15

ON AN EFFECTIVE METHOD OF SOLVING EXTREMAL PROBLEMS FOR QUADRATIC FUNCTIONALS *

We consider here a method of successive approximations applicable to a wide class of minimization problems for quadratic functionals. This method may be used for actual solution of these problems as well as for their qualitative study (existence theorems, properties of solutions, etc.).

Let us outline the general idea of the method. A functional $I(f)$ on a linear metric space is considered. $I(f)$ is assumed to be quadratic, that is $I(f+\varepsilon g)$ is a polynomial of the second degree in ε. Given a fixed element F_0, we choose the direction of the gradient of $I(f)$ at $f = f_0$, i.e., the element $g = g_1$ such that $[d/d\varepsilon I(f+\varepsilon g)]_\varepsilon = O||g||$ has an extremal value (the space can be, for instance, of type F, [1]). Further, we obtain $\varepsilon = \varepsilon_1$ from the condition that $I(f_0 + \varepsilon g_1)$ should attain an extremum, and then the process is to be reoeated again and again.

Let us consider the application of the method to some concrete cases.

1. ***Systems of linear algebraic equations.*** Let a system

$$\sum_{k=1}^{n} a_{ik}x - k = b_i, \quad i = 1, 2, \ldots, n \tag{1}$$

be given. Consider the quadratic form

$$H(X) = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik}x_k - b_i \right)^2 \tag{2}$$

that attains its minimum equal to zero at $X = \{x_i\}$, the solution of (1). Suppose that $x_i = x_i^0$ are chosen as initial values of the x's. We take

$$H(X^{(0)} + \varepsilon Z) = \sum_i \left[\sum_k a_{ik}(x^0 + \varepsilon z_k) - b_i \right]^2 =$$

L. V. Kantorovich, Ob odnom effektivnom metode dlya resheniya ekstremal'nykh zadach dlya kvadratichnykh funktsionalov, Doklady AN SSSR, 1945, 48:7, pp. 595–600.

$$= H(X^{(0)}) + 2\varepsilon \sum_i \left(\sum_k r_i^{(1)} a_{ik}\right) z_k + \sum_i \left(\sum_k a_{ik} z_k\right)^2, \tag{3}$$

where

$$r_i^{(1)} = \sum_k a_{ik}^{(0)} x_k - b_i.$$

The coefficient 2ε is maximal when $\sum z_k^2 = const$ if the z_k are proportional to their own coefficients. We simply put

$$\sum_i \left(\sum_k a_{ik} r_i^{(1)}\right)^2. \tag{4}$$

To minimize $H(X^{(0)} + \varepsilon Z)$, we set

$$\varepsilon = \varepsilon^{(1)} = \frac{\sum_k z_k^{(1)2}}{\sum_i \left(\sum_k a_{ik} z_k^{(1)}\right)^2}. \tag{5}$$

Now, we put

$$r_i^{(2)} = \sum_k a_{ik} (x_k^{(0)} + \varepsilon^{(1)} z_k^{(1)})^2 - b_i = r_i^{(1)} + \varepsilon^{(1)} \sum_k a_{ik} z_k^{(1)} \tag{6}$$

and then calculate successively $z_k^{(2)}$, $\sum_k a_{ik} z_k^{(2)}$, $\varepsilon^{(2)}$, $r_i^{(3)}$, etc. by formulas analogous to (4), (5), and (6). The solution of (1) is

$$x_i = x_i^{(0)} + \varepsilon^{(1)} z_i^{(1)} + \varepsilon^{(2)} z_i^{(2)} + \cdots. \tag{7}$$

Computations may be arranged in a simple scheme and can easily be carried out on calculators. The formulas and computations can be considerably simplified when the matrix $\|a_{ik}\|$ is symmetric.

The method of successive approximations just considered converges (i) to the solution if the latter is unique, (ii) to one of the solutions if there are several, (iii) to a solution of the least squares problem if there is no solution. In particular, it is convenient to apply the method to normal systems in using the Gauss least squares method. The order of convergence equals that of a geometrical progression (usually with a small denominator).

Let us note a simple geometric interpretation of the method. Namely, given a family of similar ellipsoids $H(X) = const$, we draw the normal to the ellipsoid which passes through the initial point $X^{(0)}$. Then we choose another ellipsoid which touches the normal at point $X^{(1)} = X^{(0)} + \varepsilon^{(1)} Z^{(1)}$, draw the normal at the point, and so on.

We notice finally that the method may be applied to infinite systems as well.

2. ***Fredholm integral equations.*** For the sake of simplicity, we consider only the case of symmetric kernels. The solution of the equation

$$L(\phi) = \phi(x) - \lambda \int_a^b K(x,y)\phi(y)\,dy - h(x) = 0 \tag{8}$$

extremizes the integral

$$H(\phi) = \int_a^b \phi^2(x)\,dx - \lambda \int_a^b \int_a^b K(x,y)\phi(x)\phi(y)\,dx\,dy - 2\int_a^b \phi(x)h(x)\,dx \tag{9}$$

provided $\lambda/\lambda_k < 1$, $k = 1, 2, \ldots$.

Starting from an arbitrary $\phi_0(x)$, we construct successive approximations in the same way as above; we put

$$\phi_1(x) = \phi_0(x) - \lambda \int_a^b K(x,y)\phi_0(y)\,dy - h(x), \tag{10}$$

$$\varepsilon_1 = -\int_a^b \phi_1^2(x)\,dx \left(\int_a^b \phi_1^2(x)\,dx - \lambda \int_a^b \int_a^b K(x,y)\phi_1(x)\phi_1(y)\,dx\,dy\right)^{-1} \tag{11}$$

The following functions $\phi_k(x)$ and numbers ε_k are defined analogously. The solution $\overline{\phi}(x)$ is given by the formula:

$$\overline{\phi}(x) = \phi_0(x) + \varepsilon_1\phi_1(x) + \cdots. \tag{12}$$

If $\phi(x)$ is a solution of (8), which by necessity has a solution, if λ is not an eigenvalue, then, as is easily seen,

$$H(\phi_0) - H(\phi_0 + \varepsilon_1\phi_1) =$$

$$= -\left(\int_a^b \phi_1^2(x)\,dx\right)^2 \left(\int_a^b \phi_1^2(x)\,dx - \lambda \int_a^b \int_a^b K(x,y)\phi_1(x)\phi_1(y)\,dx\,dy\right)^{-1}$$

$$H(\phi_0) - H(\overline{\phi}) =$$

$$== \int_a^b \eta^2(x)\,dx - \lambda \int_a^b \int_a^b K(x,y)\eta(x)\eta(y)\,dx\,dy; \quad \eta(x) = \phi_0(x) - \overline{\phi}(x).$$

Hence, if the α_k are the Fourier coefficients of $\phi_1(x)$ with respect to the eigen-functions,

$$[H(\phi_0) - H(\phi_0 + \varepsilon_1\phi_1)]/[H(\phi_0) - H(\overline{\phi})] =$$

$$= \left(\sum \alpha_k^2\right)^2 \left[\left(\sum \left(1 - \frac{\lambda}{\lambda_k}\right)\alpha_k\right)\left(\sum \left(1 - \frac{\lambda}{\lambda_k}\right)^{-1} \alpha_k^2\right)\right]^{-1}.$$

It is evident that if α and β are any numbers satisfying the inequalities

$$0 < \alpha \leq 1 - \frac{\lambda}{\lambda_k} \leq \beta < +\infty,$$

we have

$$H(\phi_0) - H(\phi_0 + \varepsilon_1 \phi_1) \geq \frac{\alpha}{\beta}[H(\phi_0) - H(\overline{\phi})].$$

We see that the functional tends to its minimum with the rate of a geometric progression. This implies that series (12) converges as rapidly (more accurately: the quotient of the progression does not exceed $(\beta - \alpha)^2/(\beta + \alpha)^2$).

The method is also applicable to the case where λ is an eigenvalue, when it is required to find the eigenfunctions and the eigenvalues. It can also be applied to equations with non-symmetric kernels, as well as to the case in which the condition $\lambda/\lambda_k < 1$ is not fulfilled. In the latter case we must consider $\int [L(\phi)]^2 \, dx$ instead of $H(\phi)$. Finally, the method can be used in solving equations of the first kind.

3. ***Ordinary differential equations.*** Consider the equation

$$L(y) = \frac{d}{dx}(p(x)y) - q(x)y - f(x) = 0, \qquad y(a) = y(b) = 0. \tag{13}$$

This equation is associated with the minimization problem for the integral

$$I(y) = \int_a^b [p(x)y'^2 + q(x)y^2 + 2f(x)y] \, dx. \tag{14}$$

Taking a function $z(x)$ that satisfies together with $y_0(x)$ the conditions $z(a) = z(b) = 0$, we easily obtain

$$I(y_0 + \varepsilon z) = I(y_0) + 2\varepsilon \int_a^b \left[p y_0' - \int (q y_0 + f) \, dx \right] z' \, dx + \varepsilon^2 \int_a^b (p z'^2 + q z^2) \, dx.$$

To find the value $z = z_1$ extremalizing the multiplier of 2ε under the condition $\int z'^2 \, dx = const$, we put

$$z_1' = p'y' - \int (q y_0 + f) \, dx + C \quad \text{or} \quad z_1'' = L(y_0). \tag{15}$$

With the value z_1 obtained from this equation, in accordance to the conditions $z(a) = z(b) = 0$ we put:

$$\varepsilon_1 = - \left(\int z'^2 \, dx \right) \left(\int [p(x) z'^2] q(x) z^2 \, dx \right)^{-1} \tag{16}$$

and, finally,

$$y_1(x) = y_0(x) + \varepsilon_1 z_1(x). \tag{17}$$

Further approximations can be determined in a similar way. For the case in which $p(x) > 0$, $q(x) \geq 0$ the following inequality, analogous to that in the point **2** holds

$$I(y_1) - I(y_0) \geq \frac{p_{\min}}{p_{\max} + \frac{(b-a)^2}{\pi^2} q_{\max}} I(y_0) - I(\overline{y}), \tag{18}$$

where $\overline{y}$ is a solution. Hence, it follows that the approximations converge to the solution and the convergence has the rate of a progression.

It is possible to apply the method to equations of higher orders as well as to other boundary problems. The method can also serve as a basis for various grapho-analytical methods. Thus beam designers will find it useful in reducing the design of trussed beams to a repeated calculation of simple beams.

4. ***Boundary problems for partial differential equations.*** Consider, for instance, the Dirichlet problem for the self-adjoint elliptic equation

$$L(u) = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right) - cu - f = 0, \quad u = \phi(s) \text{ on } \Gamma \tag{19}$$

The minimization problem for the integral

$$I(u) = \iint_D \left[a\left(\frac{\partial u}{\partial x}\right)^2 + b\left(\frac{\partial u}{\partial y}\right)^2 + cu^2 + 2fu\right] dx\, dy \tag{20}$$

is associated with (19). We have

$$I(u_0 + \varepsilon\eta) = I(u_0) - 2\varepsilon \iint_D L(u_0)\eta\, dx +$$

$$+\varepsilon^2 \iint_D \left[a\left(\frac{\partial \eta}{\partial x}\right)^2 + b\left(\frac{\partial \eta}{\partial y}\right)^2 + c\eta^2\right] dx\, dy.$$

It is natural to put

$$\|\eta\| = \iint_D \left[a\left(\frac{\partial \eta}{\partial x}\right)^2 + b\left(\frac{\partial \eta}{\partial y}\right)^2\right] dx\, dy.$$

Then, under the condition $\|\eta\| = \mathit{const}$, the multiplier of 2ε attains its extremum when $\eta = \eta_1$ is replaced by the solution of the equation

$$\Delta\eta_1 = L(u), \quad \eta_1 = 0 \quad \text{on} \quad \Gamma. \tag{21}$$

As to $I(u_0 + \varepsilon\eta)$, it is minimized by $\varepsilon = \varepsilon_1$, where

$$\varepsilon_1 = \iint_D L(u_0)\eta_1\, dx\, dy \left(\iint_D \left[a\left(\frac{\partial \eta}{\partial x}\right)^2 + b\left(\frac{\partial \eta}{\partial y}\right)^2 + c\eta^2\right] dx\, dy\right)^{-1} \tag{22}$$

Then $u_1 = u_0 + \varepsilon_1\eta_1$ will be the first approximation to u. The convergence of the process can also be proved under certain conditions.

Here the practical use of the method is hampered by the necessity of solving the Poisson equation (when determining η) in each step of the process. Thus the Green's function in the domain D is required. But seeing that the domain can be transformed into a circle (or a sphere in the case of spatial transformations) by suitably changing variables without altering the type of the equation and that the Green's function of the circle is elementary, it appears that our method may also find practical application in a number of cases.

On its basis different numerical, graphical and experimental methods can be developed. It can also be applied to other types of equation and to other graphical problems.

5. ***Functional equations in Hilbert space.*** Let H be a self-adjoint positive definite operator defined on a linear manifold R_1 in a Hilbert space R. Let T be an analogous operator which takes R_1 onto R and possesses the inverse T^{-1} on R. We suppose H to be bounded with respect to T, that is, $0 < \alpha(Tf, f) \leq (Hf, f) \leq \beta(Tf, f)$. Let us find a solution $f \in R_1$ of the equation

$$Hf - \phi = 0. \tag{23}$$

A solution $\overline{f}$ of this equation minimizes the quadratic functional

$$I(f) = (Hf, f) - 2(f, \phi). \tag{24}$$

Let $g \in R_1$ and $f_0 \in R$ be the initial value of f. Then

$$I(f_0 + \varepsilon g) = I(f_0) + 2\varepsilon(Hf_0 - \phi, g) + \varepsilon^2(Hg, g).$$

The second summand on the right-hand side attains its maximum under the condition $(Tg, g) = const$ when $Hf_0 - \phi$ is proportional to Tg, in other words, when g is proportional to

$$g_1 = T^{-1}(Hf_0 - \phi). \tag{25}$$

Having chosen the element g_1, we minimize I by equating ε_1 to

$$\varepsilon_1 = -\frac{(Hf_0 - \phi, g_1)}{(Hg_1, g_1)} = -\frac{(Tg_1, g_1)}{(Hg_1, g_1)}.$$

Now put $f_1 = f_0 + \varepsilon_1$. Then we obtain

$$\frac{I(f_1) - I(f_0)}{I(f_0) - I(\overline{f})} = \frac{(Tg_1, g_1)^2}{(Hg_1, g_1)\frac{1}{\alpha}(Hf_0 - H\overline{f}, f_0 - \overline{f})} =$$

$$= \frac{(Tg_1,g_1)^2(Hf_0 - H\overline{f}, f_0 - \overline{f})}{(Hg_1,g_1)(Tg_1, f_0 - \overline{f})^2} \geq \frac{(Tg_1,g_1)^2(Hf_0 - H\overline{f}, f_0 - \overline{f})}{(Hg_1,g_1)(Tg_1,g_1)(Tf_0 - T\overline{f}, f_0 - \overline{f})} \geq$$

$$\geq \frac{(Tg_1,g_1)(Hf_0 - H\overline{f}, f_0 - \overline{f})}{(Hg_1,g_1)\frac{1}{\alpha}(Hf_0 - H\overline{f}, f_0 - \overline{f})} \geq \frac{\alpha}{\beta}.$$

It is evident that $I(f_n)$ tends to $I(f)$ with the rate of a geometrical progression. Hence, we may conclude that $f_n \to f$ in the sense of the norm $\|g\| = (Tg, g)$.

It is possible to apply the method to non-quadratic functionals (for instance, to systems of non-linear algebraic equations). In such cases, however, the determination of ε becomes essentially more difficult, and we have to content ourselves with approximate values of ε.

It may be remarked in conclusion that this method, though developped here independently, is connected with the author's general concepts regarding extremal problems. These ideas are briefly outlined in [2].

References

[1] S. Banach. *Théorie des opérations linéaires*, Warszawa, 1932.

[2] L. V. Kantorovich. *Doklady AN SSSR*, 1940, 28, pp. 212–215. — See Paper 3 of Volume III.