Theorem 1.25: Continuity condition of stochastic processes

Let X_t be a stochastic process with index $t \in \mathbb{R}^m$ and taking value in a complete metric space. If for some $\alpha, \beta, \mathsf{L} > 0$,

$$\forall s, \forall t, \quad \mathbb{E}[\mathrm{dist}^\alpha(\mathsf{X}_s, \mathsf{X}_t)] \le \mathsf{L}\|t-s\|_\infty^{m+\beta},$$

then there exists a modification $\tilde{\mathsf{X}}_t$ such that for any $T > 0$ and $\gamma \in [0, \beta/\alpha)$,

$$\mathbb{E}\Big[\sup_{s,t\in[-T,T)^m, s\neq t} \frac{\mathrm{dist}^\alpha(\tilde{\mathsf{X}}_t, \tilde{\mathsf{X}}_s)}{\|t-s\|_\infty^{\alpha\gamma}}\Big] < \infty. \tag{1.3}$$

In particular, $\tilde{\mathsf{X}}_t$ is locally Hölder continuous of order γ.

Proof: We follow Schilling (2021, Theorem 10.1, p. 167) and apply the chaining idea due to Kolmogorov.

Consider the dyadic numbers $D = D_T := \cup_n D_n$ where $D_n := \{-T + (2T)2^{-n}\mathbb{N}_0^m\} \cap [-T, T)^m$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, as well as the restricted diagonal:

$$\Delta_n = (D_n \times D_n) \cap \{(t, s) : \|t-s\|_\infty \le (2T)2^{-n}\}.$$

Clearly, each $t \in D_n$ has at most $3^m - 1$ neighbors in D_n. Thus $|\Delta_n| \le 3^m 2^{mn}$. Find $t^n, s^n \in D_n$ such that

$$t^n \le t < t^n + (2T)2^{-n}, \quad s^n \le s < s^n + (2T)2^{-n}.$$

Since $(t^n, s^n) \to (t, s)$ as $n \to \infty$, we have

$$\begin{aligned}
\mathrm{dist}(\mathsf{X}_t, \mathsf{X}_s) &\le \sum_{k\ge n} \mathrm{dist}(\mathsf{X}_{t^{k+1}}, \mathsf{X}_{t^k}) + [\mathrm{dist}(\mathsf{X}_{t^n}, \mathsf{X}_{s^n})] + \sum_{k\ge n} \mathrm{dist}(\mathsf{X}_{s^{k+1}}, \mathsf{X}_{s^k}) \\
&\le \mathrm{dist}(\mathsf{X}_{t^n}, \mathsf{X}_{s^n}) + 2\sum_{k\ge n+1} \sigma_k, \quad \text{where} \quad \sigma_k := \sup_{(t,s)\in\Delta_k} \mathrm{dist}(\mathsf{X}_t, \mathsf{X}_s) \\
&\le 2\sum_{k\ge n} \sigma_k, \quad \text{provided that } \|t-s\|_\infty < (2T)2^{-n} \text{ so that } \|t^n - s^n\|_\infty \le 2(2T)2^{-n}.
\end{aligned}$$

Applying the above bound to all pairs $(t, s) \in D \times D$:

$$\begin{aligned}
\sup_{t,s\in D, t\neq s} \frac{\mathrm{dist}(\mathsf{X}_t, \mathsf{X}_s)}{\|t-s\|_\infty^\gamma} &= \sup\Big\{\frac{\mathrm{dist}(\mathsf{X}_t, \mathsf{X}_s)}{2^{-(n+1)\gamma}} : n \ge 0, t, s \in D, (2T)2^{-n-1} \le \|s-t\|_\infty < (2T)2^{-n}\Big\} \\
&\le \sup_{n\ge 0}\Big(2 \cdot 2^{(n+1)\gamma}\sum_{k\ge n}\sigma_k\Big) = 2^{1+\gamma}\sup_{n\ge 0}\sum_{k\ge n} 2^{n\gamma}\sigma_k \le 2^{1+\gamma}\sum_{k=0}^\infty 2^{k\gamma}\sigma_k.
\end{aligned}$$

Since $\mathbb{E}\sigma_k^\alpha \le \sum_{(t,s)\in\Delta_k} \mathbb{E}[\mathrm{dist}^\alpha(\mathsf{X}_t, \mathsf{X}_s)]$ while $|\Delta_k| \le 3^m 2^{mk}$, we have for $\alpha \ge 1$:

$$\begin{aligned}
\Bigg[\mathbb{E}\Big(\sup_{t,s\in D, t\neq s} \frac{\mathrm{dist}(\mathsf{X}_t, \mathsf{X}_s)}{\|t-s\|_\infty^\gamma}\Big)^\alpha\Bigg]^{1/\alpha} &\le 2^{1+\gamma}\sum_{k=0}^\infty 2^{k\gamma}3^{m/\alpha}2^{mk/\alpha}\mathsf{L}^{1/\alpha}[(2T)2^{-k}]^{(m+\beta)/\alpha} \\
&= 2^{1+\gamma}3^{m/\alpha}\mathsf{L}^{1/\alpha}(2T)^{(m+\beta)/\alpha} \cdot \sum_{k=0}^\infty \big(\tfrac{1}{2^{\beta/\alpha-\gamma}}\big)^k < \infty.
\end{aligned} \tag{1.4}$$

A similar result holds for $\alpha \in (0, 1]$: we simply do not take the $1/\alpha$ power above.

Letting $T = 1, 2, \ldots$ we conclude that on a countable dense set $\mathcal{D} = \cup_T D_T$, for almost all ω,

$$\forall T, \forall t, s \in D_T, \quad \mathrm{dist}(\mathsf{X}_t, \mathsf{X}_s) \le c_T(\omega)\|t-s\|_\infty^\gamma. \tag{1.5}$$

Taking limit we obtain an extension $\tilde{\mathsf{X}}_t$ such that (1.5) holds for all $t, s \in [-T, T)^m$, while

$$\mathbb{E}[\mathrm{dist}^\alpha(\mathsf{X}_t, \tilde{\mathsf{X}}_t)] = \mathbb{E}\Big[\lim_{s\in\mathcal{D}, s\to t} \mathrm{dist}^\alpha(\mathsf{X}_t, \mathsf{X}_s)\Big] \le \liminf_{s\in\mathcal{D}, s\to t} \mathbb{E}[\mathrm{dist}^\alpha(\mathsf{X}_t, \mathsf{X}_s)] \le \liminf_{s\in\mathcal{D}, s\to t} \mathsf{L}\|t-s\|_\infty^{m+\beta} = 0,$$

i.e., $\tilde{\mathsf{X}}_t$ is a modification of X_t that is locally Hölder continuous of order γ, and (1.3) follows from (1.4). ■

Schilling, R. L. (2021). "Brownian Motion: A Guide to Random Processes and Stochastic Calculus". 3rd. De Gruyter.

Definition 1.26: Brownian motion: Kolmogorov's construction

A stochastic process $\{\mathsf{X}_t : t \in \mathcal{T}_+\}$ is called Brownian motion if

- initialization: $\mathsf{X}_0 \equiv 0$;
- independent increment: $\forall n, \forall 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, $\mathsf{X}_{t_1} - \mathsf{X}_{t_0}, \ldots, \mathsf{X}_{t_n} - \mathsf{X}_{t_{n-1}}$ are independent;
- stationary increment: $\forall s \leq t$, $\mathsf{X}_t - \mathsf{X}_s \simeq \mathsf{X}_{t-s} - \mathsf{X}_0$;
- Gaussian: $\mathsf{X}_t \simeq \mathcal{N}(0, t)$;
- continuity: for almost all ω, $t \mapsto \mathsf{X}_t(\omega)$ is continuous.

In particular, Brownian motion is a Gaussian process with continuous path and covariance kernel

$$\kappa(s,t) := \mathbb{E}(\mathsf{X}_s \mathsf{X}_t) = s \wedge t, \ \forall s, t \in \mathcal{T}_+.$$

(Recall that X_t is a Gaussian process if any finite section $\mathsf{X}_{t_1}, \mathsf{X}_{t_2}, \ldots, \mathsf{X}_{t_n}$ is (jointly) Gaussian.) It follows from the Kolmogorov extension theorem that a stochastic process X_t that satisfies all conditions above except continuity exists. For any k,

$$\mathbb{E}|\mathsf{X}_s - \mathsf{X}_t|^{2k} = \mathbb{E}|\sqrt{t-s} \cdot \mathsf{X}_1|^{2k} = |t-s|^k \cdot \mathbb{E}|\mathsf{X}_1|^k.$$

Thus, identifying $\alpha = 2k, m = 1, \beta = k - m = k - 1$ in Theorem 1.25, we obtain a modification of X_t that is locally Hölder continuous of order $\gamma < \frac{k-1}{2k} \to \frac{1}{2}$ (Wiener 1930).

Wiener, N. (1930). "Generalized harmonic analysis". *Acta Mathematica*, vol. 55, pp. 117–258.

Theorem 1.27: Nondifferentiability of Brownian motion (Paley et al. 1933)

Brownian motion is nowhere Hölder continuous of order $\gamma > \frac{1}{2}$.

Proof: We follow the simple proof of Dvoretzky et al. (1961). We need only consider $t \in [0, n)$.

Suppose X_t is Hölder continuous of order γ at $t = s$, meaning

$$\exists \delta > 0, \quad \exists \mathsf{L} > 0, \quad \forall t \in \mathbb{B}(s, \delta), \quad |\mathsf{X}_t - \mathsf{X}_s| \leq \mathsf{L}|t - s|^\gamma. \tag{1.6}$$

Consider the grid $\{\frac{i}{k} : i = 1, \ldots, nk\}$ for sufficiently large k. There exists a *smallest* index $i = i(k)$ such that

$$s \leq \tfrac{i}{k}, \quad \tfrac{i}{k}, \tfrac{i+1}{k}, \cdots, \tfrac{i+p}{k} \in \mathbb{B}(s, \delta), \ p \in \mathbb{N} \text{ to be chosen later.}$$

Thus, for $l = i+1, \ldots, i+p$,

$$|\mathsf{X}_{l/k} - \mathsf{X}_{(l-1)/k}| \leq |\mathsf{X}_{l/k} - \mathsf{X}_s| + |\mathsf{X}_s - \mathsf{X}_{(l-1)/k}| \leq \mathsf{L}[|\tfrac{l}{k} - s|^\gamma + |\tfrac{l-1}{k} - s|^\gamma] \leq \tfrac{2\mathsf{L}}{k^\gamma}(p+1)^\gamma.$$

Consider the set $C_K^{\mathsf{L}} := \cap_{k=K}^\infty \cup_{i=1}^{kn} \cap_{l=i+1}^{i+p} [|\mathsf{X}_{l/k} - \mathsf{X}_{(l-1)/k}| \leq \frac{2\mathsf{L}}{k^\gamma}(p+1)^\gamma]$ and $k \geq K$, we have

$$\begin{aligned}
\Pr(C_K^{\mathsf{L}}) &\leq \Pr\left(\cup_{i=1}^{kn} \cap_{l=i+1}^{i+p} [|\mathsf{X}_{l/k} - \mathsf{X}_{(l-1)/k}| \leq \tfrac{2\mathsf{L}}{k^\gamma}(p+1)^\gamma]\right) \\
&\leq \sum_{i=1}^{kn} \Pr\left(\cap_{l=i+1}^{i+p} [|\mathsf{X}_{l/k} - \mathsf{X}_{(l-1)/k}| \leq \tfrac{2\mathsf{L}}{k^\gamma}(p+1)^\gamma]\right) \\
&\leq \sum_{i=1}^{kn} \prod_{l=i+1}^{i+p} \Pr[|\mathsf{X}_{l/k} - \mathsf{X}_{(l-1)/k}| \leq \tfrac{2\mathsf{L}}{k^\gamma}(p+1)^\gamma]
\end{aligned}$$

$$= kn[\Pr(|\mathsf{X}_{1/k}| \le \tfrac{2\mathsf{L}}{k^\gamma}(p+1)^\gamma)]^p = kn[\Pr(|\mathsf{X}_1| \le \tfrac{2\mathsf{L}}{k^{\gamma-1/2}}(p+1)^\gamma)]^p$$
$$\le k^{1+p/2-p\gamma} nc(p+1)^{\gamma p},$$

which tends to 0 when $k \to \infty$ and $(\gamma - \frac{1}{2})p > 1$. Therefore, for $\gamma > \frac{1}{2}$, we can choose p so that

$$\Pr(\cup_n \cup_K \cup_{\mathsf{L}} C_K^{\mathsf{L}}) = 0.$$

Since (1.6) implies $\cup_n \cup_K \cup_{\mathsf{L}} C_K^{\mathsf{L}}$, we conclude that (1.6) holds nowhere. ■

In particular, the sample path of Brownian motion is of infinite variation over any (nonempty) interval. With a bit more work, it can be proved that (see Schilling 2021, Theorem 10.6, p.172)

$$\Pr\left(\limsup_{h\to 0} \frac{\sup_{0\le t\le 1-h} |\mathsf{X}_{t+h} - \mathsf{X}_t|}{\sqrt{2h|\log h|}} = 1\right) = 1,$$

which implies that Brownian motion is not Hölder continuous of order $\frac{1}{2}$ (at some point t).

Paley, R. E. A. C., N. Wiener, and A. Zygmund (1933). “Notes on random functions”. *Mathematische Zeitschrift*, vol. 37, pp. 647–668.

Dvoretzky, A., P. Erdös, and S. Kakutani (1961). “Nonincrease Everywhere of the Brownian Motion Process”. In: *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*. Vol. 4. 2, pp. 103–116.

Schilling, R. L. (2021). “Brownian Motion: A Guide to Random Processes and Stochastic Calculus”. 3rd. De Gruyter.

Definition 1.28: Brownian bridge

A stochastic process $\{\mathsf{Y}_t : t \in [0,1]\}$ is called a Brownian bridge if

- initialization: $\mathsf{Y}_0 = \mathsf{Y}_1 \equiv 0$;
- independent increment: $\forall n, \forall 0 \le t_0 \le t_1 \le \cdots \le t_n \le 1$, $\mathsf{Y}_{t_1} - \mathsf{Y}_{t_0}, \ldots, \mathsf{Y}_{t_n} - \mathsf{Y}_{t_{n-1}}$ are independent;
- stationary increment: $\forall 0 \le s \le t \le 1$, $\mathsf{Y}_t - \mathsf{Y}_s \simeq \mathsf{Y}_{t-s} - \mathsf{Y}_0$;
- Gaussian: $\mathsf{Y}_t \simeq \mathcal{N}(0, t(1-t))$;
- continuity: for almost all ω, $t \mapsto \mathsf{Y}_t(\omega)$ is continuous.

In particular, a Brownian bridge is a Gaussian process with continuous path and covariance kernel

$$\kappa(s,t) := s \wedge t - st, \ \forall s,t \in [0,1].$$

Exercise 1.29: Brownian motion and Brownian bridge

Verify the following:

- If X_t is a Brownian motion, then $\mathsf{Y}_t = \mathsf{X}_t - t\mathsf{X}_1$ is a Brownian bridge.
- If Y_t is a Brownian bridge, then $\mathsf{X}_t = \mathsf{Y}_t + t\mathsf{Z}$, where $\mathsf{Z} \simeq \mathcal{N}(0,1) \perp \{\mathsf{Y}_t\}$, is a Brownian motion on $[0,1]$.
- If X_t is a Brownian motion, so is $\frac{1}{\sqrt{c}}\mathsf{X}_{ct}$ for any $c > 0$.
- If X_t is a Brownian motion on $[0,\infty)$, so is $t\mathsf{X}_{1/t}$. What about $\frac{1}{t}\mathsf{X}_t$?
- If Y_t is a Brownian bridge, so is Y_{1-t}.
- If X_t is a Brownian motion, then $(1-t)\mathsf{X}_{t/(1-t)}$ and $t\mathsf{X}_{(1-t)/t}$ are Brownian bridges.

- If Y_t is a Brownian bridge, then $(1+t)\mathsf{Y}_{t/(1+t)}$ and $(1+t)\mathsf{Y}_{1/(1+t)}$ are Brownian motions.
- If $\mathsf{X}_t^n, n \in \mathbb{N}$ are *independent* Brownian motions on $[0,1]$, then

$$\mathsf{X}_t := \mathsf{X}_{t-\lfloor t \rfloor}^{\lfloor t \rfloor + 1} + \sum_{n=1}^{\lfloor t \rfloor} \mathsf{X}_1^n$$

is a Brownian motion on $[0, \infty)$.

Example 1.30: Brownian motion through random Gaussian series (Itô and Nisio 1968)

Consider the Hilbert space $L^2(\mathrm{d}t) := L^2([0,1], \mathrm{d}t)$ with scalar product

$$\langle f, g \rangle := \int_0^1 f(t) g(t) \,\mathrm{d}t.$$

Let $\{\varphi_n : n \in \mathbb{N}\}$ be a complete orthogonal system of $L^2(\mathrm{d}t)$ and G_n be a sequence of i.i.d. standard normal random variables. Define the random Gaussian series

$$\mathsf{X}_t^n := \sum_{i=1}^n \mathsf{G}_i \left\langle \mathbf{1}_{[0,t)}, \varphi_i \right\rangle = \sum_{i=1}^n \mathsf{G}_i \int_0^t \varphi_i(s) \,\mathrm{d}s.$$

Fix $t \in [0,1]$, clearly

$$\begin{aligned}
\mathbb{E}\|\mathsf{X}_t^n - \mathsf{X}_t^m\|_2^2 = \mathbb{E}\left\| \sum_{i=m\wedge n+1}^{m\vee n} \mathsf{G}_i \left\langle \mathbf{1}_{[0,t)}, \varphi_i \right\rangle \right\|_2^2 &= \sum_{i=m\wedge n+1}^{m\vee n} \sum_{j=m\wedge n+1}^{m\vee n} \mathbb{E}(\mathsf{G}_i \mathsf{G}_j) \left\langle \mathbf{1}_{[0,t)}, \varphi_i \right\rangle \left\langle \mathbf{1}_{[0,t)}, \varphi_j \right\rangle \\
&= \sum_{i=m\wedge n+1}^{m\vee n} \left\langle \mathbf{1}_{[0,t)}, \varphi_i \right\rangle^2 \to 0, \text{ as } m, n \to \infty,
\end{aligned}$$

since $t = \|\mathbf{1}_{[0,t)}\|_2^2 = \sum_{i=1}^\infty \left\langle \mathbf{1}_{[0,t)}, \varphi_i \right\rangle^2$. Thus, $\mathsf{X}_t^n \to \mathsf{X}_t$ in mean square. It is clear $\mathsf{X}_0 \equiv 0$ and X_t has stationary increments (inherited from X_t^n). Each X_t^n is a linear combination of i.i.d. standard normal, with its coefficients approaching the following correlation:

$$\sum_{i=1}^n \left\langle \mathbf{1}_{[t_{j-1}, t_j)}, \varphi_i \right\rangle \left\langle \mathbf{1}_{[t_{k-1}, t_k)}, \varphi_i \right\rangle \to \left\langle \mathbf{1}_{[t_{j-1}, t_j)}, \mathbf{1}_{[t_{k-1}, t_k)} \right\rangle = [\![j \neq k]\!] \cdot (t_j - t_{j-1}).$$

Thus, $\mathsf{X}_t \simeq \mathcal{N}(0, t)$ and X_t has independent increments. (A more rigorous argument, through the characteristic function, can be found in e.g. Schilling (2021, p. 21).)

Clearly, for each n, X_t^n has continuous path. However, the (pointwise) limit of continuous functions may not be continuous. Thus, we have extra work to do to show the limit X_t is indeed a Brownian motion (i.e., with continuous path).

Itô, K. and M. Nisio (1968). "On the convergence of sums of independent Banach space valued random variables". *Osaka Journal of Mathematics*, vol. 5, no. 1, pp. 35–48.
Schilling, R. L. (2021). "Brownian Motion: A Guide to Random Processes and Stochastic Calculus". 3rd. De Gruyter.

Example 1.31: Brownian motion: Lévy's construction (Lévy 1940)

TBD

Lévy, P. (1940). "Le Mouvement Brownien Plan". *American Journal of Mathematics*, vol. 62, no. 1, pp. 487–550.

Example 1.32: Brownian motion: Ciesielski's construction (Ciesielski 1966)

TBD

Ciesielski, Z. (1966). "Lectures on Brownian motion, heat conduction and potential theory". Aarhus University.

Example 1.33: Brownian motion: Wiener's construction (Wiener 1923, 1924)

TBD

Wiener, N. (1923). "Differential-Space". *Journal of Mathematics and Physics*, vol. 2, no. 1-4, pp. 131–174.
— (1924). "Un problème de probabilité dénombrables". *Bulletin de la Société Mathématique de France*, vol. 52, pp. 569–578.

Example 1.34: Brownian motion: Donsker's construction (Donsker 1951, 1952)

TBD

Donsker, M. D. (1951). "An invariance principle for certain probability limit theorems". In: *Memoirs of the American Mathematical Society*. Vol. 6. Four papers on probability, pp. 1–12.
— (1952). "Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems". *Annals of Mathematical Statistics*, vol. 23, no. 2, pp. 277–281.

Example 1.35: Kolmogrov-Smirnov test (Kolmogorov 1933)

TBD

Kolmogorov, A. N. (1933). "Sulla determinazione empirica di una legge di distribuzione". *Giornale dell'Istituto italiano degli attuari*, vol. 4. English translation at `https://doi.org/10.1007/978-94-011-2260-3_15`, pp. 83–91.

Definition 1.36: Lévy process

A stochastic process $\{\mathsf{X}_t\}$ is called a Lévy process if

- initialization: $\mathsf{X}_0 \equiv 0$
- independent increment: $\forall n$, $\forall t_0 \leq t_1 \leq \cdots \leq t_n$, $\mathsf{X}_{t_1} - \mathsf{X}_{t_0}, \ldots, \mathsf{X}_{t_n} - \mathsf{X}_{t_{n-1}}$ are independent
- stationary increment: $\forall s \leq t$, $\mathsf{X}_t - \mathsf{X}_s \simeq \mathsf{X}_{t-s} - \mathsf{X}_0$
- continuity in probability: $\lim_{t\downarrow 0} \mathsf{X}_t \to \mathsf{X}_0 = 0$ (i.p.)

See Applebaum (2009) and Sato (2013).

Applebaum, D. (2009). "Lévy Processes and Stochastic Calculus". 2nd. Cambridge University Press.
Sato, K. (2013). "Lévy Processes and Infinitely Divisible Distributions". 2nd. Cambridge University Press.

Definition 1.37: Poisson process

A stochastic process N_t is called a Poisson process if

- initialization: $\mathsf{N}_0 \equiv 0$;
- independent increment: $\forall n, \forall t_0 \leq t_1 \leq \cdots \leq t_n$, $\mathsf{N}_{t_1} - \mathsf{N}_{t_0}, \ldots, \mathsf{N}_{t_n} - \mathsf{N}_{t_{n-1}}$ are independent;
- stationary increment: $\forall s \leq t$, $\mathsf{N}_t - \mathsf{N}_s \simeq \mathsf{N}_{t-s} - \mathsf{N}_0$;

- Poisson: $\mathsf{N}_t \simeq \mathrm{Pois}(\lambda t)$;
- right continuity: for almost all ω, $t \mapsto \mathsf{N}_t(\omega)$ is right continuous with left limit.

Right continuity allows us to conclude that N_t has increasing sample path that are natural number valued , whereas the existence of left limit implies that over a finite time there are only finitely many jumps .

In fact, the jumps are always of size 1. Indeed, fix s. Then, since N_t is increasing, for all large n,

$$\begin{aligned}\Pr(\exists t \in [0,s] : \mathsf{N}_t - \mathsf{N}_{t-} \geq 2) &\leq \Pr(\exists k_n \in [1, 2^n] : \mathsf{N}_{(k_n+1)s/2^n} - \mathsf{N}_{k_n s/2^n} \geq 2) \\ &= [\Pr(\mathsf{N}_{s/2^n} \geq 2)]^{2^n} \\ &= [1 - \exp(-\lambda s/2^n) - \lambda s/2^n \exp(-\lambda s/2^n)]^{2^n},\end{aligned}$$

which tends to 0 as $n \to \infty$. Taking union over each rational s we confirm that jump of size at least 2 occurs with 0 probability.