f(\mathbf{w}_t) - f_\star)}{(\mathsf{L} + 1/\eta_t)\varrho^3}}.$$

If $\beta_t = 1$ (which we deduce below can happen only at $t = 0$), then

$$f(\mathbf{w}_{t+1}) - f_\star \le \frac{\varrho^3(\mathsf{L} + 1/\eta_t)}{6}. \qquad (15.15)$$

Provided that $\frac{1}{\eta_t} \le 2\mathsf{L} + \frac{3}{\eta_{t+1}}$, we will have $\beta_{t+1} \le 1$ since, and hence for all $t \ge 1$:

$$f(\mathbf{w}_{t+1}) - f_\star \le f(\mathbf{w}_t) - f_\star - (f(\mathbf{w}_t) - f_\star)^{3/2} \sqrt{\tfrac{8}{9(\mathsf{L}+1/\eta_t)\varrho^3}}.$$

Apply Lemma 15.24 and note that $f(\mathbf{w}_1)$ satisfies (15.15). ■

For constant step size $\eta_t \equiv \eta$, the function value decreases at the rate of $O(1/t^2)$, matching that of the accelerated gradient Theorem 7.8. Moreover, the open loop step size

$$\eta_t \to 0, \quad \sum_t \sqrt{\eta_t} = \infty$$

suffices to guarantee convergence. It is, however, not possible to achieve the rate $O(1/t^2)$ with an open loop step size (at least based on our current bounds).

Nesterov, Y. and B. T. Polyak (2006). "Cubic regularization of Newton method and its global performance". *Mathematical Programming*, vol. 108, pp. 177–205.

Remark 15.26: Digesting the proof

Inspecting the proof of Theorem 15.25 carefully leads to the following observations:

- Amijo's backtracking for the step size η_t applies (see Remark 1.20), so we need only search η_t so that (15.14) holds at each iteration. Moreover, we could also increase η_t by a factor of 3 if η_t is found small.
- The function f need only be star convex w.r.t. some global minimizer, i.e., for some $\mathbf{w}_\star$ such that $f(\mathbf{w}_\star) = \inf_{\mathbf{w}} f(\mathbf{w})$, we have for all $\mathbf{w}$:

  $$f\big((1-\beta)\mathbf{w} + \beta\mathbf{w}_\star\big) \le (1-\beta)f(\mathbf{w}) + \beta f(\mathbf{w}_\star).$$

  For example, the functions $f(w) = |w|(1 - \exp(-|w|))$ and $f(w, z) = w^2z^2 + w^2 + z^2$ are star-convex but not convex.
- There is a small cost to the above generality: Theorem 15.25 is only about the gap between $f(\mathbf{w}_t)$ and the minimum value $f_\star$ while recall that in Theorem 2.21 or Theorem 7.8 we are able to prove a convergence rate for the gap between $f(\mathbf{w}_t)$ and any $f(\mathbf{w})$.

Exercise 15.27: New gun for old battles

Can you adapt the proof of Theorem 15.25 to the gradient descent Algorithm 1.4 and establish its rate of convergence for star-convex functions?

Theorem 15.28: Global superlinear rate under (γ, p)-growth (Nesterov and Polyak, 2006)

Let F denote the (global) minimizer(s) of f and suppose f have (γ, p)-growth, i.e.,

$$f(\mathbf{w}) - f_\star \ge \tfrac{\gamma}{p} \cdot \operatorname{dist}^p(\mathbf{w}, \mathsf{F}), \quad \textit{where} \quad p \in [1, 2].$$

Suppose f is star-convex and f'' is $\mathsf{L} = \mathsf{L}_2^{[2]}$-Lipschitz continuous. If the step size $\eta_t \ge \underline{\eta} > 0$ always satisfies (15.17) (e.g. $\eta_t \le \frac{1}{\mathsf{L}}$), we have at first

$$\big(f(\mathbf{w}_{t+1}) - f_\star\big)^{(3-p)/(2p)} \le \big(f(\mathbf{w}_t) - f_\star\big)^{(3-p)/(2p)} - \tfrac{3-p}{p}\left(\tfrac{\gamma}{p}\right)^{3/(2p)} \cdot \sqrt{\tfrac{2\eta_t}{1+\eta_t\mathsf{L}}}$$

and then

$$f(\mathbf{w}_{t+1}) - f_\star \le \tfrac{1}{6}(\mathsf{L} + \tfrac{1}{\eta_t}) \left(\tfrac{p}{\gamma}\right)^{3/p} [f(\mathbf{w}_t) - f_\star]^{3/p}, \tag{15.16}$$

where the transition happens when (at the latest)

$$f(\mathbf{w}_t) - f_\star \le \left[\tfrac{2\underline{\eta}}{1+\underline{\eta}\mathsf{L}} \left(\tfrac{\gamma}{p}\right)^{3/p}\right]^{p/(3-p)}.$$

Proof: Using the condition on the step size η_t and applying (15.9) we have

$$\begin{aligned} f(\mathbf{w}_{t+1}) - f_\star &\le \bar{f}_t(\mathbf{w}_{t+1}) - f_\star \\ &\le \inf_{\mathbf{w}} \; f(\mathbf{w}) - f_\star + \tfrac{1}{6}(\mathsf{L} + \tfrac{1}{\eta_t})\|\mathbf{w} - \mathbf{w}_t\|_2^3 \\ &\le \min_{\beta_t \in [0,1]} f\big((1-\beta_t)\mathbf{w}_t + \beta_t \mathbf{w}_\star\big) - f_\star + \tfrac{\beta_t^3}{6}(\mathsf{L} + \tfrac{1}{\eta_t})\|\mathbf{w}_\star - \mathbf{w}_t\|_2^3 \\ &\le \min_{\beta_t \in [0,1]} f(\mathbf{w}_t) - f_\star - \beta_t[f(\mathbf{w}_t) - f_\star] + \tfrac{\beta_t^3}{6}(\mathsf{L} + \tfrac{1}{\eta_t})[p(f(\mathbf{w}_t) - f_\star)/\gamma]^{3/p}. \end{aligned} \tag{15.17}$$

Setting the derivative w.r.t. β_t to zero we obtain (see Theorem 19.4):

$$\beta_t = 1 \wedge \sqrt{\frac{2(\frac{\gamma}{p})^{3/p}(f(\mathbf{w}_t) - f_\star)^{1-3/p}}{(\mathsf{L}+1/\eta_t)}}.$$

If $\beta_t < 1$, we have

$$f(\mathbf{w}_{t+1}) - f_\star \le f(\mathbf{w}_t) - f_\star - \sqrt{\frac{8(\frac{\gamma}{p})^{3/p}}{(\mathsf{L}+1/\eta_t)}}(f(\mathbf{w}_t) - f_\star)^{1-(3-p)/(2p)}.$$

Applying Lemma 15.24 we deduce

$$\big(f(\mathbf{w}_{t+1}) - f_\star\big)^{(3-p)/(2p)} \le \big(f(\mathbf{w}_t) - f_\star\big)^{(3-p)/(2p)} - \frac{3-p}{p} \cdot \sqrt{\frac{2(\frac{\gamma}{p})^{3/p}}{(\mathsf{L}+1/\eta_t)}}.$$

Since $\eta_t \ge \underline{\eta} > 0$, after a constant number of iterations, we must have $\beta_t \equiv 1$, i.e., the transition to the superlinear rate (15.16) happens. ■

In other words, after (at most) a constant number of iterations, cubic regularization settles to a superlinear rate.

For instance, σ-strongly convex functions are of $(\sigma, 2)$-growth. However, the (global) superlinear rate obtained here (for $p = 2$) is slower than the (local) quadratic rate in Theorem 15.21.

Nesterov, Y. and B. T. Polyak (2006). "Cubic regularization of Newton method and its global performance". *Mathematical Programming*, vol. 108, pp. 177–205.

Lemma 15.29: Recursive estimate II

Consider nonnegative sequences u_t and ξ_t, where $\bar{\xi} \ge \xi_t \ge \underline{\xi} > 0$. Let $q \in (0, 1]$. Then,

- *the recursion $u_t \ge u_{t+1}(1 + \xi_t u_{t+1}^q)$ implies that*

$$\begin{cases} \ln(u_t) & \le \left(\frac{1}{q+1}\right)^t \ln(u_0 \underline{\xi}^{1/q}) - \ln(\underline{\xi}^{1/q}) \\ u_t^{-q} & \le u_{t+1}^{-q} - \theta\xi_t, \quad \text{if } u_t \le (\mu/\underline{\xi})^{1/q} \text{ for some } \mu > 1 \end{cases},$$

where $\theta := [1 - (1+\delta)^{-q}]/\delta$ and $\delta := \mu\bar{\xi}/\underline{\xi}$.

- *the recursion* $u_t \geq u_{t+1}(1+\xi_t u_{t+1}^{-q})$ *implies that*

$$u_{t+1} \leq \tfrac{1}{1+\underline{\xi} u_0^{-q}} \cdot u_t \quad \wedge \quad \underline{\xi}^{-1/(1-q)} \cdot u_t^{1/(1-q)}.$$

Proof: For the first claim, we note that

$$u_t \geq \xi_t u_{t+1}^{q+1} \iff \ln(u_{t+1}) \leq \tfrac{1}{q+1}\ln(u_t) - \tfrac{1}{q+1}\ln(\xi_t) \implies \ln(u_t) \leq \left(\tfrac{1}{q+1}\right)^t \ln(u_0) - \sum_{\tau=0}^{t-1} \left(\tfrac{1}{q+1}\right)^{t-\tau} \ln(\xi_\tau)$$
$$\leq \left(\tfrac{1}{q+1}\right)^t \ln(u_0) - \tfrac{1}{q}\left[1 - \left(\tfrac{1}{q+1}\right)^t\right] \ln(\underline{\xi}).$$

Thus, u_t decreases below $\mu^{1/q}\underline{\xi}^{-1/q}$ for any $\mu > 1$ at a linear rate, after which we switch to the following bound:

$$u_t^{-q} \leq u_{t+1}^{-q}(1+\xi_t u_{t+1}^q)^{-q} \leq u_{t+1}^{-q}(1-\theta\xi_t u_{t+1}^q) = u_{t+1}^{-q} - \theta\xi_t,$$

where we applied the inequality $(1+x)^{-q} \leq 1-\theta x$ for $x \in [0,\delta]$, with $\theta := [1-(1+\delta)^{-q}]/\delta$ and $\delta := \mu\bar{\xi}/\underline{\xi}$.

The second claim is obvious once we note that $u_t \geq u_{t+1}$ is monotone. ■

Definition 15.30: (γ, p)-gradient growth (Polyak, 1963)

Recall that a function is of (γ, p)-gradient growth if for all $\mathbf{w}$:

$$f(\mathbf{w}) - f_\star \leq \tfrac{\gamma}{p} \cdot \|f'(\mathbf{w})\|_2^p,$$

where $\gamma > 0$ and $p \in [1,2]$. For instance, σ-strongly convex functions are of $(\frac{1}{\sigma}, 2)$-gradient growth.

Polyak, B. T. (1963). "Gradient methods for the minimization of functionals". *USSR Computational Mathematics and Mathematical Physics*, vol. 3, no. 4, pp. 643–653.

Theorem 15.31: Global convergence rate under gradient growth (Nesterov and Polyak, 2006)

Suppose f is of (γ,p)-gradient growth and f'' is $\mathsf{L} = \mathsf{L}_2^{[2]}$-Lipschitz continuous. Suppose $0 < \underline{\eta} \leq \eta_t \leq \bar{\eta}$ and η_t satisfies (15.18) for some $\alpha > 0$ (e.g., if $\eta_t \leq \frac{3(1-4\alpha)}{2\mathsf{L}}$). Let $q := \frac{3}{2p} - 1$ and $\xi_t := \sqrt{\frac{8\eta_t\alpha^2}{(1+\eta_t\mathsf{L})^3}}\left(\frac{p}{\gamma}\right)^{3/(2p)}$ with upper and lower bound $\bar{\xi}$ and $\underline{\xi}$, respectively.

- *If $p \in [1, \frac{3}{2})$, $q \in (0, \frac{1}{2}]$ and we have*

$$\begin{cases} \ln[f(\mathbf{w}_t) - f_\star] & \leq \left(\frac{1}{q+1}\right)^t \ln\left([f(\mathbf{w}_0) - f_\star]\underline{\xi}^{1/q}\right) - \ln(\underline{\xi}^{1/q}) \\ [f(\mathbf{w}_t) - f_\star]^{-q} & \leq [f(\mathbf{w}_{t+1}) - f_\star]^{-q} - \theta\xi_t, \quad \text{if } [f(\mathbf{w}_t) - f_\star] \leq (\mu/\underline{\xi})^{1/q} \text{ for some } \mu > 1 \end{cases},$$

  where $\theta := [1-(1+\delta)^{-q}]/\delta$ and $\delta := \mu\bar{\xi}/\underline{\xi}$.

- *If $p \in [\frac{3}{2}, 2]$, $q \in [-\frac{1}{4}, 0]$ and we have*

$$[f(\mathbf{w}_{t+1}) - f_\star] \leq \tfrac{1}{1+\underline{\xi}[f(\mathbf{w}_0)-f_\star]^{-q}} \cdot [f(\mathbf{w}_t) - f_\star] \quad \wedge \quad \underline{\xi}^{-1/(1-q)} \cdot [f(\mathbf{w}_t) - f_\star]^{1/(1-q)}.$$

Proof: Applying Proposition 15.16, Proposition 15.18 and gradient growth, we have

$$\begin{aligned} f(\mathbf{w}_t) - f(\mathbf{w}_{t+1}) &\geq \tfrac{\alpha}{\eta_t}\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^3 \\ &\geq \tfrac{\alpha}{\eta_t}\left[\tfrac{2\eta_t}{1+\eta_t \mathsf{L}}\right]^{3/2} \|f'(\mathbf{w}_{t+1})\|_2^{3/2} \\ &\geq \tfrac{\alpha}{\eta_t}\left[\tfrac{2\eta_t}{1+\eta_t \mathsf{L}}\right]^{3/2} \left(\tfrac{p}{\gamma}\right)^{3/(2p)} [f(\mathbf{w}_{t+1}) - f_\star]^{3/(2p)}. \end{aligned} \tag{15.18}$$

Applying Lemma 15.29 completes the proof. ■

Thus, we see some sharp transitions in the convergence rate:

- When $p \in [1, \frac{3}{2})$, cubic regularization first converges superlinearly and then settles into the sublinear rate $O(t^{-(2p)/(3-2p)})$. In particular, when $p = 1$, we obtain the familiar rate $O(t^{-2})$.
- When $p \in [\frac{3}{2}, 2]$, cubic regularization first converges linearly and then superlinearly (with exponent $2p/3$).

For convenience, we may set $\alpha = \frac{1}{12}$, in which case we need only perform backtracking to guarantee $f(\mathbf{w}_{t+1}) \leq \bar{f}_t(\mathbf{w}_{t+1})$, see Proposition 15.16.

Nesterov, Y. and B. T. Polyak (2006). "Cubic regularization of Newton method and its global performance". *Mathematical Programming*, vol. 108, pp. 177–205.

Exercise 15.32: (γ, p)-gradient growth

Prove the following:

- A convex function restricted to a domain of diameter ϱ is of $(\varrho, 1)$-gradient growth.
- Suppose φ is of (γ, p)-gradient growth, $(\mathbf{s}')^\top \mathbf{s}' \succeq \sigma \mathrm{Id}$ and $\inf \varphi = \inf \varphi \circ \mathbf{s}$. Then, $\varphi \circ \mathbf{s}$ is of $(\sigma^{p/2}\gamma, p)$-gradient growth.

Alert 15.33: Adaptation

We remark that the fast rates in both Theorem 15.28 and Theorem 15.31 are achieved without the knowledge of (γ, p), i.e. the step size of cubic regularization does not even depend on them!

Remark 15.34: Composition with a homeomorphism

All of our results immediately extends to the composite function $f \circ \mathbf{s}$ with the same conditions on f, provided that $\mathbf{s}$ is a homeomorphism whose inverse is 1-Lipschitz continuous:

$$\|\mathbf{w} - \mathbf{z}\|_2 \leq \|\mathbf{s}(\mathbf{w}) - \mathbf{s}(\mathbf{z})\|_2.$$

We remark that triangular maps form a natural family of homeomorphisms.

Example 15.35: Comparison with first-order algorithms

Let us now compare cubic-regularization with first-order gradient algorithms. Consider the class of σ-strongly

convex functions with $\mathsf{L} = \mathsf{L}^{[2]}$-Lipschitz continuous Hessian. It follows that

$$\varrho := \inf\{\|\mathbf{w} - \mathbf{w}_\star\|_2 : f(\mathbf{w}) \le f(\mathbf{w}_0)\} \le \sqrt{\tfrac{2[f(\mathbf{w}_0) - f_\star]}{\sigma}}.$$

Let $p = 2$, $\gamma = \sigma$, and $\eta_t = \frac{1}{\mathsf{L}}$. We divide the progress of cubic regularization into three stages:

- Stage 1: using Theorem 15.25 we have

  $$f(\mathbf{w}_t) - f_\star \le \tfrac{9\varrho^3\mathsf{L}}{t^2}.$$

  Thus, after $t_1 \le 3\sqrt{\varrho\mathsf{L}/\sigma}$ iterations we arrive at:

  $$f(\mathbf{w}_{t_1}) - f_\star \le \sigma\varrho^2.$$

- Stage 2: using Theorem 15.28 we have

  $$\sqrt[4]{f(\mathbf{w}_{t+1}) - f_\star} \le \sqrt[4]{f(\mathbf{w}_t) - f_\star} - \frac{1}{2}\left(\frac{\sigma}{2}\right)^{3/4} \cdot \sqrt{\frac{1}{\mathsf{L}}}.$$

  Thus, after another $t_2 \le 2^{7/4}\sqrt{\varrho\mathsf{L}/\sigma} \le 3.4\sqrt{\varrho\mathsf{L}/\sigma}$ iterations we arrive at:

  $$f(\mathbf{w}_{t_1+t_2}) - f_\star \le \tfrac{\sigma^3}{8\mathsf{L}^2}.$$

- Stage 3: using Theorem 15.28 again we then have (the transition has happened)

  $$f(\mathbf{w}_{t+1}) - f_\star \le \tfrac{\mathsf{L}}{3}\left(\tfrac{2}{\sigma}\right)^{3/2}[f(\mathbf{w}_t) - f_\star]^{3/2}.$$

  Thus, after another $t_3 \le \log_{\frac{3}{2}} \log_9 \frac{9\sigma^3}{8\epsilon\mathsf{L}^2}$ we finally obtain

  $$f(\mathbf{w}_{t_1+t_2+t_3}) - f_\star \le \epsilon.$$

The total number of iterations is bounded by $6.4\sqrt{\varrho\mathsf{L}/\sigma} + \log_{\frac{3}{2}} \log_9 \frac{9\sigma^3}{8\epsilon\mathsf{L}^2}$ (which is by no means optimized). In comparison, let $\mathsf{L}^{[1]} = \|f''(\mathbf{w}_\star)\|_{\mathrm{sp}}$ and we estimate

$$\sigma \cdot \mathrm{Id} \le f''(\mathbf{w}) \le (\mathsf{L}^{[1]} + \varrho\mathsf{L}^{[2]}) \cdot \mathrm{Id}.$$

Thus, the accelerated gradient Algorithm 7.6 needs

$$O\left(\sqrt{\tfrac{\mathsf{L}^{[1]} + \varrho\mathsf{L}^{[2]}}{\sigma}} \log \tfrac{(\mathsf{L}^{[1]} + \varrho\mathsf{L}^{[2]})\varrho^2}{\epsilon}\right)$$

iterations to get an ϵ-approximate minimizer, which is substantially worse than that of cubic regularization. (For gradient descent, remove the square root to get an even worse bound.)

Remark 15.36: Extensions of cubic regularization

Some extensions of cubic regularization include:

- Acceleration: Nesterov (2008)
- Constraints: Nesterov (2006)
- Proximal: Grapiglia and Nesterov (2017, 2019)
- Uniformly convex: Doikov and Nesterov (2021)

- Minimax: Huang et al. (2022)
- Stochastic: Tripuraneni et al. (2018)

Nesterov, Y. (2008). "Accelerating the cubic regularization of Newton's method on convex problems". *Mathematical Programming*, vol. 112, pp. 159–181.
— (2006). "Cubic regularization of Newton's method on convex problems with constraints".
Grapiglia, G. N. and Y. Nesterov (2017). "Regularized Newton Methods for Minimizing Functions with Hölder Continuous Hessians". *SIAM Journal on Optimization*, vol. 27, no. 1, pp. 478–506.
— (2019). "Accelerated Regularized Newton Methods for Minimizing Composite Convex Functions". *SIAM Journal on Optimization*, vol. 29, no. 1, pp. 77–99.
Doikov, N. and Y. Nesterov (2021). "Minimizing Uniformly Convex Functions by Cubic Regularization of Newton Method". *Journal of Optimization Theory and Applications*, vol. 189, pp. 317–339.
Huang, K., J. Zhang, and S. Zhang (2022). "Cubic Regularized Newton Method for Saddle Point Models: A Global and Local Convergence Analysis". *Journal of Scientific Computing*, vol. 90.
Tripuraneni, N., M. Stern, C. Jin, J. Regier, and M. I. Jordan (2018). "Stochastic Cubic Regularization for Fast Nonconvex Optimization". In: *32nd Conference on Neural Information Processing Systems*.

Remark 15.37: Lower bound

Lower bounds for second-order methods can be found in e.g. Arjevani et al. (2019, 2020), Birgin et al. (2017), Cartis et al. (2010), and Cartis et al. (2020). In particular, cubic regularization is not optimal and can be accelerated, in a way similar to how accelerated gradient Algorithm 7.6 accelerates the gradient descent Algorithm 1.4.

Arjevani, Y., O. Shamir, and R. Shiff (2019). "Oracle complexity of second-order methods for smooth convex optimization". *Mathematical Programming*, vol. 178, pp. 327–360.
Arjevani, Y., Y. Carmon, J. C. Duchi, D. J. Foster, A. Sekhari, and K. Sridharan (2020). "Second-Order Information in Non-Convex Stochastic Optimization: Power and Limitations". In: *Proceedings of Thirty Third Conference on Learning Theory*, pp. 242–299.
Birgin, E. G., J. L. Gardenghi, J. M. Martínez, S. A. Santos, and P. L. Toint (2017). "Worst-case evaluation complexity for unconstrained nonlinear optimization using high-order regularized models". *Mathematical Programming*, vol. 163, pp. 359–368.
Cartis, C., N. I. M. Gould, and P. L. Toint (2010). "On the Complexity of Steepest Descent, Newton's and Regularized Newton's Methods for Nonconvex Unconstrained Optimization". *SIAM Journal on Optimization*, vol. 20, no. 6, pp. 2833–2852.
Cartis, C., N. I. M. Gould, and P. L. Toint (2020). "Sharp Worst-Case Evaluation Complexity Bounds for Arbitrary-Order Nonconvex Optimization with Inexpensive Constraints". *SIAM Journal on Optimization*, vol. 30, no. 1, pp. 513–541.

Remark 15.38: Other second-order methods

Nesterov (2021) presented another second-order algorithm that converges even faster than the lower bound suggested in Remark 15.37! (Of course, the catch is on what function class we are talking about.)

See also Doikov and Nesterov (2020), Kamzolov and Gasnikov (2020), and Dvurechensky and Nesterov (2018).

Nesterov, Y. (2021). "Superfast second-order methods for unconstrained convex optimization". *Journal of Optimization Theory and Applications*, vol. 191, pp. 1–30.
Doikov, N. and Y. Nesterov (2020). "Convex optimization based on global lower second-order models". In: *34th Conference on Neural Information Processing Systems*.
Kamzolov, D. and A. Gasnikov (2020). "Near-Optimal Hyperfast Second-Order Method for convex optimization and its Sliding".
Dvurechensky, P. and Y. Nesterov (2018). "Global performance guarantees of second-order methods for unconstrained convex minimization".