Méthode générale pour la résolution des systèmes d'équations simultanées*

M. Augustine Cauchy

Comptes Rendus Hebd. Séances Acad. Sci. 25, 536–538.
OC I, 10 (383), 399–402.

Being given a system of simultaneous equations that the concern is to resolve, one begins ordinarily by reducing them to a single one, by aid of successive eliminations, save to resolve definitely, if it is able, the resulting equation. But it is important to observe, 1° that, in a great number of cases, the elimination is not able to be effected in any manner; 2° that the resulting equation is generally very complicated, even though the given equations are rather simple. For these two motives, one imagines that it would be very useful to understand a general method which may be able to serve to resolve directly a system of simultaneous equations. Such is that which I have obtained, and of which I am going to say some words here. I will limit myself for the moment to indicate the principles on which it is founded, proposing to myself to return with more details on the same subject, in a forthcoming Memoir.

Let first

$$u = f(x, y, z)$$

be a function of many variables x, y, z,... which never become negative and which remain continuous, at least between certain limits. In order to find the values of x, y, z,..., which will verify the equation

$$u = 0, \tag{1}$$

it will suffice to make decrease indefinitely the function u, until it vanishes. Now let

$$\mathrm{x},\ \mathrm{y},\ \mathrm{z},\ \ldots$$

be particular values attributed to the variables x, y, z,...; u the value corresponding to u; X, Y, Z, ... the values corresponding to $D_x u, D_y u, D_z u, \ldots$, and $\alpha, \beta, \gamma, \ldots$ some very small increments attributed to the particular values x, y, z, ... When one will put

$$x = \mathrm{x} + \alpha, \quad y = \mathrm{y} + \beta, \quad z = \mathrm{z} + \gamma, \ldots,$$

one will have sensibly

$$u = f(\mathrm{x} + \alpha, \mathrm{y} + \beta, \ldots) = \mathrm{u} + \alpha\mathrm{X} + \beta\mathrm{Y} + \gamma\mathrm{Z} + \cdots \tag{2}$$

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. July 17, 2010

[537] We imagine now that, θ being a positive quantity, one takes

$$\alpha = -\theta \mathrm{X}, \quad \beta = -\theta \mathrm{Y}, \quad \gamma = -\theta \mathrm{Z}, \ldots$$

Formula (2) will give sensibly

$$f(\mathrm{x} - \theta\mathrm{X}, \mathrm{y} - \theta\mathrm{Y}, \mathrm{z} - \theta\mathrm{Z}, \ldots) = \mathrm{u} - \theta(\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 \cdots). \tag{3}$$

It is easy to conclude from it that the value Θ of u, determined by the formula

$$\Theta = f(\mathrm{x} - \theta\mathrm{X}, \mathrm{y} - \theta\mathrm{Y}, \mathrm{z} - \theta\mathrm{Z}, \ldots), \tag{4}$$

will become inferior to u, if θ is sufficiently small. If, now, θ comes to increase, and if, as we have supposed, the function $f(x,y,z,\ldots)$ is continuous, the value Θ of u will decrease until it vanishes, or at least until it coincides with a *minimum* value, determined by the equation in a single unknown

$$D_\theta \Theta = 0. \tag{5}$$

It will suffice therefore, either to resolved this last equation, or at least to attribute to θ a sufficiently small value, in order to obtain a new value of u inferior to u. If the new value of u is not a *minimum*, one will be able to deduce, by operating always in the same manner, a third still smaller value; and, by continuing thus, one will find successively some values of u more and more small, which will converge toward a minimum value of u. If the function u, which is supposed not at all to admit negative values, offers some null values, there will be able always to be determined by the preceding method, provided that one chooses conveniently the values of x, y, z, ….

It is good to observe that, if the particular value of u represented by u is already very small, one will be able ordinarily to deduce another value Θ much smaller, by equating to zero the second member of formula (3), and by substituting the value that one will obtain thus for θ, namely

$$\theta = \frac{\mathrm{u}}{\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2 \cdots}, \tag{6}$$

in the second member of formula (4).

We suppose now that the unknowns $x,y,z,\ldots$ must satisfy no longer a single equation, but a system of simultaneous equations

$$u = 0, \quad v = 0, \quad w = 0, \ldots, \tag{7}$$

[538] of which the number will be able to surpass even the one of the unknowns. In order to restore this last case to the preceding, it will suffice to substitute into the system (7) the unique equation

$$u^2 + v^2 + w^2 + \cdots = 0. \tag{8}$$

When, by aid of the method that we just indicated, one will have determined some already well approximated values of the unknowns $x,y,z,\ldots$, one will be able, if one wishes, to obtain new very rapid approximations by aid of the linear or Newtonian method, of which I have made mention in the Memoir of 20 September.

One is able to draw from the principles exposed here a very advantageous part for the determination of the orbit of a star, by applying them no longer to the differential equations, but to the finite equations which represent the movement of this star, and by taking for unknowns the same elements of the orbit. Then the unknowns are in number six. But the number of equations to resolve is more considerable, some of among them serving to define some implicit functions of the unknowns; and besides the number of equations increases with the number of observations that one wishes to make agree to the solution of the problem. We add that the only numbers which enter into the equations to resolve are the longitudes, latitudes, etc., furnished by the observations themselves. Now these longitudes, latitudes, etc., are always more exact than their derivatives relative to time, which enter into the differential equations. Therefore, after having obtained by aid of the differential equations, just as we have explicated in the preceding Memoirs, some approximate values of the unknowns, one will be able, by departing from these approximate values, and resolving them, as we just said, the finite equations of the movement of the star, to obtain a very great precision in the results of the calculation.