$p(s,x,t,y) \geq 0$, $\quad \int p(s,x,t,y)\,dy = 1$

Chapman-Kolmogorov : $\quad p(s,x,t,y) = \int p(s,x,\tau,z)\, p(\tau,z,t,y)\,dz$

$$[T_{s,t}(f)](x) \triangleq \int p(s,x,t,y) f(y)\,dy$$

$$f(y) = \mathbb{1}_B(y), \quad T_{s,t} f(x) = \int p(s,x,t,y)\mathbb{1}_B(y)\,dy$$

$$= P_r(s,x,t,B)$$

$$T_{s,t} = T_{s,\tau} T_{\tau,t}$$

$$\|$$

$$\int p(s,x,t,y) f(y)\,dy = \int\int p(s,x,\tau,z)\, \underline{p(\tau,z,t,y) f(y)}\, dz\, \underline{dy}$$

$$= \int p(s,x,\tau,z)\; T_{\tau,t} f(z)\,dz$$

$$= T_{s,\tau}\; T_{\tau,t} f(x)$$

$$L_t = \lim_{h \downarrow 0} \frac{T_{t,t+h} - T_{t,t}}{h} \qquad T_{t,t} = id$$

generator of Markov process

$$L_t^* p = -\nabla \cdot (p\vec{b}) + \nabla^2 \cdot (\tfrac{1}{2} pA)$$

$$L_t f(x) = \frac{1}{2} A(t,x) \cdot \underbrace{\nabla^2 f(x)}_{C} + \vec{b}(t,x) \cdot \nabla f(x)$$

① $\lim_{h \downarrow 0} p(t, x, t+h, B_\varepsilon(x)) = 0$

② $\lim_{h \downarrow 0} \int_{\cancel{B_\varepsilon(x)}} (\vec{y} - \vec{x})\, p(t, x, t+h, y)\, dy = \vec{b}(t,x)$

③ $\lim_{h \downarrow 0} \int \underbrace{(\vec{y} - \vec{x})(\vec{y} - \vec{x})^T}\, \underline{p(t, x, t+h, y)}\, dy = A(t,x) \succeq 0$

① $P_r\left(\|X_{t+h} - X_t\| \geq \varepsilon \,\middle|\, X_t \right) \to 0$ as $h \downarrow 0$

② $$\frac{E[X_{t+h} - X_t \mid X_t = x]}{h} \to \vec{b}(t,x)$$

③ $$\frac{E[(X_{t+h} - X_t)(X_{t+h} - X_t)^T \mid X_t = x]}{h} \to A(t,x)$$

$$\langle L_t f ; p \rangle = \langle f ; L_t^* p \rangle$$

$$\langle \tfrac{1}{2} A \cdot \nabla^2 f + \vec{b} \cdot \nabla f ;\ p \rangle = \langle \cancel{\nabla^2} f ; \nabla^2 \tfrac{1}{2} pA \rangle \overset{-}{+} \langle \cancel{\nabla} f ; \nabla \cdot p\vec{b} \rangle$$

$$\int f' \cdot g\, dx = gf \Big| - \int g' f\, dx \qquad \langle f' ; g \rangle = -\langle f ; g' \rangle$$

$$\underbrace{\nabla \cdot p\vec{b}}_{\mathrm{div}(p\vec{b})} = \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \\ \vdots \\ \partial_{y_d} \end{pmatrix} \cdot \begin{pmatrix} pb_1 \\ pb_2 \\ \vdots \\ pb_d \end{pmatrix} = \sum_{i=1}^{d} \partial_{y_i}(pb_i)$$

$$\nabla p = \begin{pmatrix} \partial_1 p \\ \partial_2 p \\ \vdots \\ \partial_d p \end{pmatrix}$$

$$L_t = \lim_{h \downarrow 0} \frac{T_{t,t+h} - T_{t,t}}{h}$$

FPK:

$$\partial_t p(s,x,t,y) = L_t^* p(s,x,t,y) \quad \text{(forward)}$$

$$-\partial_s p(s,x,t,y) = L_s p(s,x,t,y) \quad \text{(backward)}$$

Derivation: $\boxed{\langle f, \partial_t p(s,x,t,\cdot)\rangle}$

$$\parallel$$

$$\partial_t \langle f(y); p(\underline{s,x}, t, \underline{y})\rangle = \partial_t T_{s,t} f(x) = \lim_{h \downarrow 0} \frac{T_{s,t+h} - T_{s,t}}{h} f(x)$$

$$= \lim_{h \downarrow 0} T_{s,t} \frac{T_{t,t+h} - T_{t,t}}{h} f(x) = T_{s,t} L_t f(x)$$

$$= \langle L_t f, p(s,x,t,\cdot)\rangle = \boxed{\langle f; L_t^* p(s,x,t,\cdot)\rangle}$$

$$\frac{T_{s-h,s} - T_{s,s}}{h}$$

FPK for SDE: $dX_t = \vec{b}_t(X_t)\,dt + G_t(X_t)\,dB_t$

$$L_t f = \vec{b}_t \cdot \nabla f + \frac{1}{2}[G_t G_t^T] \cdot \nabla^2 f$$

$$\partial_t p(s,x,t,y) = L_t^* p = \boxed{-\nabla \cdot (p\vec{b}_t) + \frac{1}{2}\nabla^2 \cdot (p\,\overset{I}{\cancel{G_t G_t^T}})}$$

Derivation: take a function f

$$df(X_t) = \nabla f(X_t)\,dX_t + \frac{1}{2}\nabla^2 f(X_t) \cdot G_t G_t^T\,dt$$

$$= \Big[\underbrace{\vec{b}_t \cdot \nabla f + \frac{1}{2}\nabla^2 f \cdot G_t G_t^T}\Big]dt + \underbrace{\nabla f \cdot G_t \cdot dB_t}_{= 0}$$

$$\cancel{d}\langle f, \partial_t p(s,x,t,\cdot)\rangle = \langle \cancel{L_t} f, L_t^* p(s,x,t,\cdot)\rangle\,dt$$

Continuity equation $dX_t = \vec{b}_t(X_t)\,dt$

$$\partial_t p = -\nabla \cdot (pb)$$

Heat equation: $\boxed{\partial_t p = \Delta p = \sum_{i=1}^{d} \partial_i^2 p}$

$$\int_0^t dX_t = \int_0^t \sqrt{2}\,dB_t = \sqrt{2}B_t \sim N(0, 2t)$$

$$p(t,x) = p(t,\cdot)$$

$$p : t \longmapsto p(t,\cdot)$$

$$\frac{dp_t}{dt} = -\nabla f(p_t) = \Delta p_t$$

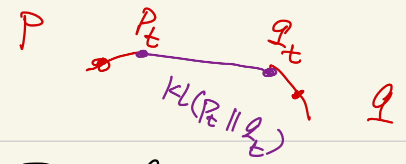
$$f(p) = \frac{1}{2}\|\nabla p\|_2^2 = \int \langle \nabla p(x), \nabla p(x)\rangle dx$$

$$\left.\frac{d\, f(p+\varepsilon\cdot q)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{d\,\frac{1}{2}\|\nabla(p+\varepsilon q)\|_2^2}{d\varepsilon}$$

$$= \frac{d\,\frac{1}{2}\int\langle \nabla p+\varepsilon\nabla q, \nabla p+\varepsilon\nabla q\rangle}{d\varepsilon}$$

$$= \int\langle \nabla p, \nabla q\rangle dx = \langle \nabla p ; \nabla q\rangle$$

$$= -\langle \Delta p ; q\rangle$$



score matching: two densities p, q

Fisher divergence: $F(p \| q) = \mathbb{E}_{x \sim \cancel{q}\, p} \| \underbrace{\nabla \log p(x)}_{\vec{S}_p(x)} - \underbrace{\nabla \log q(x)}_{\vec{S}_q(x)} \|^2$

$$p_t = p * N(0, 2t), \quad q_t = q * N(0, 2t)$$

$$\boxed{\frac{d}{dt} KL(p_t \| q_t)\Big|_{t=0} = -F(p_t \| q_t)\Big|_{t=0}}$$

$$\| \quad \int p_t \log \frac{p_t}{q_t}\, dx$$

$$\boxed{\partial_t p = \Delta p} \leftarrow N(0, 2t)$$

$$\partial_t [p * N(0,2t)] = p * \partial_t N(0,2t) \qquad \Delta (p * N(0,t)) = p * \Delta N(0,t)$$

$$\int \frac{dp_t}{dt} \cdot \log p_t / q_t \, dx + \cancel{\int p_t \cdot \frac{\partial_t p_t}{p_t} dx} - \int p_t \frac{\partial_t q_t}{q_t} dx$$

$$\int \Delta p_t \cdot \log p_t \, dx - \underbrace{\int \Delta p_t \log q_t \, dx}_{= \int p_t \langle \vec{S}_{p_t}, \vec{S}_{q_t} \rangle} - \int p_t \frac{\Delta q_t}{q_t} dx$$

$$\int \Delta p_t \cdot \log p_t \, dx = \langle \Delta p_t ; \log p_t \rangle = - \langle p_t \frac{\nabla p_t}{p_t} ; \nabla \log p_t \rangle = - \int p_t \| \vec{S}_{p_t} \|^2 dx$$

$$\int p_t \frac{\Delta q_t}{q_t} dx = \langle \frac{p_t}{q_t}, \Delta q_t \rangle = \langle \nabla \frac{p_t}{q_t}, \nabla q_t \rangle$$

$$-\int p_t \|\vec{S}_{p_t}\|_2^2 + \int p_t \langle \vec{S}_{p_t}, \vec{S}_{q_t} \rangle + \int \nabla\left(\frac{p_t}{q_t}\right) \cdot \nabla q_t$$

$$\Vert$$

$$\int \frac{\nabla p_t \cdot q_t - p_t \cdot \nabla q_t}{q_t^2} \cdot \nabla q_t$$

$$\Vert$$

$$\int \frac{\nabla p_t}{p_t} \cdot \frac{\nabla q_t}{q_t} \cdot p_t - p_t \cdot \|S_{q_t}\|^2$$

$$= \int p_t \cdot \langle S_{p_t}, S_{q_t} \rangle - p_t \|S_{q_t}\|^2$$

$$-\int p_t \|\vec{S}_{p_t}\|_2^2 + \int p_t \cdot 2 \langle \vec{S}_{p_t}, \vec{S}_{q_t} \rangle - \int p_t \cdot \|S_{q_t}\|^2$$

$$= -\int p_t \cdot \|\vec{S}_{p_t} - \vec{S}_{q_t}\|_2^2 \, dx$$

$$= -F(p_t \| q_t)$$

$N(0,I)$ data

$\wr \quad \wr$

$p, \; q$ find T s.t. $q = T^{-1}_{\#} p$

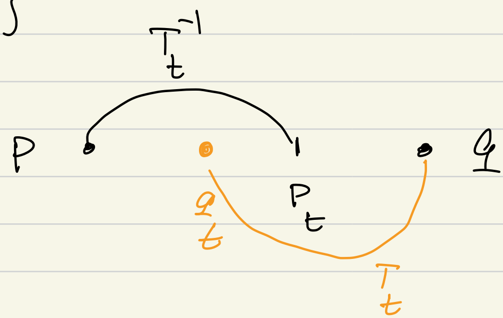
$x \sim p, \; T^{-1}(x) \sim q$

also $p = T_{\#} q$

build T through $\{T_t\}$

$p_t := (T_t^{-1})_{\#} p$

$q_t := (T_t)_{\#} q$



$p_t(x) = p(T_t x) \cdot |\det T_t'(x)|$

$$\overset{T_t x}{\uparrow} \qquad \overset{T_t x}{\uparrow}$$

$$p_t(x)\,dx = p(z)\,dz$$

$$\min_{T_t} KL(q \,\|\, p_t) = \int q \log q/p_t$$

$$\equiv -\int q \cdot \log p_t$$

$$\approx -\sum_{i=1}^{n} \log p_t(x_i)$$

$$KL(p_t \,\|\, q) = \int p_t \log p_t - p_t \log q$$