CS794/CO673: Optimization for Data Science

Lec 07: Mirror Descent

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Problem

Constrained minimization:

$$f_\star = \inf_{\mathbf{w} \in C \subseteq \mathsf{V}} f(\mathbf{w})$$

- f: convex and possibly nonsmooth
- C: convex constraint
- V: vector space that $\mathbf{w}$ lives in, e.g. $\mathbb{R}^d$ with Euclidean norm $\|\cdot\|_2$
- When f' is L-Lipschitz, (projected) gradient descent yields $\frac{\mathsf{L}\|\mathbf{w}_0 - \mathbf{w}\|_2^2}{2t}$
- When f is L-Lipschitz, (projected) subgradient yields $\frac{\mathsf{L}\|\mathbf{w}_0 - \mathbf{w}\|_2}{\sqrt{t}}$

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How We Measure Things Matters

$$\min_{\mathbf{w}\in\Delta} \sum_{j=1}^{d} f_j(w_j),$$

- Each univariate function $f_j : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz continuous.
- The sum $f : \mathbb{R}^d \to \mathbb{R}$ is $\sqrt{d}$-Lipschitz continuous w.r.t. the Euclidean norm:

$$\|f'\|_2^2 = \sum_j (f_j')^2 \le d.$$

- The diameter $\|\mathbf{w}_0 - \mathbf{w}\|_2 \le \sqrt{2}$.
- Applying subgradient we obtain a convergence rate of $\sqrt{\frac{2d}{t}}$
- But, we also have $\|f'\|_\infty = \max_j |f_j'| \le 1$
- The diameter $\|\mathbf{w}_0 - \mathbf{w}\|_1 = \|\mathbf{w}_0 - \mathbf{w}\|_\infty^\circ \le 2$
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What Makes Incremental Update Possible?

So far, all of our updates are of the following (additive) incremental form:

$$\mathbf{w} \leftarrow \mathbf{w} - \eta \cdot \mathbf{g},$$

which is so natural that we often forget what makes it even mathematically possible:

- The scalar multiplication of the step size η to $\mathbf{g}$
- The negation $-$
- And the addition of $\mathbf{w}$ with $-\eta \cdot \mathbf{g}$

These operations are possible because $\mathbf{w}$ and $\mathbf{g}$ are from the same vector space

- From now on $f'(\mathbf{w})$ lives in a dual space V^*
- Need a way to pull things back and forth: $\mathsf{J} : \mathsf{V} \to \mathsf{V}^*, \quad \mathsf{J}^{-1} : \mathsf{V}^* \to \mathsf{V}$
- With the Euclidean norm $\|\cdot\|_2$, we may simply take $\mathsf{J} = \mathsf{J}^* = \mathrm{Id}$

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Algorithm 1: Winnow

Input: $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{R}^{\mathsf{p} \times \mathsf{n}}$, threshold $\delta \geq 0$, step size $\eta > 0$, initialize $\mathbf{w} \in \mathrm{int}\Delta_{\mathsf{p}-1}$

Output: approximate solution $\mathbf{w}$

```
for t = 1, 2, ... do
    receive training example index I_t ∈ {1, ..., n}      // index I_t can be random
    if ⟨a_{I_t}, w⟩ ≤ δ then
        w ← w ⊙ exp(η a_{I_t})                            // update only when making a mistake
        w ← w / ||w||_1                                    // normalize
```

$$\ln \mathbf{w} \leftarrow \ln \mathbf{w} + \eta \cdot \mathbf{a}_{I_t}, \quad \text{where} \quad \mathsf{J}(\mathbf{w}) = \ln(\mathbf{w})$$

N. Littlestone (1988). "Learning quickly when irrelevant attributes abound: A new linear-threshold algorithm". *Machine Learning*, vol. 2, pp. 285–318.

Algorithm 2: Winnow

Input: $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{R}^{\mathsf{p}\times\mathsf{n}}$, threshold $\delta \geq 0$, step size $\eta > 0$, initialize $\mathbf{w} \in \mathrm{int}\Delta_{\mathsf{p}-1}$

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for $t = 1, 2, \ldots$ **do**
    receive training example index $I_t \in \{1, \ldots, \mathsf{n}\}$ // index I_t can be random
    if $\langle \mathbf{a}_{I_t}, \mathbf{w} \rangle \leq \delta$ **then**
        $\mathbf{w} \leftarrow \mathbf{w} \odot \exp(\eta \mathbf{a}_{I_t})$ // update only when making a mistake
        $\mathbf{w} \leftarrow \mathbf{w} / \|\mathbf{w}\|_1$ // normalize

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Online Prediction

- n experts, each of whom provides a prediction x_i, collectively as $\mathbf{x} \in \mathbb{R}^n$
- Form our own opinion by averaging $\hat{y} = \langle \mathbf{w}, \mathbf{x} \rangle$, $\mathbf{w} \in \Delta$
- Suffer a loss, say the square loss $\ell(\mathbf{w}; \mathbf{x}, y) = (y - \hat{y})^2$
- Repeat the game for $t = 1, \ldots, T$ rounds

$$\text{Regret} := \frac{1}{T} \sum_{t=1}^{T} (y_t - \hat{y}_t)^2 - \min_{\mathbf{w} \in \Delta} \frac{1}{T} \sum_{t=1}^{T} (y_t - \langle \mathbf{w}, \mathbf{x}_t \rangle)^2, \quad \text{where} \quad \hat{y}_t = \langle \mathbf{w}_t, \mathbf{x}_t \rangle .$$

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Exponentiated Gradient (EG)

$$\tilde{\mathbf{w}}_{t+1} = \mathbf{w}_t \odot \exp(-\eta_t \cdot \ell'(\hat{y}_t - y_t)\mathbf{x}_t)$$

$$\mathbf{w}_{t+1} = \frac{\tilde{\mathbf{w}}_{t+1}}{\langle \mathbf{1}, \tilde{\mathbf{w}}_{t+1} \rangle}$$

- Diminishing regret on the order of $O(\sqrt{\frac{\ln n}{T}})$, assuming $\|\mathbf{x}_t\|_\infty \leq 1$ and $y_t \in [0, 1]$
- No assumption on how the sequence $(\mathbf{x}_t, y_t)$ is generated; can even be adversarial
- Setting $\mathbf{w} = \mathbf{e}_i$: EG performs no worse than the best expert in hindsight for big T
- Can consult a large number of experts: dependence on n is only logarithmic
- Gradient descent achieves $O(\frac{1}{\sqrt{T}})$ under the assumption $\|\mathbf{x}_t\|_2 \leq 1$

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J. Kivinen and M. K. Warmuth (1997). "Exponentiated Gradient versus Gradient Descent for Linear Predictors". *Information and Computation*, vol. 132, no. 1, pp. 1–63.

Exponentiated Gradient (EG)

$$\tilde{\mathbf{w}}_{t+1} = \mathbf{w}_t \odot \exp(-\eta_t \cdot \ell'(\hat{y}_t - y_t)\mathbf{x}_t)$$

$$\mathbf{w}_{t+1} = \frac{\tilde{\mathbf{w}}_{t+1}}{\langle \mathbf{1}, \tilde{\mathbf{w}}_{t+1} \rangle}$$

- Diminishing regret on the order of $O(\sqrt{\frac{\ln n}{T}})$, assuming $\|\mathbf{x}_t\|_\infty \leq 1$ and $y_t \in [0, 1]$
- No assumption on how the sequence $(\mathbf{x}_t, y_t)$ is generated; can even be adversarial
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Two Choices

- We have a mismatch between $\mathbf{w} \in \mathsf{V}$ and $f'(\mathbf{w}) \in \mathsf{V}^*$
- We use a duality (mirror) map $\mathsf{J} : \mathsf{V} \to \mathsf{V}^*, \quad \mathsf{J}^{-1} : \mathsf{V}^* \to \mathsf{V}$

1. Update in the gradient space V^* and pull the update back to the input space V:

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Legendre function

We call a continuous convex function h Legendre if

- Its domain has nonempty interior, i.e., $\mathrm{int}(\mathrm{dom}\, h) \neq \emptyset$
- h is differentiable on $\mathrm{int}(\mathrm{dom}\, h)$
- $\|h'(\mathbf{w})\| \to \infty$ as $\mathbf{w} \to \partial\, \mathrm{dom}\, h$
- h is strictly convex on $\mathrm{int}(\mathrm{dom}\, h)$

Theorem: $\mathsf{J} = h'$

h' is a topological isomorphism, i.e. it is continuous and its inverse is also continuous.

Below, we will choose a norm $\|\cdot\|$ and a Legendre function h that is 1-strongly convex w.r.t. $\|\cdot\|$, i.e.

$$\mathrm{D}_h(\mathbf{w}, \mathbf{z}) := h(\mathbf{w}) - h(\mathbf{z}) - \langle \mathbf{w} - \mathbf{z}; \nabla h(\mathbf{z}) \rangle \geq \frac{1}{2}\|\mathbf{w} - \mathbf{z}\|^2.$$

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Example: (Squared) Euclidean distance

Let $h(\mathbf{w}) = \frac{1}{2}\|\mathbf{w}\|_2^2$. Then, h is Legendre and its induced Bregman divergence $\mathsf{D}_h(\mathbf{w}, \mathbf{z}) = \frac{1}{2}\|\mathbf{w} - \mathbf{z}\|_2^2$ is the (square) Euclidean distance. We have $\mathsf{J}(\mathbf{w}) = h'(\mathbf{w}) = \mathbf{w}$ and of course $\mathsf{J}^{-1} = \mathsf{J}$.

Example: KL and Pinsker

Consider the KL function $h(\mathbf{w}) = \sum_j w_j \ln w_j - w_j$, where $0 \ln 0 := 0$. It is Legendre and its induced Bregman divergence D_h is known as the KL divergence:

$$\forall \mathbf{w}, \mathbf{z} \geq \mathbf{0}, \quad \mathsf{KL}(\mathbf{w}, \mathbf{z}) = \sum_j w_j \ln \frac{w_j}{z_j} - w_j + z_j,$$

which is 1-strongly convex w.r.t. the ℓ_1 norm (restricted to the simplex):

$$\forall \mathbf{w}, \mathbf{z} \in \Delta, \quad \mathsf{KL}(\mathbf{w}, \mathbf{z}) \geq \tfrac{1}{2}\|\mathbf{w} - \mathbf{z}\|_1^2,$$

also known as Pinsker's inequality in information theory.

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Algorithm 3: Mirror descent

Input: $\mathbf{w}_0 \in C$, Legendre function h

for $t = 0, 1, \ldots$ **do**
  compute (sub)gradient $f'(\mathbf{w}_t)$
  choose step size $\eta_t > 0$
  $h'(\mathbf{z}_{t+1}) = h'(\mathbf{w}_t) - \eta_t \cdot f'(\mathbf{w}_t)$ `// update in the gradient space`
  $\mathbf{w}_{t+1} \leftarrow \operatorname*{argmin}_{\mathbf{w} \in C} \mathsf{D}_h(\mathbf{w}, \mathbf{z}_{t+1})$ `// projecting back to the constraint`

Key insight (note the similarity as before):

$$\begin{aligned}\mathbf{w}_{t+1} = \operatorname*{argmin}_{\mathbf{w} \in C} \ & f(\mathbf{w}_t) + \langle \mathbf{w} - \mathbf{w}_t; f'(\mathbf{w}_t) \rangle + \tfrac{1}{\eta_t} \mathsf{D}_h(\mathbf{w}, \mathbf{w}_t) \\ \geq \ & f(\mathbf{w}_t) + \langle \mathbf{w} - \mathbf{w}_t; f'(\mathbf{w}_t) \rangle + \tfrac{1}{2\eta_t} \|\mathbf{w} - \mathbf{w}_t\|^2 \\ = \operatorname*{argmin}_{\mathbf{w} \in C} \ & \mathsf{D}_h(\mathbf{w}, \mathbf{z}_{t+1}), \quad \text{where} \quad h'(\mathbf{z}_{t+1}) = h'(\mathbf{w}_t) - \eta_t \cdot f'(\mathbf{w}_t),\end{aligned}$$

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$$\geq \; f(\mathbf{w}_t) + \langle \mathbf{w} - \mathbf{w}_t; f'(\mathbf{w}_t) \rangle + \frac{1}{2\eta_t} \|\mathbf{w} - \mathbf{w}_t\|^2$$

$$= \operatorname*{argmin}_{\mathbf{w} \in C} \; \mathsf{D}_h(\mathbf{w}, \mathbf{z}_{t+1}), \quad \text{where} \quad h'(\mathbf{z}_{t+1}) = h'(\mathbf{w}_t) - \eta_t \cdot f'(\mathbf{w}_t),$$

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EG $\in$ MD

- Let $C = \Delta$ and h be KL
- We compute the Bregman projection:

$$\underset{\mathbf{w}\in\Delta}{\text{argmin}}\ \mathsf{KL}(\mathbf{w},\mathbf{z}) = \sum_j w_j \log\frac{w_j}{z_j} - w_j + z_j$$
$$= \sum_j w_j \log\frac{w_j}{z_j/\langle\mathbf{1},\mathbf{z}\rangle} - \log\langle\mathbf{1},\mathbf{z}\rangle - 1 + \langle\mathbf{1},\mathbf{z}\rangle$$
$$\equiv \mathsf{KL}(\mathbf{w}, \tfrac{\mathbf{z}}{\langle\mathbf{1},\mathbf{z}\rangle})$$

- $h'(\mathbf{w}) = \ln\mathbf{w}$ while $(h')^{-1}(\mathbf{g}) = \exp(\mathbf{g})$, all component-wise
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$$\mathbf{z}_{t+1} = (h')^{-1}(h'(\mathbf{w}_t) - \eta_t\cdot f'(\mathbf{w}_t)) = \mathbf{w}_t \odot \exp(-\eta_t f'(\mathbf{w}_t)), \quad \mathbf{w}_{t+1} = \tfrac{\mathbf{z}_{t+1}}{\langle\mathbf{1},\mathbf{z}_{t+1}\rangle}$$

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choose a Legendre function h that matches the "geometry" (i.e. norm) of the constraint set C, so that projection is trivial

Theorem: convergence of mirror descent for smooth function

Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex and L-smooth (w.r.t. some norm $\|\cdot\|$), $C \subseteq \mathbb{R}^d$ be closed convex, and η_t is chosen suitably, then for all $\mathbf{w} \in C$ and $t \geq 1$, the mirror descent iterates $\{\mathbf{w}_t\} \subseteq C$ satisfy:

$$f(\mathbf{w}_t) \leq f(\mathbf{w}) + \frac{\mathsf{D}_h(\mathbf{w}, \mathbf{w}_0)}{t\bar{\eta}_t}, \quad \text{where} \quad \bar{\eta}_t := \frac{1}{t}\sum_{s=0}^{t-1} \eta_s,$$

$\mathsf{D}_h(\mathbf{w}, \mathbf{w}_0) \geq \frac{1}{2}\|\mathbf{w} - \mathbf{w}_0\|^2$ for some 1-strongly convex Legendre function h.

- Again, the rate of convergence does not depend on d, the dimension!
- Proof is literally the same as that of projected gradient
- Choosing $\eta_t \equiv 1/\mathsf{L}$ we obtain $f(\mathbf{w}_t) - f(\mathbf{w}) \leq \frac{\mathsf{L}\mathsf{D}_h(\mathbf{w},\mathbf{w}_0)}{t}$
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\begin{aligned}
f(\mathbf{w}_{t+1}) &\le f(\mathbf{w}_t) + \langle \mathbf{w}_{t+1} - \mathbf{w}_t; f'(\mathbf{w}_t)\rangle + \tfrac{1}{\eta_t}\mathsf{D}_h(\mathbf{w}_{t+1}, \mathbf{w}_t) \\
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\end{aligned}
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where the second inequality follows from $\mathbf{w}_{t+1}$ being the Bregman projection to the convex set C, and the last inequality is due to the convexity of f.

Take $\mathbf{w} = \mathbf{w}_t$ we see that

$$f(\mathbf{w}_{t+1}) \le f(\mathbf{w}_t),$$

i.e., the algorithm is descending. Summing from $t = 0$ to $t = T - 1$:

$$T\bar{\eta}_T \cdot [f(\mathbf{w}_T) - f(\mathbf{w})] \le \sum_{t=0}^{T-1} \eta_t [f(\mathbf{w}_{t+1}) - f(\mathbf{w})] \le \mathsf{D}_h(\mathbf{w}, \mathbf{w}_0).$$

Dividing both sides by $T\bar{\eta}_T$ completes the proof.

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Theorem: convergence of mirror descent for nonsmooth function

Let $C \subseteq \mathbb{R}^d$ be closed convex and $f : C \to \mathbb{R}$ be L-Lipschitz continuous convex (w.r.t. some norm $\|\cdot\|$). Start with $\mathbf{w}_0 \in C$, for any $\mathbf{w} \in C$, we have:

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- The bound on the right-hand side vanishes iff $\sum_t \eta_t \to \infty$ and $\eta_t \to 0$
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Thus,

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Extending to Composite

$$\min_{\mathbf{w}} \; f(\mathbf{w}), \quad \text{where} \quad f(\mathbf{w}) = \ell(\mathbf{w}) + r(\mathbf{w})$$

Algorithm 5: Composite mirror descent

Input: $\mathbf{w}_0$, functions ℓ and r, Legendre function h

for $t = 0, 1, \ldots$ **do**

  compute (sub)gradient $\ell'(\mathbf{w}_t)$ `// can be stochastic`

  choose step size $\eta_t > 0$

  $h'(\mathbf{z}_{t+1}) = h'(\mathbf{w}_t) - \eta_t \cdot \ell'(\mathbf{w}_t)$ `// gradient step w.r.t.` ℓ

  $\mathbf{w}_{t+1} \leftarrow \operatorname*{argmin}_{\mathbf{w}} \frac{1}{\eta_t} \mathsf{D}_h(\mathbf{w}, \mathbf{z}_{t+1}) + r(\mathbf{w})$ `// proximal step w.r.t.` r

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J. C. Duchi et al. (2010). "Composite Objective Mirror Descent". In: *Proceedings of the 23rd Annual Conference on Learning Theory*; J. C. Duchi et al. (2012). "Ergodic Mirror Descent". *SIAM Journal on Optimization*, vol. 22, no. 4, pp. 1549–1578.