ON THE THEORY OF GAMES OF STRATEGY

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INTRODUCTION

1. The present paper is concerned with the following question:

n players $S_1, S_2, \ldots, S_n$ are playing a given game of strategy, $\omega$. How must one of the participants, $S_m$, play in order to achieve a most advantageous result?

The problem is well known, and there is hardly a situation in daily life into which this problem does not enter. Yet, the meaning of this question is not unambiguous. For, as soon as $n > 1$ (i.e., $\omega$ is a game of strategy in the proper sense), the fate of each player depends not only on his own actions but also on those of the others, and their behavior is motivated by the same selfish interests as the behavior of the first player. We feel that the situation is inherently circular.

Hence we must first endeavor to find a clear formulation of the question. What, exactly, is a game of strategy? A great many different things come under this heading, anything from roulette to chess, from baccarat to bridge. And after all, any event - given the external conditions and the participants in the situation (provided the latter are acting of their own free will) - may be regarded as a game of strategy if one looks at the effect it has on the participants. What element do all these things have in common?

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1 A shortened version of this paper has been presented to the Goettingen Mathematical Society on December 7, 1926.

2 This is the principal problem of classical economics: how is the absolutely selfish "homo economicus" going to act under given external circumstances?
We may assume that it is the following:

A game of strategy consists of a certain series of events each of which may have a finite number of distinct results. In some cases, the outcome depends on chance, i.e., the probabilities with which each of the possible results will occur are known, but nobody can influence them. All other events depend on the free decision of the players $S_1$, $S_2$, ..., $S_n$. In other words, for each of these events it is known which player, $S_n$, determines its outcome and what is his state of information with respect to the results of other ("earlier") events at the time when he makes his decision. Eventually, after the outcome of all events is known, one can calculate according to a fixed rule what payments the players $S_1$, $S_2$, ..., $S_n$ must make to each other.

It is easy to bring this somewhat qualitative explanation into a precise form. The definition of a game of strategy would then be the following:

For a complete description of a game $\Theta$, the following data are necessary, which in their entirety are the "rules of the game."

a) The number of events or "draws" depending on chance, and the number of events or "steps" depending on the free decision of the individual players must be specified. Let these numbers be $z$ and $s$ respectively, and let us denote the "draws" by $E_1$, $E_2$, ..., $E_z$, the "steps" by $F_1$, $F_2$, ..., $F_s$.

b) The number of possible results of each "draw", $E_\mu$, and of each "step", $F_\nu$, must be specified. Let these numbers be $M_\mu$ and $N_\nu$ respectively. ($\mu = 1, 2, ..., z$, $\nu = 1, 2, ..., s$.) We shall denote the results, for short, by their numbers $1, 2, ..., M_\mu$ and $1, 2, ..., N_\nu$ respectively.

c) For each "draw" $E_\mu$ the probabilities $\alpha_{\mu}^{(1)}$, $\alpha_{\mu}^{(2)}$, ..., $\alpha_{\mu}^{(M_\mu)}$ of the different results $1, 2, ..., M_\mu$ must be given. Obviously, we have

$$\alpha_{\mu}^{(1)} \geq 0, \alpha_{\mu}^{(2)} \geq 0, ..., \alpha_{\mu}^{(M_\mu)} \geq 0$$

$$\alpha_{\mu}^{(1)} + \alpha_{\mu}^{(2)} + ... + \alpha_{\mu}^{(M_\mu)} = 1.$$
5) For every "step" \( F_v \), the player \( S_m \) who determines the outcome of this "step" ("whose step" \( F_v \) is) must be specified. In addition, the numbers of all "draws and steps" must be specified of whose outcome the player is informed at the time he makes his decision concerning \( F_v \). (These "draws" and "steps" we shall call "earlier" than \( F_v \).)

In order that this whole scheme be consistent and permit of a temporal-causal interpretation, there must be no cycles \( F_{v_1}, F_{v_2}, \ldots, F_{v_p}, F_{v_p} = F_{v_1} \), and \( F_{v_q} \) must always be "earlier" than \( F_{v_{q+1}} \) (\( q = 1, 2, \ldots, p \)).

c) Finally, \( n \) functions \( f_1, f_2, \ldots, f_n \) must be given. Each of them depends on \( z + s \) variables which take on the values

\[
\begin{align*}
1, 2, \ldots, M_1; & \quad 1, 2, \ldots, M_2; & \quad \ldots & \quad 1, 2, \ldots, M_s; \\
1, 2, \ldots, N_1; & \quad 1, 2, \ldots, N_2; & \quad \ldots & \quad 1, 2, \ldots, N_s
\end{align*}
\]

respectively. These functions are real-valued and

\[
f_1 + f_2 + \ldots + f_n = 0
\]

holds identically. If in the course of a play which has been completed the results of the \( z \) "draws" and \( s \) "steps" were \( x_1, x_2, \ldots, x_z, y_1, y_2, \ldots, y_s \) respectively, \( (x_\mu = 1, 2, \ldots, M_\mu, y_v = 1, 2, \ldots, N_v) \) \( \mu = 1, 2, \ldots, z, \nu = 1, 2, \ldots, s \), the players \( S_1, S_2, \ldots, S_n \) obtain from each other the amounts

\[
\begin{align*}
f_1(x_1, \ldots, x_z, y_1, \ldots, y_s), & \quad f_2(x_1, \ldots, x_z, y_1, \ldots, y_s), \\
& \quad \ldots, f_n(x_1, \ldots, x_z, y_1, \ldots, y_s)
\end{align*}
\]

(On closer inspection it can be seen that in spite of this somewhat lengthy description we are dealing here with quite simple matters. Actually, in several respects our definition might have been somewhat more

\[\text{The identity } f_1 + f_2 + \ldots + f_n = 0 \]

expresses that the players make payments to each other only, but collectively they neither gain nor lose.
2. With this definition the concept of a game of strategy is precisely defined. But it also becomes clear that, as we indicated at the beginning of 1., the expression "S_m tries to achieve a result as advantageous as possible" is rather obscure. What constitutes the most advantageous result for the player S_m is obviously the largest possible value of f_m, but how can any value of f_m be "achieved" by S_m? By himself, S_m is in no position to fix the value of f_m! The value of f_m depends on the variables x_1, ..., x_n, y_1, ..., y_m, only part of which are determined by S_m's decision (viz. those y_i for which S_m has the "step" F_y, i.e., those for which S(F_y) = S_m). All other variables y_i depend on the decisions of the participants and all variables x_i depend on chance.

In our case, the "unforeseeable" chance event is actually the factor which it is easiest to deal with. In fact, assume that a particular f_m depends only on those y_i which are decided on by S_m (S(F_y) = S_m), and in addition on the x_i (which are determined by chance). In that case, S_m can at least anticipate this much: If I make certain moves, I can expect such and such results (i.e., values of f_m) with such and such probabilities (since the probabilities p_i are given) - regardless of how the other players act! If we now assume that "a most advantageous result" is the highest possible value of the expectation (and this assumption or a similar one has to be made in order to apply the methods of the theory of probability) we have, in principle, solved our problem. For, we have here a simple maximum problem: The values of those variables y_i which S_m has to determine must be so chosen by him that the expected value of f_m (which depends only on these variables y_i) becomes as large as possible.

It is this type of game which in the theory of probability is treated in the so-called "theory of games of chance." A typical example is roulette: Let the number of players be k + 1 (S_1, ..., S_k are the "pointers", S_{k+1} is the "banker"), S_k has no influence whatsoever on the game, and the result achieved by S_k, f_k, (k = 1, 2, ..., k) only depends on chance.

The nature of the game is put on the basis of the variables y_i. Thus it is exactly the effects which the results of all other

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depends on chance and his own actions.\footnote{As may be guessed, on the basis of the preceding footnote, the unambiguous result in this case for the behavior of the pointers is rather trivial: if possible they should have the turn-over zero; the closer they approximate it, so much the better!}

The name alone, "game of chance", indicates that the main emphasis is put on the variables \(x_1\), which are dependent on chance, and not on the variables \(y_1\), which are subject to the decisions of the players. But this is exactly what we are interested in. We shall try to investigate the effects which the players have on each other, the consequences of the fact (so typical of all social happenings!) that each player influences the results of all other players, even though he is only interested in his own.

§1. GENERAL SIMPLIFICATIONS

1. The definition of a game of strategy given in the Introduction is rather complicated, which may appear justified in view of the fact that games of strategy may be arbitrarily complex. Nevertheless it is possible to bring all games falling under this definition into a much simpler more normal form, in a way, into the simplest form that is at all conceivable. We contend that it is sufficient to consider games of the following kind:

\[ z = 1 \text{ (i.e., only one "draw" takes place).} \]

\[ s = n, \text{ the } v\text{-th "step" being that of the player } S_v (S(F_v) = S_v). \]

The relation "earlier" is never realized (i.e., each player must make his dispositions without knowing anything about the other participants or about the "draw").

The play thus takes the following course: Each player \(S_m\) \((m = 1, 2, \ldots, n)\) chooses a number \(1, 2, \ldots, N_m\) without knowing the choices of the others. Now a "draw" takes place in which the numbers \(1, 2, \ldots, M\) will appear with the probabilities \(\alpha_1, \alpha_2, \ldots, \alpha_M\). The results achieved by the players are (if the "draw" and the \(n\) "steps" have resulted in \(x, y_1, y_2, \ldots, y_n\))

\[ f_1(x, y_1, \ldots, y_n), f_2(x, y_1, \ldots, y_n), \ldots, f_n(x, y_1, \ldots, y_n). \]

This apparently far-reaching restriction is actually not essential, for the following reason:

Let the "steps" of the players \(S_m (S(F_v) = S_m)\) be those with the numbers \(v_1^{(m)}, v_2^{(m)}, \ldots, v_m^{(m)}\). Obviously, it is inadmissible to make the assumption that \(S_m\) might be able to tell, before the start of the
game, what his choices for all these steps are going to be. It would mean a restriction of his free will and change his chances (for the worse).
For, $S_m$'s decision in each of these "steps" will generally be significantly influenced by the results of the "draws" and "steps" known to him at the moment of his decision.

On the other hand, it may well be assumed that before the play has started he knows how to answer the following question: What will be the outcome of the $v_k^{(m)}$-th "step" ($k = 1, 2, \ldots, \sigma_m$) provided the results of all "draws" and "steps" "earlier" than $v_k^{(m)}$ are available? In other words, the player knows beforehand how he is going to act in a precisely defined situation; he enters the play with a theory worked out in detail. Even if this may not be the case for a particular player, it is clear that such an assumption will certainly not spoil his chances.

Accordingly, we define the "strategy" of a player $S_m$ as follows:

In order to describe completely the "strategy" of a player $S_m$ ($m = 1, 2, \ldots, n$) the following specifications are necessary:

As before, let $S_m$ have the "steps" with the numbers $v_1^{(m)}, v_2^{(m)}, \ldots, v_{\sigma_m}^{(m)}$ and assume that at the moment when he decides on the $v_k^{(m)}$-th "step" ($k = 1, 2, \ldots, \sigma_m$) the results of the "draws" and "steps" with the numbers $v_1^{(m)}, v_2^{(m)}, \ldots, v_{\sigma_m}^{(m)}$ are "earlier" than $v_k^{(m)}$. For each possible combination of results of the "draws" and "steps" mentioned above (obviously, there is only a finite number of such combinations) it must be specified what $S_m$'s decision with respect to the $v_k^{(m)}$-th "step" is going to be (i.e., what will be the outcome of this step).

One sees immediately that only a finite number of strategies is available to $S_m$, which we shall call $S_1^{(m)}, S_2^{(m)}, \ldots, S_{\sigma_m}^{(m)}$.

It can now easily be shown (using the assumption on the absence of cycles in the Introduction, i., definition of a game of strategy, (5)) that the course of a play is described in a permissible and unambiguous manner if we specify
1. which strategies $S_1^{(1)}, S_2^{(2)}, \ldots, S_n^{(n)}$ are being used by the players $S_1, S_2, \ldots, S_n$ respectively,

2. what are the results of the "draws" $E_1, E_2, \ldots, E_z$.

Two points should be noted here. In the first place, it is inherent in the concept of "strategy" that all the information about the actions of the participants and the outcome of "draws" a player is able to obtain or to infer is already incorporated in the "strategy." Consequently, each player must choose his strategy in complete ignorance of the choices of the rest of the players and of the results of the "draws."

Secondly, it has become entirely immaterial that the "draws" $E_1, E_2, \ldots, E_z$ are separate events (where for $E_\mu$, $\mu = 1, 2, \ldots, z$, the numbers $1, 2, \ldots, M_\mu$ will occur with the respective probabilities $a_\mu^{(1)}, a_\mu^{(2)}, \ldots, a_\mu^{(M_\mu)}$) since the players must act, i.e., choose their "strategies" without knowing the outcome of the "draws." But if this is the case nothing prevents us from combining all $z$ "draws" into a single "draw", $H$, the outcome of which will be the aggregates of numbers

$x_1, x_2, \ldots, x_z \ (x_\mu = 1, 2, \ldots, M_\mu, \mu = 1, 2, \ldots, z)$

with their respective probabilities $a_1, a_2, \ldots, a_z$, or, what amounts to the same thing, the numbers $1, 2, \ldots, M$ ($M = M_1 \cdot M_2 \cdot \ldots \cdot M_z$) with their respective probabilities, which we shall call $\beta_1, \beta_2, \ldots, \beta_M$.

Thus we can modify 2. in the following way:

2'. The result of the "draw" $H$ must be specified.

($H$ may have the results $1, 2, \ldots, M$ with the respective probabilities $\beta_1, \beta_2, \ldots, \beta_M$).

The choices (interpreted as "steps") of the players $S_1, S_2, \ldots, S_n$ in 1. together with the "draw" in 2'. are fully equivalent to the original game $G$ (if one takes the fact into account that each "step" is taken in complete ignorance of all other circumstances), and they evidently constitute a game $G'$ which, indeed, is of the simple form mentioned at the beginning of this section.

2. The last element to be eliminated from the game, since from our point of view it is inessential, is the "draw." This is done by replacing the actual results for the individual players by their expected values. To be exact:

If the players $S_1, S_2, \ldots, S_n$ have chosen the "strategies"
of generality. If each of the players determine if the outcome of the "draw" $H$ has been the number $v (1, 2, \ldots, M)$, then let the results for the players $S_1, S_2, \ldots, S_n$ be

\[
f_1(v, u_1, \ldots, u_n), f_2(v, u_1, \ldots, u_n), \ldots, f_n(v, u_1, \ldots, u_n)
\]

(We may disregard the fact that we are dealing with "strategies" and not with actual "steps" and simply speak of the choices $u_1, u_2, \ldots, u_n$)

If only the choices of $u_1, u_2, \ldots, u_n$ are known, but not yet the "draw" $v$, the expected values of $f_1, f_2, \ldots, f_n$ will be

\[
g_m(u_1, \ldots, u_n) = \sum_{v=1}^{M} \beta_v f_m(v, u_1, \ldots, u_n) \quad (m = 1, 2, \ldots, n)
\]

($f_1 + f_2 + \ldots + f_n = 0$ implies $g_1 + g_2 + \ldots + g_n = 0$). It is entirely in the spirit of the probabilistic method to discount the "draw" altogether and to deal exclusively with the expected values $g_1, g_2, \ldots, g_n$.

In doing so we obtain the following basic type of a game of strategy which is even more schematized and simplified.

Each of the players $S_1, S_2, \ldots, S_n$ chooses a number, $S_m$ choosing one of the numbers $1, 2, \ldots, \xi_m$ ($m = 1, 2, \ldots, n$). Each player must make his decision without being informed about the choices of the other participants.

After having made their choices $x_1, x_2, \ldots, x_n$ ($x_m = 1, 2, \ldots, \xi_m, m = 1, 2, \ldots, n$) the players receive the following amounts respectively:

\[
g_1(x_1, \ldots, x_n), g_2(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)
\]

(where identically $g_1 + g_2 + \ldots + g_n = 0$).

The rules of the game have thus been obtained in a form which retains only those characteristics of a game of strategy which are essential to our consideration - and as we have just shown, essentially without loss of generality.
of generality. Nothing is left of a "game of chance". The actions of the players determine the result completely (since everything takes place as if each of the players has his eye on the expected value only). As a result, the feature which was emphasized at the end of the Introduction emerges in a particularly clear form: each \( g_m \) depends on all \( x_1, x_2, \ldots, x_n \).

The standard case of probability theory that \( g_m \) depends on \( x_m \) only (which, of course, cannot hold for all \( m \)) now appears to be entirely trivial.

\[ \$2. \ \text{THE CASE} \ \ n = 2 \]

1. Since we cannot proceed any further with the same generality, it is now appropriate to consider the simplest case for \( n \). The case \( n = 0 \) is meaningless, and so is the case \( n = 1 \) (since \( g_1 + g_2 + \ldots + g_n = 0 \)); neither involves an actual game of strategy. So we shall now investigate the case \( n = 2 \).

Since \( g_1 + g_2 = 0 \), we can put \( g_1 = g_2 = -g \). The description of a general two-person game is then as follows:

The players \( S_1, S_2 \) choose arbitrary numbers among the numbers \( 1, 2, \ldots, \Sigma_1 \) and \( 1, 2, \ldots, \Sigma_2 \) respectively, each one without knowing the choice of the other. After having chosen the numbers \( x \) and \( y \) respectively, they receive the sums \( g(x, y) \) and \( -g(x, y) \) respectively.

\( g(x, y) \) may be any function (defined for \( x = 1, 2, \ldots, \Sigma_1, y = 1, 2, \ldots, \Sigma_2 \)).

It is easy to picture the forces struggling with each other in such a two-person game. The value of \( g(x, y) \) is being tugged at from two sides, by \( S_1 \) who wants to maximize it, and by \( S_2 \) who wants to minimize it. \( S_1 \) controls the variable \( x \), \( S_2 \) the variable \( y \). What will happen?

2. After \( S_1 \) has chosen the number \( x \) (\( x = 1, 2, \ldots, \Sigma_1 \)), his result \( g(x, y) \) still depends on the choice \( y \) of \( S_2 \), but in any event \( g(x, y) \geq \text{Min}_y g(x, y) \). And by an appropriate choice of \( x \) this lower limit can be made equal to \( \text{Max}_x \text{Min}_y g(x, y) \) (and not any larger!). I.e., if \( S_1 \) so wishes, he certainly can make \( g(x, y) \)

\[ \geq \text{Max}_x \text{Min}_y g(x, y) \]

(irrespective of what \( S_2 \) does!). The same argument holds for \( S_2 \). If
S₂ so wishes, he certainly can make \( g(x, y) \)

\[ \leq \min_y \max_x g(x, y) \]

(irrespective of what \( S_1 \) does!).

If now

\[ \max_x \min_y g(x, y) = \min_y \max_x g(x, y) = M \]

it follows from the above, as well as from the fact that \( S_1 \) wants to maximize \( g(x, y) \) and \( S_2 \) wants to minimize it, that \( g(x, y) \) will have the value \( M \). For, \( S_1 \) is interested in making it large and can keep it from becoming smaller than \( M \). \( S_2 \), on the other hand, is interested in making it small and can keep it from becoming larger than \( M \). Hence it will have the value \( M \).

Though, in general,

\[ \max_x \min_y g(x, y) \leq \min_y \max_x g(x, y) \]

it is not at all true that the = sign always holds. Actually, it is easy to exhibit such \( g(x, y) \) for which the < sign holds, that is, for which the above consideration breaks down. The simplest example of this kind is the following:

\[ S_1 = S_2 = 2, \quad g(1, 1) = 1, \quad g(1, 2) = -1, \]
\[ g(2, 1) = -1, \quad g(2, 2) = 1. \]

(Evidently, \( \max \min = -1 \) and \( \min \max = 1 \).)

Another example is the so-called game of "Morra":

\[ S_1 = S_2 = 3, \quad g(1, 1) = 0, \quad g(1, 2) = 1, \quad g(1, 3) = -1, \]
\[ g(2, 1) = -1, \quad g(2, 2) = 0, \quad g(2, 3) = 1, \]
\[ g(3, 1) = 1, \quad g(3, 2) = -1, \quad g(3, 3) = 0. \]

(Here, too, \( \max \min = -1 \) and \( \min \max = 1 \).)

The fact that this difficulty comes up can also be realized in the following way:

\[ \max_x \min_y g(x, y) \] is the best result that \( S_1 \) can achieve if

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8 Also called "gangster baccarat." In the usual formulation, 1, 2, 3 are called "Paper", "Stone", "Scissors" ("Paper covers the stone, scissors cut the paper, stone grinds the scissors").
he is "found out" by $S_2$, if whenever $S_1$ plays $x$, $S_2$ plays a $y$ such that $g(x, y) = \min_y g(x, y)$. (According to the rules of the game $S_2$ was not supposed to know how $S_1$ was going to play, he would have to infer it in some other way. This is what we mean to indicate by the expression "finding out". In the same way, the best result that $S_2$ can achieve if $S_1$ has found him out is $\max_x \min_y g(x, y)$. If the two numbers are equal this means: it makes no difference which of the two players is the better psychologist, the game is so insensitive that the result is always the same. It is obvious that this is not the case for the two games just mentioned: here, everything depends on finding the adversary out, on guessing whether he is going to choose 1 or 2 (or 1 or 2 or 3).

The fact that the two quantities $\max \min$ and $\min \max$ are different means that it is impossible for each of the two players, $S_1$ and $S_2$, to be cleverer than the other.

3. Still, it is possible, by use of an artifice, to force the equality of the two above-mentioned expressions.

To this purpose, the possibilities of action for the two players $S_1$ and $S_2$ are extended as follows: At the beginning of the game, $S_1$ is not asked to choose one of the numbers 1, 2, ..., $\Sigma_1$. He only has to specify $\Sigma_1$ probabilities $t_1, t_2, \ldots, t_{\Sigma_1}$ ($t_1 \geq 0$, $t_2 \geq 0$, ..., $t_{\Sigma_1} \geq 0$, $t_1 + t_2 + \ldots + t_{\Sigma_1} = 1$) and then draw the numbers 1, 2, ..., $\Sigma_1$ from an urn containing these numbers with the probabilities $t_1, t_2, \ldots, t_{\Sigma_1}$. He then chooses the number drawn. This may look like a restriction of his free will: it is not he who determines $x$. But this is not so. Because if he really wants to get a particular $x$, he can specify $t_x = 1$, $t_u = 0$ (for $u \neq x$). On the other hand, he is protected against his adversary "finding him out"; for, if, e.g., $t_1 = t_2 = 1/2$, nobody (not even he himself!) can predict whether he is going to choose 1 or 2!

$S_2$ is supposed to act in the same way. He also chooses $\Sigma_2$ probabilities $\eta_1, \eta_2, \ldots, \eta_{\Sigma_2}$ and proceeds accordingly.

Let us denote the sequence $t_1, t_2, \ldots, t_{\Sigma_1}$ by $\xi$ and the sequence $\eta_1, \eta_2, \ldots, \eta_{\Sigma_2}$ by $\eta$. If $S_1$ chooses $\xi$ and $S_2$ chooses $\eta$, $S_1$ has the expected value

$$h(\xi, \eta) = \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g(p, q) \xi_p \eta_q$$
and $S_2$ has the expected value $-h(t, \eta)$. The new function $h(t, \eta)$ includes the old one, $g(x, y)$, in the following sense: if $t_x = \eta_y = 1$ and $t_u = \eta_v = 0$ (for $u \neq x, v \neq y$), then $h(t, \eta) = g(x, y)$.

We can now apply the same consideration to $h(t, \eta)$ we applied to $g(x, y)$. If $S_1$ has made the choice $t$, his expected value is at least $\min_{\eta} h(t, \eta)$. Hence, he is in a position to obtain the minimal expected value $\max_t \min_{\eta} h(t, \eta)$ (irrespective of what $S_2$ does!). In the same way, $S_2$ can keep the expected value of $S_1$ from exceeding the maximal value $\min_{t} \max_{\eta} h(t, \eta)$. Again we have

$$\max_t \min_{\eta} h(t, \eta) \leq \min_{\eta} \max_t h(t, \eta)$$

and the question is whether the equality sign always holds.

Evidently, in this case our chances are better than they were for $g(x, y)$; for, $g(x, y)$ could be any function, whereas $h(t, \eta)$ is a bilinear form! Even though $h(t, \eta)$ is essentially a generalization of $g(x, y)$, yet it is a function of a much simpler type than $g(x, y)$. In fact, we shall prove in Section 3 that the relation

$$\max_t \min_{\eta} h(t, \eta) = \min_{\eta} \max_t h(t, \eta)$$

holds for all bilinear forms $h(t, \eta)$ (where $\max_t$ is taken for all $t$ for which $\eta_1 \geq 0, \ldots, \eta_{\Sigma_1} \geq 0, t_1 + \ldots + t_{\Sigma_1} = 1$, and $\min_{\eta}$ is taken for all $\eta$ for which $\eta_1 \geq 0, \ldots, \eta_{\Sigma_2} \geq 0, \eta_1 + \ldots + \eta_{\Sigma_2} = 1$).

4. Anticipating the result we put

$$\max_t \min_{\eta} h(t, \eta) = \min_{\eta} \max_t h(t, \eta) = M.$$

Let $\mathcal{S}$ be the set of all $t$ for which $\min_{\eta} h(t, \eta)$ assumes its maximal value $M$, and let $\mathcal{W}$ be the set of all $\eta$ for which $\max_t h(t, \eta)$ assumes its minimal value $M$. From these definitions the relations below follow immediately.

1. If $t$ belongs to $\mathcal{S}$, then always $h(t, \eta) \geq M$
2. If $\eta$ belongs to $\mathcal{W}$, then always $h(t, \eta) \leq M$
3. If $t$ does not belong to $\mathcal{S}$, there exists an $\eta$ for which $h(t, \eta) < M$
4. If $\eta$ does not belong to $\mathcal{W}$, there exists a $t$ for which $h(t, \eta) > M$
5. If $t$ belongs to $\mathcal{S}$ and $\eta$ belongs to $\mathcal{W}$, then $h(t, \eta) = M$. 

On the other hand, it seems justified:

Clear.

For any $S_1$ and $S_2$.

Evidently, it can be called skew-symmetric, i.e., if on interchanging $h(t, \eta)$ and $-h(t, \eta)$ or, equivalently,

$$\max_t \min_{\eta} h(t, \eta) = -\min_{\eta} \max_t h(t, \eta),$$

i.e., if the bilinear form $h(t, \eta)$ can be seen as follows:

$$-\max_t \min_{\eta} h(t, \eta) = \min_{\eta} \max_t h(t, \eta),$$

i.e.,

One can easily see that $\mathcal{S}$ contains only those $t$ for which $h(t, \eta)$ assumes its maximal value $M$ and $\mathcal{W}$ contains only those $\eta$ for which $h(t, \eta)$ assumes its minimal value $M$. I.e., both games are possible and in both examples $M$.

Use is made of our rather deep theorems to prove

$$\max \min \leq \min \max.$$ 

While this paper is short, E. Borel in the C. R. present a complete proof of the question of bilinear forms, no examples for $M = 0$.

Our result above
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On the basis of the relations (1) to (5) the following statement seems justified:

Clearly, \( S_1 \) must choose a complex \( \xi \) belonging to \( \mathcal{H} \), and \( S_2 \) must choose a complex \( \eta \) belonging to \( \mathcal{H}_2 \).

For any such choice, a play has the value \( M \) or \( -M \) for \( S_1 \) and \( S_2 \) respectively.

Evidently, a two-person game can be called "fair" if \( M = 0 \); and it can be called "symmetric" if the players \( S_1, S_2 \) have the same roles. I.e., if on interchanging \( \xi \) and \( \eta \) (which presupposes that \( \Sigma_1 = \Sigma_2 \)) \( h(\xi, \eta) \) and \( -h(\eta, \xi) \) are also interchanged, in other words, if

\[
h(\xi, \eta) = -h(\eta, \xi)
\]

or, equivalently,

\[
g(x, y) = -g(y, x)
\]

i.e., if the bilinear form \( h(\xi, \eta) \), or else the matrix \( g(x, y) \) is skew-symmetric. In this case, the game is, of course, also "fair", as can be seen as follows:

\[
-\max_{\xi} \min_{\eta} h(\xi, \eta) = \min_{\eta} \max_{\xi} h(\xi, \eta) = \min_{\eta} \max_{\xi} h(\eta, \xi)
\]

\[
= \min_{\eta} \max_{\xi} h(\xi, \eta)
\]

i.e.,

\[
-M = M, M = 0. \quad (9)
\]

One can easily see that in our two examples (in 2.) \( M = 0 \) since \( \mathcal{H} \) contains only \( \xi_1 = \xi_2 = 1/2 \) and \( \xi_1 = \xi_2 = \xi_3 = 1/3 \) respectively, and \( \mathcal{H}_2 \) contains only \( \eta_1 = \eta_2 = 1/2 \) and \( \eta_1 = \eta_2 = \eta_3 = 1/3 \) respectively. I.e., both games are "fair" ("Paper, Scissors, Stone" is even symmetric), and in both examples each player must choose all numbers at random, all of

\[
(\xi, \eta) \geq M
\]

(\( \xi, \eta \) \leq M exists an

exists a

to \( \mathcal{H} \),

\[
\max \min \leq 0, \quad \min \max \geq 0.
\]

While this paper was put into its final form, I learned of the note of E. Borel in the Comptes Rendus of Jan. 10, 1927 ("Sur les systèmes de formes linéaires...et la théorie du jeu," pp. 52-55). Borel formulates the question of bilinear forms for a symmetric two-person game and states that no examples for \( \max \min < \min \max \) are known.

Our result above answers his question.
them with the same probability.

The following point should be emphasized: Although in Section 1 chance was eliminated from the games of strategy under consideration (by introducing expected values and eliminating "draws"), it has now made a spontaneous reappearance. Even if the rules of the game do not contain any elements of "hazard" (i.e., no draws from urns) - as e.g., the two examples in 2. - in specifying the rules of behavior for the players it becomes imperative to reconsider the element of "hazard". The dependence on chance (the "statistical" element) is such an intrinsic part of the game itself (if not of the world) that there is no need to introduce it artificially by way of the rules of the game: even if the formal rules contain no trace of it, it still will assert itself.

§3. PROOF OF THEOREM "Max Min = Min Max"

1. Let us slightly change our notation by replacing \( \Sigma_1 \) by \( M + 1 \), \( \Sigma_2 \) by \( N + 1 \) and \( g(p, q) \) by \( a_{pq} \). We then have

\[ h(\xi, \eta) = \sum_{p=1}^{M+1} \sum_{q=1}^{N+1} a_{pq} \xi^p \eta^q . \]

Because of the conditions

\[ \xi_1 + \ldots + \xi_M + \xi_{M+1} = 1, \quad \eta_1 + \ldots + \eta_N + \eta_{N+1} = 1 \]

the complex \( \xi \) is already determined by \( \xi_1, \ldots, \xi_M \) and so is the complex \( \eta \) by \( \eta_1, \ldots, \eta_N \). Thus

\[ h(\xi, \eta) = \sum_{p=1}^{M} \sum_{q=1}^{N} u_{pq} \xi^p \eta^q + \sum_{p=1}^{M} v_{p} \xi^p + \sum_{q=1}^{N} w_{q} \eta^q + r . \]

(There is no need to specify explicitly the coefficients in terms of \( a_{pq} \).) We shall make use of only some of the properties of \( h(\xi, \eta) \) and investigate continuous functions of two sets of variables \( f(\xi, \eta) \) which satisfy the following conditions:

(K). If \( f(\xi', \eta) \geq A, f(\xi'', \eta) \geq A \), then \( f(\xi, \eta) \geq A \) for every \( 0 \leq \varphi \leq 1, \xi = \varphi \xi' + (1 - \varphi) \xi'' \) (i.e., \( \xi_p = \varphi \xi_p' + (1 - \varphi) \xi_p'' \), \( p = 1, 2, \ldots, M \)). If \( f(\xi, \eta') \leq A, f(\xi, \eta'') \leq A \), then \( f(\xi, \eta) \leq A \) for every \( 0 \leq \varphi \leq 1, \eta = \varphi \eta' + (1 - \varphi) \eta'' \) (i.e., \( \eta_q = \varphi \eta_q' + (1 - \varphi) \eta_q'' \), \( q = 1, 2, \ldots, N \)).

Clearly, \( \xi^p \) and \( \eta^q \) respectively. We set (in 1.), then

\[ M^p f(\xi_1, \ldots, \xi_M, \eta_1, \ldots, \eta_N) \]

Evidently, we need the property (K), i.e.,
(It is clear that \( h(t, \eta) \), being linear in the \( t \) as well as in the \( \eta \), has the property (K).) For these functions \( f(t, \eta) \) we are going to prove that

\[
\text{Max}_t \text{Min}_\eta f(t, \eta) = \text{Min}_\eta \text{Max}_t f(t, \eta)
\]

where \( \text{Max}_t \) is taken over the range \( t_1 \geq 0, \ldots, t_M \geq 0, t_1 + \ldots + t_M \leq 1 \) and \( \text{Min}_\eta \) is taken over the range \( \eta_1 \geq 0, \ldots, \eta_N \geq 0, \eta_1 + \ldots + \eta_N \leq 1 \).

We can also write

\[
\begin{align*}
\text{Max}_t & \quad \text{Max}_{t_2} \quad \ldots \quad \text{Max}_{t_M} \quad \text{Min}_\eta & \quad \text{Min}_{\eta_2} \quad \ldots \quad \text{Min}_{\eta_N} f(t, \eta) \\
& \quad t_1 \geq 0 \quad t_2 \geq 0 \quad \ldots \quad t_M \geq 0 \quad \eta_1 \geq 0 \quad \eta_2 \geq 0 \quad \ldots \quad \eta_N \geq 0 \\
& \quad t_1 \leq 1 \quad t_1 + t_2 \leq 1 \quad \ldots \quad t_1 + t_2 + \ldots + t_M \leq 1 \\
& = \text{Min}_\eta & \quad \text{Min}_{\eta_2} \quad \ldots \quad \text{Min}_{\eta_N} & \quad \text{Max}_t & \quad \text{Max}_{t_2} \quad \ldots \quad \text{Max}_{t_M} f(t, \eta) \\
& \quad \eta_1 \geq 0 \quad \eta_2 \geq 0 \quad \ldots \quad \eta_N \geq 0 \quad \eta_1 \leq 1 \quad \eta_1 + \eta_2 \leq 1 \quad \ldots \quad \eta_1 + \eta_2 + \ldots + \eta_N \leq 1
\end{align*}
\]

2. We introduce the following notation:

\[
M^t f(t_1, \ldots, t_r, \eta_1, \ldots, \eta_s) = \text{Max}_t f(t_1, \ldots, t_r, \eta_1, \ldots, \eta_s) \quad t_1 \geq 0 \quad \ldots \quad t_r \leq 1
\]

\[
M^s f(t_1, \ldots, t_r, \eta_1, \ldots, \eta_s) = \text{Min}_s f(t_1, \ldots, t_r, \eta_1, \ldots, \eta_s) \quad \eta_1 \geq 0 \quad \ldots \quad \eta_s \leq 1
\]

Clearly, \( M^r \) and \( M^s \) eliminate the dependence of \( f \) on \( t_r \) and \( \eta_s \) respectively. We are going to prove: if \( f \) satisfies the condition (K) (in 1.), then

\[
M^r f = \text{Max}_t M^r f(t_1, \ldots, t_r, \eta_1, \ldots, \eta_s) = M^r M^1 \ldots M^r \eta_1 M^2 \ldots M^r \eta_s f
\]

\[
\quad = M^r M^1 M^2 \ldots M^r \eta_1 M^2 \ldots M^r \eta_s f
\]

Evidently, we need only prove the following two assertions:

a) If \( f = f(t_1, \ldots, t_r, \eta_1, \ldots, \eta_s) \) is continuous and has the property (K), the same holds for \( M^r f \) and \( M^s f \).
a) If \( f = f(t_1, \ldots, t_n, \eta_1, \ldots, \eta_s) \) is continuous and has the property (K), then
\[
M^* f = M^* f = M^* f = M^* f
\]

We first prove (a). It is sufficient to consider \( M^* f \); the same considerations apply to \( M^* f \).

We have
\[
M^* f(t_1, \ldots, t_n, \eta_1, \ldots, \eta_s) = f^*(t_1, \ldots, t_{n-1}, \eta_1, \ldots, \eta_s)
= \max_{\xi_1 \geq 0} f(t_1, \ldots, t_n, \eta_1, \ldots, \eta_s)
\]

\[
\xi_1 + \ldots + \xi_n \leq 1
\]

It is obvious that the continuity of \( f \) implies that of \( f^* \). We still have to examine the two properties in (K).

First, let
\[
f^*(t_1', \ldots, t_{n-1}', \eta_1, \ldots, \eta_s) \geq A, \quad f^*(t_1'', \ldots, t_{n-1}'', \eta_1, \ldots, \eta_s) \geq A
\]

The \( f^* \) represent maximal values of \( f \) on finite intervals. Since \( f \) is continuous, they are actually assumed, say, for \( t_1' \) and \( t_1'' \) respectively.

Then
\[
f(t_1', \ldots, t_n, \eta_1, \ldots, \eta_s) \geq A, \quad f(t_1'', \ldots, t_n, \eta_1, \ldots, \eta_s) \geq A
\]

and since \( f \) satisfies (K) (we put \( t_1 = \alpha t_1' + (1 - \alpha)t_1'' \), \( \ldots, t_{n-1} = \alpha t_{n-1}' + (1 - \alpha)t_{n-1}'' \)),
\[
f(t_1, \ldots, t_n, \eta_1, \ldots, \eta_s) \geq A
\]

The inequalities
\[
\xi_1' \geq 0, \quad \xi_1'' \leq 1, \quad \xi_1'' \geq 0, \quad \xi_1'' \leq 1
\]

imply directly
\[
\xi_1' + \ldots + \xi_n' \leq 1, \quad \xi_1'' + \ldots + \xi_n'' \leq 1
\]

Hence, a fortiori, for the maximum \( f^* \)

Second, let
\[
f^*(t_1, \ldots, t_{n-1}', \eta_1', \ldots, \eta_s')
\]

In virtue of the maximum
\[
f(t_1, \ldots, t_n, \eta_1', \ldots, \eta_s')
\]

for all \( t_n \) for which
\[
(\eta_1' = \alpha \eta_1 + (1 - \alpha)\eta_1''
\]

Since \( f \) satisfies
\[
(\eta_1 = \alpha \eta_1' + (1 - \alpha)\eta_1''
\]

This completes the proof.

3. We will say \( \eta_1, \ldots, \eta_{s-1} \) are \( r \) times \( \xi_1, \ldots, \xi_n \) if we keep the variables \( \eta_1, \ldots, \eta_{s-1} \) as a function of \( \xi_1, \ldots, \xi_n \) and if
\[
f(\xi_1, \ldots, \xi_n) \leq A.
\]

It remains for us to prove
\[
\max_{\xi_1 + \ldots + \xi_n \leq 1} f(\xi_1, \ldots, \xi_n) \geq A.
\]

(We write \( a \) and \( b \) respectively.)
continuous and has

Consider \( M^n f \); the

\( f^n(\tau_1, \ldots, \tau_{n-1}, \eta_1', \ldots, \eta_{n-1}') \leq A, \quad f^n(\tau_1, \ldots, \tau_{n-1}, \eta_1'', \ldots, \eta_{n-1}'') \leq A \).

In virtue of the maximum property of \( f^n \) we have

\[ f(\tau_1, \ldots, \tau_n, \eta_1', \ldots, \eta_n') \leq A, \quad f(\tau_1, \ldots, \tau_n, \eta_1'', \ldots, \eta_n'') \leq A \]

for all \( \tau_n \) for which

\[ \tau_n > 0, \quad \tau_1 + \ldots + \tau_n \leq 1. \]

Since \( f \) satisfies (K), this again implies

\[ f(\tau_1, \ldots, \tau_n, \eta_1', \ldots, \eta_n') \leq A \]

\((\eta_1 = \phi \eta_1' + (1 - \phi) \eta_1'', \ldots, \eta_n = \phi \eta_n' + (1 - \phi) \eta_n'')\). And since this holds for all \( \tau_n \) mentioned above, we have

\[ f^n(\tau_1, \ldots, \tau_{n-1}, \eta_1, \ldots, \eta_n) \leq A. \]

This completes the proof of our assertion (a).

3. We will now show that always (i.e., for all \( \tau_1, \ldots, \tau_{n-1}, \eta_1, \ldots, \eta_{n-1} ) \)

\( M^n f = M^n f^n \). If in \( f(\tau_1, \ldots, \tau_n, \eta_1, \ldots, \eta_n) \)

we keep the variables \( \tau_1, \ldots, \tau_{n-1}, \eta_1, \ldots, \eta_{n-1} \) fixed, then \( f \), as

a function of \( \tau_n, \eta_n \) alone, evidently still satisfies the condition (K).

It remains for us to prove (writing \( \xi \), \( \eta \), \( \tau_n, \eta_n )\):

If \( f(\xi, \eta) \) is a continuous function and if

\[ f(\xi', \eta) \geq A, f(\xi'', \eta) \geq A \]

for \( \xi' \leq \xi \leq \xi'' \) implies that

\( f(\xi, \eta) \geq A \), and if \( f(\xi', \eta') \leq A \),

\( f(\xi, \eta'') \leq A \) for \( \eta' \leq \eta \leq \eta'' \) implies that

\( f(\xi, \eta) \leq A \), then

\[ \max_{\xi} \min_{\eta} f(\xi, \eta) = \min_{\eta} \max_{\xi} f(\xi, \eta) \]

(Note a and b for \( 1 - \xi_1 - \ldots - \xi_{n-1} \) and \( 1 - \eta_1 - \ldots - \eta_{n-1} \)

respectively.)
The assertion can also be formulated in the following way: There exists a "saddle point" \( t_0, \eta_0 \) (0 \( \leq t_0 \leq a, 0 \leq \eta_0 \leq b \)), i.e., \( f(t_0, \eta) \) assumes its minimum for \( \eta = \eta_0 \) in \( 0 \leq \eta \leq b \) and \( f(t, \eta_0) \) assumes its maximum for \( t = t_0 \) in \( 0 \leq t \leq a \).

First of all, we evidently have

\[
\max_t \min_\eta f(t, \eta) \leq \min_\eta \max_t f(t, \eta)
\]

and secondly, the assertion just formulated implies that

\[
\max_t \min_\eta f(t, \eta) \geq \min_\eta f(t_0, \eta) = f(t_0, \eta_0)
\]

\[
\min_\eta \max_t f(t, \eta) \leq \max_t f(t, \eta_0) = f(t_0, \eta_0)
\]

denote:

\[
\max_t \min_\eta f(t, \eta) = \min_\eta \max_t f(t, \eta) = f(t_0, \eta_0)
\]

Our task is now to find a pair \( t_0, \eta_0 \) with the desired properties.

Let \( t \) be fixed. For which values of \( \eta, 0 \leq \eta \leq b \), does \( f(t, \eta) \) assume its minimum? The answer is easy. Since \( f \) is continuous, this set is closed, and because of the second assumption about \( f \) \( (f(t, \eta) \leq A, f(t, \eta) \leq A \) implies \( f(t, \eta) \leq A \) for all \( \eta' \leq \eta \leq \eta'' \) \) it is convex. But the only closed and convex sets of real numbers are closed intervals. Therefore, this set is a subinterval of the interval \( 0, b \); we call it \( K(t), K''(t) \).

If \( \eta \) is fixed, we conclude in the same way that those \( t \), \( 0 \leq t \leq a \), for which \( f(t, \eta) \) assumes its maximum form a closed sub-interval of \( 0, a \); we call it \( L'(\eta), L''(\eta) \).

Evidently, always \( K'(t) \leq K''(t) \) and \( L'(\eta) \leq L''(\eta) \). Furthermore, the continuity of \( f(t, \eta) \) implies that \( K'(t), L'(\eta) \) and \( K''(t), L''(\eta) \) are lower and upper semi-continuous functions respectively.\(^{10}\)

Let now \( t^* \) be fixed. We form the set of all \( t^{**} \) with the following property: There exists an \( \eta^* \) such that \( f(t^{**}, \eta) \) assumes its

\(^{10}\) Let us indicate the proof for \( K'(t) \). It will be the same for the other three functions. If \( K'(t) = 0 \), the assertion is trivial since always \( K'(t) \geq 0 \). Let \( K'(t) > 0 \). For \( 0 \leq \eta \leq K'(t) - \epsilon (\epsilon > 0) \), there exists a \( \delta > 0 \) such that \( f(t, \eta) \geq \min_\eta f(t, \eta) + \delta \), by the definition of \( K'(t) \). Hence, if \( t \) is sufficiently close to \( t^* \), we still have \( f(t, \eta) \geq \min_\eta f(t, \eta) + \delta/2 \) (because both \( f(t, \eta) \) and \( \min_\eta f(t, \eta) \) are continuous); i.e., \( f(t, \eta) \) does not assume its minimum (in \( \eta \), for \( 0 \leq \eta \leq b \)) in \( 0 \leq \eta \leq K'(t) - \epsilon \). Therefore \( K'(t) > K'(t) - \epsilon \), and \( K'(t) \) is lower semi-continuous as we have asserted.
minimal value (in \(0 \leq \eta \leq b\)) at \(\eta = \eta^*\) and \(f(\xi, \eta^*)\) assumes its
maximal value (in \(0 \leq \xi \leq a\)) at \(\xi = \xi^*\). I.e., we form the union of
all intervals \(L'(\eta^*) \leq \xi \leq L''(\eta^*)\) where \(\eta^*\) assumes all values in the
interval \(K'(\xi^*) \leq \eta^* \leq K''(\xi^*)\).

In the interval \(K'(\xi^*) \leq \eta \leq K''(\xi^*)\) the lower semi-continuous
function \(L'(\eta^*)\) assumes its minimum and the upper semi-continuous
function \(L''(\eta^*)\) assumes its maximum. Hence the set of \(\xi^*\) contains a
minimal as well as a maximal element. It also contains every intermediate
\(\xi'\), which can be demonstrated in the following way: If it were
not so, every interval \(L'(\eta^*), L''(\eta^*)\) would lie either entirely to the
left or entirely to the right of \(\xi'\), and both kinds would exist (those
belonging to the smallest as well as to the largest \(\xi^*\)). Since \(\eta^*\)
runs over an interval, both kinds of \(\eta^*\) would have a common limit-point.
Since both \(L'(\eta^*) \leq \xi'\) and \(L''(\eta^*) \geq \xi'\) occur arbitrarily close to \(\eta^*
\)
(and \(L', L''\) are lower and upper semi-continuous respectively), it follows
that \(L'(\eta^*) \leq \xi', L''(\eta^*) \geq \xi';\) i.e., \(\xi'\) belongs indeed to one of the
intervals, namely, to \(L'(\eta^*), L''(\eta^*)\).

Our \(\xi^*\) thus form a closed subinterval of \(0, a\), which we shall
call \(H'(\xi^*), H''(\xi^*)\). \(H'(\xi^*)\) is the minimum of the \(L'(\eta^*), H''(\xi^*)\)
the maximum of the \(L''(\eta^*),\) for \(K'(\xi^*) \leq \eta^* \leq K''(\xi^*)\). It is easy to
see that \(H'(\xi^*)\) and \(H''(\xi^*)\) are again lower and upper semi-continuous
respectively (this is implied by the corresponding properties of \(K'(\xi^*),
K''(\xi^*),\) and \(L'(\eta^*), L''(\eta^*)\)).

It remains to find a \(\xi^*(0 \leq \xi^* \leq a)\) which at the same time is
a \(\xi^*\), i.e., a \(\xi^*\) for which \(H'(\xi^*) \leq \xi^* \leq H''(\xi^*)\).

If no such \(\xi^*\) existed, every interval \(H'(\xi^*), H''(\xi^*)\) would
lie either entirely to the left or entirely to the right of \(\xi^*\), and both
kinds would exist (\(\xi^* = a\) and \(\xi^* = 0\)). Since \(\xi^*\) runs over an
interval, both kinds of \(\xi^*\) would have a common limit-point \(\xi'\). Since
arbitrarily close to \(\xi\) both \(H'(\xi^*) \leq \xi^*\) and \(H''(\xi^*) \geq \xi^*\) occur (and
\(H', H''\) are lower and upper semi-continuous respectively), it follows that
\(H'(\xi^*) \leq \xi', H''(\xi^*) \geq \xi';\) i.e., \(\xi'\) belongs indeed to the interval
\(H'(\xi^*), H''(\xi^*)\).

We have now proved our last assertion (and hence the assertion
(8)), which concludes the proof of our theorem.

§4. THE CASE \(n = 3\)

Having dealt in Sections 2 and 3 with the case \(n = 2\) we now
proceed to the next case, \(n = 3\).

Consider a three-person game characterized, according to the
description at the end of Section 1, by three functions $g_1, g_2, g_3$ of three variables $x, y, z (x = 1, 2, \ldots, \Sigma_1, y = 1, 2, \ldots, \Sigma_2,$ $z = 1, 2, \ldots, \Sigma_3)$, where identically

$$g_1 + g_2 + g_3 = 0$$.

In the case $n = 2$ it was possible to strictly determine the value of a play for each of the players $S_1$ and $S_2$ with the results

$$\text{value for } S_1 = \max_x \min_y \left( \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g(p, q) t_{pq} \eta_q \right) = \max_x \min_y \left( \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g_1(p, q) t_{pq} \eta_q \right)$$

$$\text{value for } S_2 = -\min_y \max_x \left( \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g(p, q) t_{pq} \eta_q \right) = \max_x \min_y \left( \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g_2(p, q) t_{pq} \eta_q \right)$$

where

$$\text{value for } S_1 + \text{value for } S_2 = 0$$.

Let us now try in the case $n = 3$ to compute the values of a play for the three players $S_1, S_2, S_3$. Assume these values to be $w_1, w_2, w_3$ respectively. For these values to be satisfactory under any conditions and without further discussion they clearly should have the following property: No two players must be able, by forming a coalition, to achieve an expected value exceeding the sum of the "values of a play" assigned to them. Furthermore, $w_1 + w_2 + w_3 = 0$ must hold, since the players make payments only to each other.

By putting

$$\max_x \min_y \left( \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} (g_1(pqr) + g_2(pqr)) t_{pq} \eta_r \right) = M_{1,2}$$

$$\max_x \min_y \left( \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} (g_1(pqr) + g_3(pqr)) t_{pq} \eta_r \right) = M_{1,3}$$

$$\max_x \min_y \left( \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} (g_2(pqr) + g_3(pqr)) t_{pq} \eta_r \right) = M_{2,3}$$

and it is easy to get an example is provided by

$$\Sigma_1 = \Sigma_2$$

among the $x, y, z$, notably by $x = y = \mu$, and $x = y = \nu$,

there will be $w_1, w_2, w_3$ all.

If now there is $w_1 = w_2 = w_3 = 0$,

the third

$\eta_\mu = \eta_\nu$

In this game, $S_\nu$ in coalition case couple" and thus $S_\nu$ in coalition case couple" and thus $\eta_\mu, \eta_\nu$ in coalition case couple"

$M_{1,2} + M_{1,3} + M_{2,3}$

The reason the fall is the following, necessary for any two players to hold a position to rob the rules of the game.
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... g_1, g_2, g_3 of

\[
\sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g_1(p, q) \xi_p \eta_q
\]

\[
\sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g_2(p, q) \xi_p \eta_q
\]

The \( \xi_p \) and \( \eta_q \) form a system of probabilities, as do the \( \eta_p \); similarly \( \xi_p \) and \( \eta_q \) for \( S_1 \) and \( S_2 \) by forming a coalition can play an ordinary
two-person game against \( S_3 \), thereby procuring for themselves (in
accordance with the above) the expected value \( M_{1,2} \). The same holds for
\( S_1 \) and \( S_3 \), and for \( S_2 \) and \( S_3 \) regarding the expected values \( M_{1,3} \)
and \( M_{2,3} \) respectively. Hence we must have

\[
w_1 + w_2 \geq M_{1,2} \quad w_1 + w_3 \geq M_{1,3} \quad w_2 + w_3 \geq M_{2,3}
\]

\[
w_1 + w_2 + w_3 = 0
\]

Clearly, this is possible if and only if

\[
M_{1,2} + M_{1,3} + M_{2,3} \leq 0
\]

As we are going to show in 2., it is always true that

\[
M_{1,2} + M_{1,3} + M_{2,3} \geq 0
\]

and it is easy to give examples in which the > sign holds. Such an ex-
ample is provided by the following three-person game:

\[
\Sigma_1 = \Sigma_2 = \Sigma_3 = 3. \text{ If among the } x_1, x_2, x_3 (i.e.,}
\text{ among the choices of } S_1, S_2, S_3, \text{ formerly also de-
noted by } x, y, z \text{ there are two for which } x_\mu = v, 
x_\nu = \mu, \text{ then } \mu, \nu \text{ form a "true couple". Clearly,}
\text{ there will be precisely one true couple, or none at all.}
\]

\text{If no "true couple" exists, let } g_1 = g_2 = g_3 = 0.
\text{If there is a "true couple", let it be } \mu, \nu \text{ and let}
\text{the third of the numbers } 1, 2, 3 \text{ be } \lambda. \text{ Let}
\text{ } g_\mu = g_\nu = 1, g_\lambda = -2.

In this game, evidently \( M_{1,2} = M_{1,3} = M_{2,3} = 2 \) (any two \( S_\mu, S_\nu \)
in coalition can choose \( \nu \) and \( \mu \) respectively to form a "true
couple" and thus take from the third player, \( S_\lambda \), the amount 2!). Hence
\( M_{1,2} + M_{1,3} + M_{2,3} = 6 > 0 \).

The reason that in this game any attempt at valuation is bound to
fail is the following: In order to gain the amount 2, it is only nec-
essary for any two of the three players to get together. They are then in
a position to rob the third one without any ado, in spite of the fact that
the rules of the game are strictly symmetrical, i.e., the game is formally
fair. The symmetry would imply the value for each player to be zero, but this is obviously wrong. If two players only want to, they can procure for themselves a gain of \(2\). How is this contradiction to be resolved?

2. Let us proceed systematically.

\[ M_{1,2} + M_{1,3} + M_{2,3} \geq 0 \]

always holds. For, evidently,

\[ M_{1,2} = \max_{\eta} \min_{\xi} \sum_{p=1}^{\xi_1} \sum_{q=1}^{\xi_2} \sum_{r=1}^{\xi_3} (g_1(pqr) + g_2(pqr)) \eta_{pq} \xi_r \]

(according to our theorem on two-person games)

\[ = \max_{\eta} \min_{\xi} \sum_{p=1}^{\xi_1} \sum_{q=1}^{\xi_2} \sum_{r=1}^{\xi_3} (g_1(pqr) + g_2(pqr)) \eta_{pq} \xi_r \]

We have to prove, therefore, that

\[ \max_{\eta} \min_{\xi} \sum_{p,q,r} g_3(pqr) \eta_{pq} \xi_r + \max_{\eta} \min_{\xi} \sum_{p,q,r} g_2(pqr) \eta_{pq} \xi_r \]

\[ + \max_{\eta} \min_{\xi} \sum_{p,q,r} g_1(pqr) \eta_{pq} \xi_r \leq 0 \]

This shows that our example is anything but a "pathological" game. Actually, it is a fairly frequent and typical case. Accordingly, we shall see in Section 4, 3 and Section 5, 1 that it is even the general case of a three-person game.

11 Intuitively, this is immediately clear. \(S_2\) and \(S_3\) in coalition against \(S_1\) can at best obtain \(M_{2,3}\), hence \(S_1\) for himself (against all others) can at best obtain \(-M_{2,3}\) (because of our theorem on two-person games). Likewise, \(S_2\) for himself can at best obtain \(-M_{1,3}\). \(S_1\) and \(S_2\) in coalition can at best obtain \(M_{1,2}\). "Unity is strength," i.e.,

\[ -M_{2,3} - M_{1,3} \leq M_{1,2} \]

\[ M_{1,2} + M_{1,3} + M_{2,3} \geq 0 \]
i.e., for all systems \( \tilde{\eta}, \tilde{\eta}', \tilde{\eta}'' \)

\[
\min \sum_{p, q, r} g_3(pqr)\tilde{\eta}'_{pq} + \min \sum_{p, q, r} g_2(pqr)\tilde{\eta}''_{pr} + \min \sum_{p, q, r} g_1(pqr)\tilde{\eta}'''_{qr} \leq 0.
\]

This is actually the case. For, if we put

\[
\tilde{\eta}'_{pq} = \tilde{\eta}'_{q} \tilde{\eta}'_{p}, \quad \tilde{\eta}''_{pr} = \tilde{\eta}''_{r} \tilde{\eta}''_{p}, \quad \tilde{\eta}'''_{qr} = \tilde{\eta}'''_{q} \tilde{\eta}'''_{r},
\]

we obtain (because of \( g_1 + g_2 + g_3 = 0 \))

\[
\sum_{p, q, r} g_3(pqr)\tilde{\eta}'_{pq} + \sum_{p, q, r} g_2(pqr)\tilde{\eta}''_{pr} + \sum_{p, q, r} g_1(pqr)\tilde{\eta}'''_{qr} = 0.
\]

We know already that the > sign actually occurs. The case of equality must, therefore, be regarded as a degenerate limiting case.

Let us assume that the player \( S_1 \) makes a claim on a gain of \( v_1 \) per play. How can he enforce it? Evidently, in two ways.

First of all, he can try to play alone. Essentially, this amounts to setting up a two-person game in which he is on one side and \( S_2, S_3 \) (in coalition) on the other. The value of a play is for him \(-M_{2,3}\). This solution is acceptable only if \( v_1 \leq -M_{2,3} \). Let us, therefore, assume \( v_1 > -M_{2,3} \).

Then only the second possibility remains. \( S_1 \) must try to get \( S_2 \) or \( S_3 \) as an ally. In coalition with \( S_2 \) or \( S_3 \) he can win the sum \( M_{1,2} \) or \( M_{1,3} \) per play. Since he wants to keep \( v_1 \) for himself, he can offer the sum \( M_{1,2} - v_1 \) to \( S_2 \), or \( M_{1,3} - v_1 \) to \( S_3 \) as the price of the alliance. It is out of the question, however, that \( S_2 \) or \( S_3 \) will accept this offer if in coalition with each other they can win more than \( (M_{1,2} - v_1) + (M_{1,3} - v_1) \) per play, i.e., if

\[
(M_{1,2} - v_1) + (M_{1,3} - v_1) < M_{2,3}, \quad v_1 > 1/2 (M_{1,2} + M_{1,3} - M_{2,3}).
\]

Hence we can say that \( S_1 \) has no hope at all of satisfying a claim \( v_1 \) which is

\[
> -M_{2,3}, \quad > 1/2 (M_{1,2} + M_{1,3} - M_{2,3}).
\]
The second number is \( \geq \) the first (because \( M_{1,2} + M_{1,3} + M_{2,3} \geq 0 \)), hence

\[
\bar{w}_1 \leq 1/2 (M_{1,2} + M_{1,3} - M_{2,3}) = \bar{w}_1
\]

must hold. In the same way, it can be shown that

\[
\begin{align*}
\bar{w}_2 & \leq 1/2 (M_{1,2} + M_{2,3} - M_{1,3}) = \bar{w}_2 \\
\bar{w}_3 & \leq 1/2 (M_{1,3} + M_{2,3} - M_{1,2}) = \bar{w}_3
\end{align*}
\]

must hold.

These upper limits for the claims \( \bar{w}_1, \bar{w}_2, \bar{w}_3 \) of the three players can easily be obtained. If, say, \( S_1, S_2 \) enter into a coalition, they can achieve the gain \( M_{1,2} = \bar{w}_1 + \bar{w}_2 \) (against \( S_3 \)). In the same way, \( S_1, S_3 \) or \( S_2, S_3 \) can, by entering an alliance, make sure of the gain \( M_{1,3} = \bar{w}_1 + \bar{w}_3 \) and \( M_{2,3} = \bar{w}_2 + \bar{w}_3 \) respectively. Hence, the highest possible and still completely justified claims of the three players \( S_1, S_2, S_3 \) in a play are \( \bar{w}_1, \bar{w}_2, \bar{w}_3 \) respectively.

3. How is this valuation compatible with the impossibility we found in 1. to make a general valuation? If \( M_{1,2} + M_{1,3} + M_{2,3} = 0 \), no difficulty arises. In this case

\[
\begin{align*}
\bar{w}_1 &= -M_{2,3}, \quad \bar{w}_2 = -M_{1,3}, \quad \bar{w}_3 = M_{1,2}
\end{align*}
\]

i.e., each player can push his claims all by himself, without the help of another (and in the face of a possible coalition of his opponents). All three players can satisfy their claims simultaneously, and accordingly

\[
\bar{w}_1 + \bar{w}_2 + \bar{w}_3 = 0
\]

The situation is different for \( M_{1,2} + M_{1,3} + M_{2,3} > 0 \). Since

\[
\bar{w}_1 > -M_{2,3}, \quad \bar{w}_2 > -M_{1,3}, \quad \bar{w}_3 > -M_{1,2}
\]

no player alone can satisfy his claim, and since

\[
\bar{w}_1 + \bar{w}_2 + \bar{w}_3 = 1/2 (M_{1,2} + M_{1,3} + M_{2,3}) > 0
\]

it is impossible for all three of them to obtain their desired gains at the same time. But because

\[
\begin{align*}
\bar{w}_1 + \bar{w}_2 &= M_{1,2} \\
\bar{w}_1 + \bar{w}_3 &= M_{1,3} \\
\bar{w}_2 + \bar{w}_3 &= M_{2,3}
\end{align*}
\]

each pair of players assured of its success, while the third player is respectively per pair of players justified claim by

This can be done if \( S_1, S_2, S_3 \) must equal the claims of the coalition, he receives

\[
1/2 (M_{1,2} + M_{1,3} + M_{2,3}) = \bar{w}_1 + \bar{w}_2 + \bar{w}_3
\]

respectively. If \( S_1, S_2 \) have a coalition, he receives

\[
1/3 (M_{1,2} + M_{1,3} + M_{2,3}) = \bar{w}_1 + \bar{w}_2 + \bar{w}_3
\]

respectively. Still the most concise one is

\[
\alpha) \quad \bar{v}_1 = 1/3 (M_{1,2} + M_{1,3} + M_{2,3})
\]

Since \( M_{1,2} + M_{1,3} + M_{2,3} \) no player alone can satisfy his claim, and since

\[
\bar{v}_1 > -M_{2,3}, \quad \bar{v}_2 > -M_{1,3}, \quad \bar{v}_3 > -M_{1,2}
\]

the result

\[
\bar{v}_1 + \bar{v}_2 + \bar{v}_3 = 1/2 (M_{1,2} + M_{1,3} + M_{2,3}) > 0
\]

(13) Incidentally
each pair of players having entered an alliance (to rob the third) is assured of its success. Both players can completely satisfy their claims, while the third player will receive only \(-M_{2,3}' - M_{1,3}' - M_{1,2}\) respectively per play, and therefore, his gain will fall short of his justified claim by the amount of \(1/2 (M_{1,2} + M_{1,3} + M_{2,3})\).

This can be formulated as follows: Each of the three players \(S_1, S_2, S_3\) must endeavor to ally himself with another player. If he succeeds, he receives per play

\[
\frac{1}{2} (M_{1,2} + M_{1,3} - M_{2,3}), \quad \frac{1}{2} (M_{1,2} - M_{2,3} - M_{1,3}), \quad \frac{1}{2} (M_{1,3} + M_{2,3} - M_{1,2})
\]

respectively. If he does not succeed (i.e., if the two others form a coalition), he receives only

\[-M_{2,3}' - M_{1,3}' - M_{1,2}\]

respectively. Still another way of describing the situation, and possibly the most concise one, would be the following:

a) A play has for the players \(S_1, S_2, S_3\) the respective "basic values"

\[
v_1 = \frac{1}{3} (M_{1,2} + M_{1,3} - 2M_{2,3}), \quad v_2 = \frac{1}{3} (M_{1,2} + M_{2,3} - 2M_{1,3}), \quad v_3 = \frac{1}{3} (M_{1,3} + M_{2,3} - 2M_{1,2})
\]

Since \(v_1 + v_2 + v_3 = 0\), this is a proper valuation.

b) But for each of any two players entering an alliance against the third there exists the possibility to win \(1/6\) \(D\) in excess of the above "basic values", while the third one sustains a loss of \(1/3\) \(D\) (also in excess of his "basic value"). We have

\[
D = M_{1,2} + M_{1,3} + M_{2,3} > 0 . \quad 13
\]

(The first case, \(D = M_{1,2} + M_{1,3} + M_{2,3} = 0\), can also be included in this formulation. Here, (a) is the result, and -- since \(D = 0\) -- (b) is vacuous.)

13 Incidentally

\[
v_1 = -M_{2,3} + 1/3 D, \quad v_2 = -M_{1,3} + 1/3 D, \quad v_3 = -M_{1,2} + 1/3 D
\]
This solution shows immediately that the three-person game is essentially different from a game between two persons. The actual game strategy of the individual player recedes into the background. It does not offer anything new since the formation of coalitions (which is bound to take place) makes the play a two-person game. But the value of a play for the player does not only depend on the rules of the game. Rather, it is a question -- at least as soon as \( D > 0 \) -- of which of the three equally possible coalitions \( S_1, S_2, S_3 \) has been formed. A new element enters, which is entirely foreign to the stereotyped and well-balanced two-person game: struggle.

§5. PRELIMINARY REMARKS ON GAMES FOR \( n > 3 \)

1. For \( n > 3 \) it has not yet been possible to obtain results of general validity. It may be that the best way to proceed will be in analogy to the cases \( n = 2, 3 \) which we have already dealt with. Let us recapitulate.

\[ n = 2 \text{. We define} \]
\[ M = \max_x \min_y \sum_{p, q} g_1(pq) p^x q^y \]

An individual play has for the players \( S_1, S_2 \) the values \( M_1 - M_2 \) respectively.

\[ n = 3 \text{. We define} \]
\[ M_{1,2} = \max_x \min_y \sum_{p, q, r} (g_1(pqr) + g_2(pqr)) p^x q^y r^z \]
\[ M_{1,3} = \max_x \min_y \sum_{p, q, r} (g_1(pqr) + g_3(pqr)) p^x q^y r^z \]
\[ M_{2,3} = \max_x \min_y \sum_{p, q, r} (g_2(pqr) + g_3(pqr)) p^x q^y r^z \]
\[ D = M_{1,2} + M_{1,3} + M_{2,3} \]

From this, \( n = 3 \) with \( D = 0 \), however, as was already of the same two types, even in this case. In particular, the possibilities of cooperation among all players. Possible asymmetric "basic values" of the type of forming coalitions are equally possible.

2. In order to conclude the following

\[ M_{(\mu)} = \sum_{p_{n=1}}^n (g_{\mu_1}(p_1, \ldots, \mu_n) + \sum_{p_{n=1}}^n (g_{\mu_1}(p_1, \ldots, \mu_n) \]

\[ \text{and similarly, all} \quad \eta_p \geq 0, \quad \eta_p^p = 1; \quad \text{etc.} \]

\[ \text{all} \quad \eta_p \geq 0, \quad \eta_p^p = 1; \quad \text{etc.} \]
D > 0, and we distinguish two cases, viz. D = 0
and D > 0.

D = 0. In this case, a play has for the players
S, S, S the values -M, -M, -M, respectively.
1 2 3 2 3 1

D > 0. In this case, a play has for the players
S, S, S the "basic values" -M + 1/3 D,
1 2 3 2 3 1
-M + 1/3 D, -M + 1/3 D respectively. In
1 2 3
order to obtain the correct values, however, another
term has to be added to the "basic values". This is
due to the fact that each of any two players who form
a coalition against the third (no matter which two)
can procure an additional gain of 1/6 D, while the
third player sustains a loss of 1/3 D per play
(in excess of his basic value).

From this summary it becomes clear that the cases n = 2 and
n = 3 with D = 0 are of the same type. The case n = 3, D > 0, how-
ever, as was already established at the end of Section 4, represents a new
type. We shall denote these two types by the terms strictly-determined
and symmetric non-strictly determined respectively, which are self-
explanatory.

Is there any chance of reducing all games of strategy to these
same two types, even if n > 3? Or are new complications to be anticipated?
In particular, the possibility of asymmetrical non-strictly determined
types has to be contemplated, i.e., types for which the significant
possibilities of coalition-formation are not symmetrically distributed
among all players. For n = 3 this possibility cannot arise. Any
possible asymmetries of the rules of the game are entirely absorbed by the
"basic values" of the three players. But all players are equally capable
of forming coalitions, all three coalitions S, S, S; S, S, S; S, S, S; S, S, S
1 2 3 1 2 3 1 2 3 1 2 3
are equally possible. Let us investigate this question somewhat more closely.

2. In order to characterize a general n-person game we intro-
duce the following constants:

\[ M(\mu_1, \mu_2, \ldots, \mu_k) = \max_i \min_j \sum_{P_1=1}^{L_1} \sum_{P_2=1}^{E_2} \cdots \]

\[ \sum_{n=1}^{E_n} (g_{\mu_1}(p_1, \ldots, p_n) + \cdots + g_{\mu_k}(p_1, \ldots, p_n)) p_{\mu_1} \cdots p_{\mu_k} p_{v_1} \cdots p_{v_{n-k}} \]
where \( \mu_1, \mu_2, \ldots, \mu_k \) are any \( k \) distinct numbers among the numbers 1, 2, \ldots, \( n \), and \( \nu_1, \nu_2, \ldots, \nu_{n-k} \) are the remaining numbers (Max is to be taken for all \( \sum p_{\mu_1} \cdots p_{\mu_k} \geq 0 \), \( \sum p_{\nu_1} \cdots p_{\nu_{n-k}} = 1 \), and \( \min \) for all \( \sum p_{\nu_1} \cdots p_{\nu_{n-k}} \geq 0 \), \( \sum p_{\nu_1} \cdots p_{\nu_{n-k}} = 1 \)). Clearly, \( M(\mu_1, \ldots, \mu_k) \) is the sum per play which the coalition of the players \( S_{\mu_1}, \ldots, S_{\mu_k} \) is able to obtain from the coalition of the players \( S_{\nu_1}, \ldots, S_{\nu_{n-k}} \) (since, in fact, the game is a two-person game).

Evidently, \( M(\) = 0. Our theorem on two-person games further implies that \( M(\mu_1, \ldots, \mu_k) = -M(\nu_1, \ldots, \nu_{n-k}) \). Finally, let \( \mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_{n-k} \) be three subsets of 1, 2, \ldots, \( n \), complementary to each other. If the players \( S_{\mu_1}, \ldots, S_{\mu_k} \) as well as \( S_{\nu_1}, \ldots, S_{\nu_{n-k}} \) and \( S_{\rho_1}, \ldots, S_{\rho_{n-k}} \) form fixed coalitions, this is a three-person game, and we have (putting primes on the quantities involved in this kind of game)

\[
M_{1,2}' = M(\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_{n-k})
\]
\[
M_{1,3}' = M(\mu_1, \ldots, \mu_k, \rho_1, \ldots, \rho_{n-k}) = -M(\nu_1, \ldots, \nu_{n-k})
\]
\[
M_{2,3}' = M(\nu_1, \ldots, \nu_{n-k}, \rho_1, \ldots, \rho_{n-k}) = -M(\mu_1, \ldots, \mu_k).
\]

But according to our results for three-person games we have

\[
M_{1,2}' + M_{1,3}' + M_{2,3}' \geq 0
\]

i.e.,

\[
M(\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_{n-k}) \geq M(\mu_1, \ldots, \mu_k) + M(\nu_1, \ldots, \nu_{n-k}).
\]

We recapitulate:

A given \( n \)-person game assigns to each subset \( \mu_1, \mu_2, \ldots, \mu_k \) of 1, 2, \ldots, \( n \) a constant \( M(\mu_1, \ldots, \mu_k) \) (that is the sum per play which the coalition of the players \( S_{\mu_1}, \ldots, S_{\mu_k} \) is able to obtain from the coalition of the other players). The system of the constants \( M(\mu_1, \ldots, \mu_k) \) always satisfies the following:

1. \( M(\) = 0.
2. \( M(\mu_1, \ldots, \mu_k) \) is non-negative.
3. \( M(\mu_1, \ldots, \mu_k) \) is determined by the following system of \( n \) equations:

\[
\sum_{\nu_1} p_{\nu_1} \cdots p_{\nu_{n-k}} = 1,
\]

for each system of numbers \( \mu_1, \ldots, \mu_k \). It is not possible that \( M(\) = 0.

It is not possible that every possible coalition of \( n \) players is "three against one," since we have the above condition from discussing how the game is modified when \( n \) is increased.

If our system is to be consistent, we must add to each equation for \( M(\) an arbitrary constant \( M(\) to satisfy the condition of consistency.

Intuitively, footnote 12, p. 3

For \( n = 2 \)

\[
M(\) = 0, \ M(1,2)
\]

and for \( n = 3 \)

\[
M(\) = 0, \ M(1,2,3)
\]
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the following three conditions:

1. \[ M(\emptyset) = 0 \]

2. \[ M(\{\mu_1, \ldots, \mu_k\} + M(\nu_1, \ldots, \nu_{k-n}) = 0 \text{ if } \mu_1, \ldots, \mu_k \]
   \[ \text{and } \nu_1, \ldots, \nu_{k-n} \text{ are complementary subsets of } \]
   \[ 1, 2, \ldots, n. \]

3. \[ M(\{\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_j\} \geq M(\nu_1, \ldots, \nu_j) \]
   \[ \text{if } \mu_1, \ldots, \mu_k \text{ and } \nu_1, \ldots, \nu_j \text{ are disjoint } \]
   \[ \text{subsets of } 1, 2, \ldots, n. \]

It is not difficult to prove the converse, i.e., to specify, for each system of numbers \( M(\emptyset) \) \( (\emptyset \) ranging over all \( 2^n \) subsets of \( 1, 2, \ldots, n) \) satisfying the conditions 1. to 3., a game of strategy for which the above constants have precisely these values \( M(\emptyset) \). We refrain from discussing here such an example, which is not deep at all.

3. I venture the conjecture that the complex of valuations and coalitions in a game of strategy is determined by these \( 2^n \) constants alone. We have seen that this is true for \( n = 2, 3 \); \( n > 3 \) a general proof has yet to be found. While in the case \( n = 2 \) no coalition at all is possible and for \( n = 3 \) only one type of coalition is conceivable (i.e., "two against one"), the number of possibilities increases rapidly for \( n > 3 \). If \( n = 4 \) one must already decide whether a coalition "three against one" or "two against two" is going to be formed, i.e., which alliance offers the best chances to the participants. If \( n = 4 \) it is still possible to discuss the principal cases (on the basis of the \( M(\emptyset) \) alone!), but a satisfactory general theory is as yet lacking.

If our conjecture is correct, we have brought all games of strategy into a natural and final normal form. Each system of \( 2^n \) constants \( M(\emptyset) \) satisfying the conditions 1. to 3. represents a class of

---

\( ^{15} \) Intuitively, this assertion is as clear as the one considered in
the footnote 12, p. 34.

\( ^{16} \) For \( n = 2 \)

\[ M(\emptyset) = 0, \quad M(1) = M, \quad M(2) = -M, \quad M(1, 2) = 0 \]

and for \( n = 3 \)

\[ M(1) = 0, \quad M(1) = -M(2), \quad M(2) = -M(3), \quad M(3) = -M(1, 2), \quad M(1, 2) = M(1), \quad M(1, 3) = M(2, 3) = M(1, 2, 3) = 0. \]
"tactically equivalent" games of strategy.\footnote{Another possibility of normalizing the $M_{(w)}$ consists of introducing "basic values" $v_1, v_2, \ldots, v_n$ for the players $S_1, S_2, \ldots, S_n$ -- in analogy to the "values" (of a play for $n = 2$) and the "basic values" for $n = 3$. For the values exceeding the $v_1$ one obtains, of course, the new constants

$$M^*_i = M_{(w)} - \sum_{p \in w} v_p$$

Appropriately, the $v_p$ are chosen such that

$$M^*_1 = M^*_2 = \ldots = M^*_n, \quad v_1 + v_2 + \ldots + v_n = 0$$

i.e., all players playing for themselves are equally strong, and differences are due only to the various possibilities of coalitions.

(From 1. - 3. it follows easily that the common value of

$$M^*_1, M^*_2, \ldots, M^*_n \leq 0$$

If it is zero, all $M^*_{(w)} = 0$, i.e., after payment of the "basic values" the play is strictly determined. Hence this common value represents a kind of measure of how non-strictly determined the game is, i.e., a measure of the tactical possibilities the game offers.)

We consider (where $I$ is a core function of an essential characteristic function in our case; see \cite{1}, p. 264).

We shall work with countably many $\Sigma g_i = 0$. The conclusion 

\footnote{The preparation for the Naval Research.}