ON NON-COOPERATIVE GAMES WITH RESTRICTED DOMAINS OF ACTIVITIES* ‡ †

WU WEN-TSÜN (吳文俊)
(Institute of Mathematics, Academia Sinica)

I. INTRODUCTION

Let Γ be an n-person game with strategy space S_i and payoff function $H_i(x_1,\cdots,x_n)$, $x_i \in S_i$, for the player "i", $i=1,\cdots,n$. Let S_i^* be a set of "mixed strategies" of player "i", and $H_i^*(\mu_1,\cdots,\mu_n)$, $\mu_i \in S_i^*$, the corresponding mathematical expectations of "i". A system $\mu^*=(\mu_1^*,\cdots,\mu_n^*)$ is then, according to Nash[5], an *equilibrium point* of the game $\Gamma=\langle I,\{S_i\},\{H_i\}\rangle$ where $I=\{1,2,\cdots,n\}$ is the set of players, if

$$H_i^*(\mu_1^*,\cdots,\mu_n^*) \geqslant H_i^*(\mu_1^*,\cdots,\mu_i,\cdots,\mu_n^*),\quad i=1,\cdots,n,$$

for every $\mu_i \in S_i^*$, $i=1,\cdots,n$. In other words, the choice of strategy μ_i^* for player "i" is such that no tendency to alternate his strategy is required in order to increase his expectations so far as the other players stick to their choices. It is a fundamental theorem of the theory of games due to Nash[5] that equilibrium points exist always if each S_i is a finite set and S_i^* is the set of all possible mixed strategies. It has been generalized by Glicksberg[3] that the theorem remains true in the case that S_i are all bicompact Hausdorff spaces, that S_i^* are the sets of all regular probability measures defined over the σ-field of all Borel sets of S_i, and that H_i are all continuous functions over the product space $S_1\times\cdots\times S_n=S$.

In the case of Nash and Glicksberg the choice of (mixed) strategies as well as their alternations is thus quite arbitrary. It seems to be more realistic to suppose that both the choice and the alternations of strategies are restricted in some manner. The object of the present paper is to investigate the existence of equilibrium points of such games with restricted domains of activities of which the precise definition will be given in VIII. As we shall show in this paper, the equilibrium points of such games may be non-existent and moreover,

* Received May 15, 1960.
‡ First published in Chinese in *Acta Mathematica Sinica*, Vol. XI, No. 1, pp. 47–62, 1961.
† A facsimile of the translation in Scientia Sinica, Vol. X, No. 5, 1961, 387–409.

it is rather the interrelations between the various domains of restrictions than the strategy spaces themselves which are responsible for the existence of equilibrium points. Thus the existence of equilibrium points is ensured in the case of Nash-Glicksberg just because the domains of restrictions are extremely simple, viz., the single domain consists of the whole strategy space for each player, though the strategy spaces may themselves be arbitrary bicompact Hausdorff spaces.

The proof of our Main Theorem in VIII follows somewhat the usual reasoning based on generalizations of Kakutani's fixed point theorem. However, much deeper topological tools should be used in our case for which we refer to the original work of J. Leray[4]. For functional analysis we shall often refer to the work of Dunford-Schwartz[2].

II. The Support of a Probability Measure

Let X be a bicompact Hausdorff space and $B(X)$ the σ-field of all Borel sets of X. For any regular probability measure μ defined over $B(X)$ we shall denote by $[\mu]$ the set of all points $x \in X$ such that $\mu(U) \neq 0$ for every neighbourhood U of x and call this set the *support* of μ.

Lemma. *The support* $[\mu]$ *of a regular probability measure* μ *over* $B(X)$ *in* X *verifies the following properties*:

1°. $[\mu]$ *is a closed set of* X.

2°. $[\mu]$ *is the intersection of all closed sets* F *in* X *for which* $\mu(F)=1$.

3°. $\mu(U)=1$ *for any open set* U *containing* $[\mu]$.

4°. $\mu([\mu])=1$.

5°. $\mu(X-[\mu])=0$.

6°. $[\alpha\mu+\beta\nu] \subset [\mu] \cup [\nu]$ *for any regular probability measures* μ, ν *over* $B(X)$ *and any real numbers* α, β *with* $\alpha, \beta \geqslant 0$, $\alpha+\beta=1$.

Proof. For, by definition, if $x \notin [\mu]$, x would have a neighbourhood U_x with $\mu(U_x)=0$. Then any x' in U_x is not in $[\mu]$ so that $X-[\mu]$ is open or $[\mu]$ is closed. This proves 1°.

Let G be the intersection of all closed sets F in X for which $\mu(F)=1$. If $x \notin G$, then there is a closed set $F \subset X$ with $x \notin F$ and $\mu(F)=1$. Hence for any neighbourhood U_x of x which is disjoint from F we would have $\mu(U_x)=0$. By definition we have then $x \notin [\mu]$. Therefore $[\mu] \subset G$. On the other hand, if $x \notin [\mu]$, then the open set U_x containing x exists for which $\mu(U_x)=0$. Whence $\mu(F)=1$ for the closed set $F=X-U_x$. As $x \notin F$ we have *a fortiori* $x \notin G$. Whence $[\mu] \supset G$. It follows that $[\mu]=G$ and 2° is proved.

Let U be any open set containing $[\mu]$. For any $x \in U$ there exists, by definition and 1° above, an open set U_x containing x with $\mu(U_x)=0$ and $U_x \cap [\mu]=\emptyset$. The system of all these sets U_x forms an open covering of $X-U$ which is bicompact since X is bicompact. Let $U_i=U_{x_i}$, $i=1, \cdots, n$, be a finite number of them which together covers the set $X-U$. Then we have $\mu(X-U) \leqslant \sum \mu(U_i)=0$. Hence $\mu(X-U)=0$ or $\mu(U)=1$. This proves 3°.

Suppose now that $\mu([\mu])<1$. Since μ is regular there exists an open set U containing $[\mu]$ for which $\mu(U)<1$. As this contradicts 3°, the assertion 4° is proved. The assertion 5° follows then from 4°.

As 6° is evident from definition, our lemma is proved.

III. The Set of Probability Measures with Support in a Given Set

Let X be a bicompact Hausdorff space and $B(X)$ the σ-field of all Borel sets of X. For any regular countably additive bounded set function μ over $B(X)$, let $v(\mu, X)$ be the total variation of μ on X defined as

$$v(\mu, X) = \sup \sum_{i=1}^{n} |\mu(E_i)|,$$

where sup is to be taken over all systems of finite number of mutually disjoint sets $E_1, \cdots, E_n$ in $B(X)$. Under the norm $\|\mu\|=v(\mu, X)$ the linear space of all regular countably additive bounded set functions μ over $B(X)$ forms a Banach space which will be denoted by $R(X)$. Denote by $C(X)$ the Banach space of all bounded continuous functions f over X with the norm $\|f\|=\sup_{x \in X} |f(x)|$. Then by the Riesz representation theorem, $R(X)$ is isomorphic to the conjugate space $C^*(X)$ of $C(X)$ under the isomorphism $\mu \longleftrightarrow x^*$ such that

$$x^*(f) = \int_X f(x)\, \mu(dx),$$

which will also be denoted by $\mu(f)$ or $f(\mu)$. Let $R^{\omega}(X)$ be the space with the same underlying set as $R(X)$ but with $C(X)$-topology, i.e., topology with the sets

$$N(\mu; A, \varepsilon) = \{\nu / |f(\mu) - f(\nu)| < \varepsilon, f \in A\},$$

as basis, where $\mu \in R(X)$, $A \subset C(X)$ finite, and $\varepsilon>0$ are arbitrary. It is known that $R^{\omega}(X)$ is a locally convex, Hausdorff linear topological space (cf. e.g. [2] V. 3).

Consider now a set F of the space X and denote by $m(F)$ the set of all regular probability measures μ over $B(X)$ with $[\mu]$ contained in the closed set F. By the very definition, $m(F)$ will be the topological space with topology induced as a subspace of the topological space $R^{\omega}(X)$.

The following assertion, though elementary in character, plays an important role in our development. So we state it explicitly:

Any subset C of $R^{\omega}(X)$ convex with respect to the linear structure of the Banach space $R(X)$ is contractible over itself into a point of which the contraction is continuous with respect to the topological structure of $R^{\omega}(X)$.

To prove this, let C be the convex subset in $R(X)$ and μ_0 a fixed point in C. For any $\mu \in C$ and $0 \leqslant t \leqslant 1$ let μ_t be the point $t\mu + (1-t)\mu_0 \in C$ with $\mu_1 = \mu$. Define now a mapping h of $C \times [0, 1]$ into C by

$$h(\mu, t) = \mu_t, \qquad \mu \in C, \qquad t \in [0, 1].$$

Set $h_t : C \to C$ by $h_t(\mu) = h(\mu, t)$ so that h_t is a contraction of C over itself into the point μ_0. To see that the contraction is continuous, i.e., h is continuous in μ and t in the topology of $R^{\omega}(X)$, let us consider a fixed pair (μ, t) and an arbitrary neighbourhood N of μ_t in $R^{\omega}(X)$ given by

$$N = N(\mu_t; A, \varepsilon) = \{\nu / |f(\nu) - f(\mu_t)| < \varepsilon, f \in A\}.$$

Let $M > 0$ be a number greater than the maximum of $|f(\mu) - f(\mu_0)|$ for all f in the finite set A. Let us consider the neighbourhood U of (μ, t) in $R^{\omega}(X) \times [0, 1]$ given by

$$U = N' \times J,$$

$$N' = N\left(\mu; A, \frac{\varepsilon}{2}\right) = \left\{\nu / |f(\nu) - f(\mu)| < \frac{\varepsilon}{2}, f \in A\right\},$$

$$J = \left\{t' / |t' - t| < \frac{\varepsilon}{2M}, t' \in [0, 1]\right\}.$$

For any $(\nu, t) \in U$, we have then

$$\begin{aligned} f(\nu_{t'}) - f(\mu_t) &= t' f(\nu) + (1 - t') f(\mu_0) - t f(\mu) - (1 - t) f(\mu_0) = \\ &= (t' - t)[f(\mu) - f(\mu_0)] + t'[f(\nu) - f(\mu)]. \end{aligned}$$

Consequently

$$\begin{aligned} |f(\nu_{t'}) - f(\mu_t)| &\leqslant |t' - t| \cdot |f(\mu) - f(\mu_0)| + t' \cdot |f(\nu) - f(\mu)| \leqslant \\ &\leqslant \frac{\varepsilon}{2M} \cdot M + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and $\nu_{t'} = h(\nu, t') \in N$. This proves the continuity of h at (μ, t) and hence the contraction h is continuous in the topology of $R^{\omega}(X)$.

From the definition and the above assertion the following lemma is quite evident.

Lemma 1. (i) *For any set F of X the set $m(F)$ is convex (w. r. t. the linear structure of $R(X)$) and is continuously contractible over itself into a point (w. r. t. the topological structure of $R^{\omega}(X)$).*

(ii) *For any sets F_1 and F_2 of X we have*

$$m(F_1 \cap F_2) = m(F_1) \cap m(F_2).$$

Lemma 2. *$m(F)$ is closed in $m(X)$ if F is a closed set of X.*

Proof. For any $x \bar{\in} F$ let us take open sets U_x, V_x containing x such that

$$x \in U_x \subset \bar{U}_x \subset V_x \subset X - F.$$

There exists by Urysohn's lemma a continuous function over X, or an $f \in C(X)$, with $f=0$ on $X - V_x$, $f=1$ on $\bar{U}_x$ and $0 \leqslant f \leqslant 1$ over X.

Consider now any $\mu \in \overline{m(F)} \cap m(X)$ (the bar means closure in the topological space $R^{\omega}(X)$). For any $\varepsilon>0$, let $N(\mu; f, \varepsilon)$ be the neighbourhood of μ in $R^{\omega}(X)$ given by

$$N(\mu; f, \varepsilon) = \{\nu / |f(\mu) - f(\nu)| < \varepsilon\}.$$

There exists then a $\nu \in m(F) \cap N(\mu; f, \varepsilon)$ so that $|f(\mu) - f(\nu)| < \varepsilon$. But

$$f(\nu) = \nu(f) = \int_X f(x)\nu(dx) \leqslant \nu(V_x) = 0.$$

It follows that $f(\nu) = 0$ and

$$\mu(U_x) = \int_{U_x} f(x)\, \mu(dx) \leqslant \int_X f(x) \mu(dx) = \mu(f) = f(\mu) < \varepsilon.$$

As $\varepsilon>0$ is arbitrary, we have $\mu(U_x)=0$. Hence $x \bar{\in} [\mu]$. As $x \bar{\in} F$ is arbitrary we have $[\mu] \subset F$ or $\mu \in m(F)$. This proves that $m(F)$ is closed in $m(X)$ and hence our lemma.

Lemma 3. *$m(X)$ is closed in $R^{\omega}(X)$.*

Proof. Let $\mu \in \overline{m(X)}$, the bar meaning closure in the space $R^{\omega}(X)$. The lemma will then be true if we prove that μ is a regular probability measure over $B(X)$, or, as it is sufficient, prove that (i) $\mu(E) \geqslant 0$ for all $E \in B(X)$, and (ii) $\mu(X)=1$.

To prove (i), let us first suppose on the contrary that $\mu(E)<0$ for some closed set $E \in B(X)$. As μ is regular, there exists an open set $U \supset E$ with

$$v(\mu, U - E) < \frac{1}{2} |\mu(E)|$$

(cf. e.g. [2] III 5.11 and III 1.5). By Urysohn's lemma there is an $f \in C(X)$ with $f=1$ on E, $f=0$ on $X-U$ and $0 \leqslant f \leqslant 1$ on X. Then

$$f(\mu) = \int_X f\mu\,(dx) = \int_E \mu\,(dx) + \int_{U-E} f\mu(dx) \leqslant \mu(E) +$$

$$+ \int_{U-E} |f|\, v(\mu, dx) \leqslant \mu(E) + v(\mu, U - E) < -\frac{1}{2} |\mu(E)| < 0.$$

Take now a neighbourhood N of μ in $R^{\omega}(X)$ given by

$$N = N(\mu; f, \varepsilon) = \{\nu / |f(\mu) - f(\nu)| < \varepsilon\},$$

where $0 < \varepsilon < |f(\mu)|$. As $\mu \in \overline{m(X)}$ there exists a $\nu \in m(X) \cap N$. Then

$$\nu(E) = \int_E f\nu(dx) \leqslant \int_X f\nu(dx) = f(\nu) < f(\mu) + \varepsilon < 0,$$

contrary to $\nu \in m(X)$, $\nu(E) \geqslant 0$.

It follows that $\mu(E) \geqslant 0$ for all closed sets $E \in B(X)$. Suppose that $E \in B(X)$, not necessarily closed, is such that $\mu(E) < 0$. As μ is regular we have again by [2] III 5.11 an open set $U \supset E$ and a closed set $W \subset E$ such that $|\mu(C)| < \frac{1}{2} |\mu(E)|$ for any $C \in B(X)$ with $C \subset U - W$. In particular we have $|\mu(E - W)| < \frac{1}{2} |\mu(E)|$ so that $\mu(W) = \mu(E) - \mu(E - W) < 0$ which has been shown to be impossible as W is closed. Thus (i) is proved.

To prove (ii), let us suppose on the contrary that $\mu(X) \neq 1$. Let us take $\varepsilon > 0$ with $\varepsilon < |1 - \mu(X)|$. Consider the function $f \equiv 1$ on X and the neighbourhood N of μ given by

$$N = N(\mu; f, \varepsilon) = \{\nu / |f(\mu) - f(\nu)| < \varepsilon\}.$$

There exists then a $\nu \in m(X) \cap N$ with

$$\nu(X) = f(\nu) < f(\mu) + \varepsilon = \mu(X) + \varepsilon < 1 \text{ for } \mu(X) < 1$$

and $$\nu(X) = f(\nu) > f(\mu) - \varepsilon = \mu(X) - \varepsilon > 1 \text{ for } \mu(X) > 1,$$

contrary to $\nu \in m(X)$, $\nu(X) = 1$. This proves (ii).

Our lemma is thus proved.

Let W be the closed unit sphere in the topological space $R(X)$:

$$W = \{\mu/\|\mu\| = v(\mu, X) \leqslant 1, \mu \in R(X)\}.$$

By a theorem of Alaoglu (cf. e.g.[2] V 4.2), W, considered as a set in the space $R^{\omega}(X)$, is bicompact. As $m(X) \subset W$ and $m(X)$ is closed in $R^{\omega}(X)$ by Lemma 3, it follows that $m(X)$ is also bicompact in $R^{\omega}(X)$. As $m(F)$ is closed in $m(X)$ for any closed set F of X, we have the following

Theorem. *For any closed set F of X the set $m(F)$ is closed and bicompact in the topological space $R^{\omega}(X)$.*

IV. The Set of Probability Measures with Support Subordinate to a Covering

Let X be a bicompact Hausdorff space and $B(X)$ the σ-field of all Borel sets of X as before. For any set F of X we have defined in III $m(F)$ as the set of all regular probability measures μ over $B(X)$ with support $[\mu] \subset F$, considered as subspace of the topological space $R^{\omega}(X)$. Consider now a finite closed covering $\mathfrak{F} = \{F_1, \cdots, F_r\}$ of X consisting of closed sets F_i, $1 \leqslant i \leqslant r$. Define $m(\mathfrak{F})$ as the set of regular probability measures over $B(X)$ with support in at least one of the closed sets F_i, $1 \leqslant i \leqslant r$, i.e.,

$$m(\mathfrak{F}) = \sum_{i=1}^{r} m(F_i)$$

considered again as a subspace of $R^{\omega}(X)$.

Lemma. *For the finite closed covering $\mathfrak{F} = \{F_1, \cdots, F_r\}$ of X the subspace $m(\mathfrak{F})$ of $R^{\omega}(X)$ has the following properties:*

(i) *$m(\mathfrak{F})$ is closed in $R^{\omega}(X)$.*

(ii) *$m(\mathfrak{F})$ is a bicompact set of $R^{\omega}(X)$.*

(iii) *$\{m(F_1), \cdots, m(F_r)\}$ is a closed convexoidal covering of $m(\mathfrak{F})$ in the sense of Leray*[4].

(iv) *$m(\mathfrak{F})$ is convexoidal in the sense of Leray*[4] *if the nerve complex $K(\mathfrak{F})$ of the covering $\mathfrak{F}$ is connected.*

Proof. The properties (i) and (ii) are direct consequences of Lemmas 2, 3 and the Theorem in III. In order to prove (iii) and (iv), let us recall first some definitions of Leray[4].

A covering of a bicompact Hausdorff space is, according to Leray, convexoidal if it verifies the following properties:

(a) Each set U of the covering is closed and "simple", i.e., with the same Cêch-Alexander cohomology as that of a point.

(b) The intersection of any finite number of sets in the covering is either empty or "simple".

A space is then, according to Leray, *convexoidal* if it is bicompact Hausdorff connected, and possesses a convexoidal covering which verifies (a) and (b) as well as the further property (c) below:

(c) For any point of the space and any neighbourhood V of x, there is a set U of the covering contained in V and containing x in its interior.

The assertion (iii) is now immediate from the definition and Lemma 1 in III. To prove (iv) let us first note that $m(\mathfrak{F})$ is bicompact Hausdorff and also connected since $K(\mathfrak{F})$ is supposed to be so. Consider now any $\mu \in m(\mathfrak{F})$. Let $F_{i_1}, \cdots, F_{i_k}$ be the totality of sets in the covering $\mathfrak{F}$ which contains the support $[\mu]$ of μ. As each $m(F_i)$ is closed in $R^{\omega}(X)$ there exist neighbourhoods of μ in $R^{\omega}(X)$ disjoint from all $m(F_i)$, $i \neq i_1, \cdots, i_k$; among them there are convex ones since the space $R^{\omega}(X)$ is known to be locally convex. Let $\mathfrak{U}(\mu)$ be the system of all closed convex neighbourhoods of μ in $R^{\omega}(X)$ disjoint from all $m(F_i)$, $i \neq i_1, \cdots, i_k$. Let $\mathfrak{B}(\mu)$ be the family of all subsets of $m(\mathfrak{F})$ which are intersections of sets in $\mathfrak{U}(\mu)$ with $m(\mathfrak{F})$. Then the totality $\mathfrak{B}$ of all sets in $\mathfrak{B}(\mu)$ for all $\mu \in m(\mathfrak{F})$ constitutes a closed covering of the space $m(\mathfrak{F})$ verifying the properties (a), (b), (c) above necessary for $m(\mathfrak{F})$ to be convexoidal. For (c) follows from the fact that each $\mathfrak{B}(\mu)$ forms a neighbourhood system about μ in $m(\mathfrak{F})$, (a) is evident since each V of $\mathfrak{B}$ is closed convex and hence simple, and (b) too since any finite intersection of closed convex sets is also closed and convex, and hence simple if not empty.

V. Generalization of the Preceding Notions

Let X and $B(X)$ be as in the preceding sections. Let c be any fixed number. For any set F of X we shall denote by $m_c(F)$ the set of all regular probability measures μ over $B(X)$ such that $\mu(F) \geqslant c$. For $c > 1$ the set $m_c(F)$ is empty. For $c = 1$ the set $m_1(F)$ is simply the set $m(F)$ introduced in III. For $c \leqslant 0$ the set $m_c(F)$ coincides with $m(X)$. In the general case it may be characterized as the set of all regular probability measures μ over $B(X)$ with $\mu([\mu] \cap F) \geqslant c$.

Lemma 1. *The sets $m_c(F)$ have the following properties*:

(i) *for $c \leqslant 1$, $m_c(X)$ coincides with $m(X)$ and hence is closed and bicompact in the topology of $R^{\omega}(X)$.*

(ii) $m(F) \subset m_c(F) \subset m_d(F) \subset m(X)$ *for* $d < c \leqslant 1$.

(iii) *for* $c \leqslant 1$, $m_c(F)$ *is convex with respect to the linear structure of* $R(X)$ *and is continuously contractible over itself into a point in the topology of* $R^{\omega}(X)$.

(iv) $m_{c_1}(F_1) \cap m_{c_2}(F_2) \subset m_{c_1+c_2-1}(F_1 \cap F_2)$, *for any* c_1, c_2, F_1 *and* F_2.

Proof. Immediate from the definitions (cf. III).

Lemma 2. (i) *The closure* $\overline{m_c(F)}$ *of* $m_c(F)$ *with respect to the topology of* $R^{\omega}(X)$ *is a bicompact subset of* $R^{\omega}(X)$ *contained in* $m(X)$.

(ii) $m_c(F)$ *is a closed and bicompact subset of the topological space* $R^{\omega}(X)$ *if* F *is closed in* X.

Proof. As $m_c(F) \subset m(X)$ and $m(X)$ is closed and bicompact in $R^{\omega}(X)$ by Lemma 3 and the Theorem of III, we have $\overline{m_c(F)} \subset m(X)$ and is bicompact in $R^{\omega}(X)$. This proves (i). To prove (ii) let F be closed in X and $\nu \in \overline{m_c(F)}$ where $0<c<1$. Then ν is a regular probability measure by (i). If $\nu \bar{\in} m_c(F)$ we have $\nu(F)<c<1$ so that $[\nu] \not\subset F$ and $\nu([\nu]-F)>1-c>0$. As ν is regular there exists a closed set C contained in $[\nu]-F$ such that

$$\nu([\nu] - F - C) < \nu([\nu] - F) - 1 + c.$$

We have then

$$\nu(C) > 1 - c > 0.$$

As C and F are both closed and disjoint there is a continuous function f over X with $f \equiv 1$ on C, $f \equiv 0$ on F and $0 \leqslant f \leqslant 1$ everywhere. Set $\varepsilon = \nu(C) - 1 + c > 0$ and consider the following neighbourhood of ν in $R^{\omega}(X)$:

$$N = N(\nu; f, \varepsilon) = \{\mu / |f(\mu) - f(\nu)| < \varepsilon\}.$$

As $\nu \in \overline{m_c(F)}$ there is a $\mu \in m_c(F) \cap N$. We have

$$\mu(X - F) \geqslant \int_X f(x)\,\mu(dx) = f(\mu) > f(\nu) - \varepsilon = \int_X f(x)\,\nu(dx) - \varepsilon \geqslant$$

$$\geqslant \int_G f(x)\,\nu(dx) - \varepsilon = \nu(C) - \varepsilon = 1 - c.$$

Whence $\mu(F)<c$ and $\mu \bar{\in} m_c(F)$ contrary to the above choice of μ. This proves (ii) for $0<c<1$. The case $c \geqslant 1$ or $c \leqslant 0$ is evident.

Consider now a finite closed covering $\mathfrak{F} = \{F_1, \cdots, F_r\}$ of X and a system of numbers $c = \{c_1, \cdots, c_r\}$. We shall set

$$m_c(\mathfrak{F}) = \sum_{i=1}^{r} m_{c_i}(F_i),$$

considered as a subspace of the space $R^{\omega}(X)$.

Theorem. *Let* $0 \leqslant c_i \leqslant 1$ *for each* $i=1,\cdots,r$. *Then the subspace* $m_c(\mathfrak{F})$ *of* $R^{\omega}(X)$ *has the following properties*:

(i) $m_c(\mathfrak{F})$ *is closed in* $R^{\omega}(X)$.

(ii) $m_c(\mathfrak{F})$ *is a bicompact set of* $R^{\omega}(X)$.

(iii) $C_c(\mathfrak{F}) = \{m_{c_1}(F_1),\cdots, m_{c_r}(F_r)\}$ *is a closed convexoidal covering of* $m_c(\mathfrak{F})$ *in the sense of Leray.*

(iv) $m_c(\mathfrak{F})$ *is convexoidal in the sense of Leray if the nerve complex* $K_c(\mathfrak{F})$ *of the coverings* $C_c(\mathfrak{F})$ *of* $m_c(\mathfrak{F})$ *by the sets* $m_{c_i}(F_i)$, $1 \leqslant i \leqslant r$, *is connected.*

(v) *The nerve complex* $K_c(\mathfrak{F})$ *of the covering* $C_c(\mathfrak{F})$ *of* $m_c(\mathfrak{F})$ *is isomorphic to the nerve complex* $N(\mathfrak{F})$ *of the covering* $\mathfrak{F}$ *of* X *if*

$$c_{i_1} + \cdots + c_{i_s} > s - 1$$

for any set of indices $i_1,\cdots,i_s$, *among* 1 *to* r, *in particular if* $c_i > 1 - \frac{1}{r}$ *for all* i.

Proof. For (i)−(iv) the proof is analogous to that of the Lemma in IV. To see (v) let us remark first that the correspondence $F_i \longleftrightarrow m_{c_i}(F_i)$, $1 \leqslant i \leqslant r$, makes $N(\mathfrak{F})$ a subcomplex of $K_c(\mathfrak{F})$. Let us consider now any set of indices $i_1,\cdots,i_s$ for which

$$F_{i_1} \cap \cdots \cap F_{i_s} = \emptyset,$$
$$F_{i_1} \cap \cdots \cap \hat{F}_{i_j} \cap \cdots \cap F_{i_s} \neq \emptyset, \quad 1 \leqslant j \leqslant s,$$

in which the symbol $\hat{F}_{i_j}$ means that this set is not to be counted in the intersection. Suppose that $m_{c_{i_1}}(F_{i_1}) \cap \cdots \cap m_{c_{i_s}}(F_{i_s}) \neq \emptyset$ and μ is in this common intersection. By Lemma 1 (iv) we have

$$\mu \in m_{c_{i_1}}(F_{i_1}) \cap \cdots \cap \hat{m}_{c_{i_j}}(F_{i_j}) \cap \cdots \cap m_{c_{i_s}}(F_{i_s}) \subset$$
$$\subset m_d(F_{i_1} \cap \cdots \cap \hat{F}_{i_j} \cap \cdots \cap F_{i_s}),$$

where

$$d = c_{i_1} + \cdots + c_{i_s} - c_{i_j} - s + 2.$$

Whence

$$\mu([\mu] \cap F_{i_1} \cap \cdots \cap \hat{F}_{i_j} \cap \cdots \cap F_{i_s}) \geqslant c_{i_1} + \cdots + c_{i_s} - c_{i_j} - s + 2.$$

As the sets $F_{i_1} \cap \cdots \cap \hat{F}_{i_j} \cap \cdots \cap F_{i_s}$ are mutually disjoint we should have

$$1 = \mu([\mu]) \geqslant \sum_{j=1}^{s} \mu([\mu] \cap F_{i_1} \cap \cdots \cap \hat{F}_{i_j} \cap \cdots \cap F_{i_s}) \geqslant$$

$$\geqslant \sum_{j=1}^{s} (c_{i_1} + \cdots + c_{i_s} - c_{i_j} - s + 2) =$$

$$= (s-1)(c_{i_1} + \cdots + c_{i_s}) - s(s-2).$$

Whence $c_{i_1} + \cdots + c_{i_s} \leqslant s-1$, contrary to hypothesis. This proves the isomorphism of $K_c(\mathfrak{F})$ and $N(\mathfrak{F})$.

VI. Uniform Closedness of Multiple-Valued Mappings

Let T be a multiple-valued mapping of a bicompact Hausdorff space X into a bicompact Hausdorff space Y. The subset of the product space $X \times Y$ constituted by all points (x, y) for which $y \in T(x)$ is called the *graph* of T and will be denoted by $G(T)$. T is said to be *closed* if $G(T)$ is closed in $X \times Y$. The closed mapping T will be said to be *uniformly closed* if for any $(x, y) \in G(T)$ and any neighbourhood V of y in Y there exists a neighbourhood U of x in X such that for any $x' \in U$, the set $T(x') \cap V$ is non-empty.

Lemma. *Let the multiple-valued mapping T of a bicompact Hausdorff space X into a bicompact Hausdorff space Y be closed as well as uniformly closed. Then for any continuous function f over $X \times Y$ the multiple-valued mapping T_f of X into Y defined by $T_f(x) = \{y / y \in T(x),\ f(x,y) = \sup_{\bar{y} \in T(x)} f(x,\bar{y})\}$ is also a closed mapping.*

Proof. Set $\sup_{\bar{y} \in T(x)} f(x, \bar{y}) = m_x$, $x \in X$. Let $(x, y) \in \overline{G(T_f)}$ and $y_0 \in T_f(x)$ such that $f(x, y_0) = m_x$. As $G(T)$ is closed, we have $(x, y) \in G(T)$. If $(x, y) \bar{\in} G(T_f)$, then $f(x, y) < m_x$. Put $\varepsilon = m_x - f(x, y) > 0$ and let U, V, V_0 be neighbourhoods of x and y, y_0 in X and Y respectively such that for any $x' \in U$, $y' \in V$, $y_0' \in V_0$ we have $|f(x', y') - f(x, y)| < \frac{\varepsilon}{2}$ and $|f(x', y_0') - f(x, y_0)| < \frac{\varepsilon}{2}$. As T is uniformly closed, there is a neighbourhood $W \subset U$ of x in X such that for any $x' \in W$, $T(x') \cap V \neq \emptyset$ and $T(x') \cap V_0 \neq \emptyset$. As $(x, y) \in \overline{G(T_f)}$, there exists $(\bar{x}', \bar{y}') \in G(T_f)$ with $\bar{x}' \in W$, $\bar{y}' \in V$. For this $\bar{x}'$ we have also a $\bar{y}_0' \in V_0 \cap T(x_0')$. Then we have

$$f(\bar{x}', \bar{y}') = m_{\bar{x}'} \geqslant f(\bar{x}', \bar{y}_0').$$

On the other hand we have

$$f(\bar{x}', \bar{y}_0') = f(\bar{x}', \bar{y}') + (f(x, y) - f(\bar{x}', \bar{y}')) + (f(x, y_0) - f(x, y)) +$$

$$+ (f(\bar{x}', \bar{y}_0') - f(x, y_0)) > f(\bar{x}', \bar{y}') - \frac{\varepsilon}{2} + \varepsilon - \frac{\varepsilon}{2} = f(\bar{x}', \bar{y}')$$

which leads to a contradiction. Hence $(x, y) \in G(T_f)$ or $G(T_f)$ is closed in $X \times Y$, i.e., T_f is closed.

VII. Some Topological Theorems about Multiple-Valued Mappings

For a bicompact Hausdorff space X we shall denote by $H(X)$ the Cêch-Alexander cohomology ring based on rational coefficients. Such a space will be said to be *simple* (more exactly, *simple* with respect to rational coefficients), if it has the same cohomology ring as that of a point. The two following general theorems will be required in what follows.

Lemma 1. (Leray Theorem)[4]. *If the bicompact Hausdorff space X has a finite closed convexoidal covering $\mathfrak{F}$ in the sense of Leray, then X has the same cohomology ring as that of nerve complex N of $\mathfrak{F}$: $H(X) \approx H(N)$. In particular, we have $\chi(X) = \chi(N)$.*

Lemma 2. (Vietoris-Begle Theorem)[1]. *If f is a continuous mapping of a bicompact Hausdorff space X into a bicompact Hausdorff space Y such that $f^{-1}(y)$ is simple for each $y \in Y$, then $H(Y) \approx H(X)$ under the isomorphism f^* induced by the mapping f.*

Let φ, ψ be two continuous mappings of a bicompact Hausdorff space X into a convexoidal space Y for which $\varphi^{-1}(y)$ is simple for any point y of Y. As Y is convexoidal, the cohomology ring $H(Y)$ has a finite basis, say Z_i^p, in dimension p, $0 \leqslant p \leqslant N$, $1 \leqslant i \leqslant \alpha_p$. By the theorem of Vietoris-Begle, we know that $H(X)$ of X has also a finite basis constituted by $\varphi^*(Z_i^p)$, where $\varphi^*: H(Y) \to H(X)$ is the isomorphism induced by φ. It follows that

$$\psi^*(Z_i^p) = \sum_j b_{ij}^p \, \varphi^*(Z_j^p).$$

The number $\sum_p (-1)^p S_p B^p$ where $S_p B^p$ denotes the trace of the matrix $B^p = (b_{ij}^p)$ is independent of the choice of the basis $\{Z_i^p\}$ and will be denoted by $\Lambda(\varphi, \psi)$.

Theorem A. *Let φ, ψ be two continuous mappings of a bicompact Hausdorff space X into a convexoidal space Y for which $\varphi^{-1}(y)$ is simple for any point y of Y. If $\Lambda(\varphi, \psi) \neq 0$, then φ, ψ have a coincidence point, i.e., some point $x \in X$ for which $\varphi(x) = \psi(x)$.*

The proof of the above theorem, which is analogous to that of Leray concerning fixed points of a map (see [4] Th. 17), will be omitted. The next theorem follows directly from the definition.

Theorem B. *Let φ be a continuous mapping of a bicompact Hausdorff space X into a convexoidal space Y for which $\varphi^{-1}(y)$ is simple for any point y of Y. Then $\Lambda(\varphi, \varphi) = \chi(Y)$, where $\chi(Y)$ denotes* *the Euler-Poincaré characteristic of* Y.

Now let T be a multiple-valued mapping of a convexoidal space Y into itself such that

(i) T is closed and

(ii) $T(y)$ is simple for each $y \in Y$.

Denote the graph $G(T)$ of T by X and define the two maps φ and ψ of X into Y by the projections of $Y \times Y$ onto Y, viz.,

$$\begin{aligned} \varphi(y, y') &= y, \\ \psi(y, y') &= y', \end{aligned} \qquad (y' \in T(y) \text{ or } (y, y') \in X).$$

By (i), the graph $X = G(T)$ of T is closed in $Y \times Y$ and hence it is bicompact Hausdorff.

As $\varphi^{-1}(y)$ is homeomorphic to $T(y)$ under ψ and is simple by (ii) for each $y \in Y$, we see that the number $\Lambda(\varphi, \psi)$ is well defined. We define now:

$$\Lambda(T) = \Lambda(\varphi, \psi).$$

Theorem C. *Let* T *be a multiple-valued closed mapping of a convexoidal space* Y *into itself such that* $T(y)$ *is simple for each* $y \in Y$. *If* $\Lambda(T) \neq 0$, *then* T *has a fixed point, i.e., some point* $y \in Y$ *such that* $y \in T(y)$.

Proof. Define $X = G(T)$ and $\varphi, \psi : X \to Y$ as before. As $\Lambda(\varphi, \psi) = \Lambda(T) \neq 0$, the pair φ, ψ has by Theorem A some coincidence point $x = (y, y') \in X$ such that $\varphi(x) = \psi(x)$, i.e., $y = y' \in T(y)$, q.e.d.

Theorem D. *Let* T *be the identical mapping of a convexoidal space* Y *onto itself. Then* $\Lambda(T) = \chi(Y)$.

Proof. This follows directly from Theorem B.

Theorem E. *Let* T_0, T_1 *be multiple-valued closed mappings of a convexoidal space* Y *into itself such that*

(i) *There exists a multiple-valued closed mapping* $\tilde{T}$ *of* $\tilde{Y} = Y \times [0, 1]$ *into* $\tilde{Y}$ *with* $\tilde{T}(y, k) = T_k(y)$, *where* $k = 0, 1$, $y \in Y$, *and* $\tilde{T}(Y \times (t)) \subset Y \times (t)$, $t \in [0, 1]$.

(ii) *Set* T_t: $Y \to Y$ *by* $(T_t(y), t) = \tilde{T}(y, t)$, $t \in [0, 1]$, *then* $T_t(y)$ *is simple for each* $y \in Y$ *and* $t \in [0, 1]$.

Then $\Lambda(T_0) = \Lambda(T_1)$.

Proof. Let $\tilde{X} = G(\tilde{T})$, $X_0 = G(T_0)$, $X_1 = G(T_1)$ be the graphs of $\tilde{T}$, T_0 and T_1 respectively. Define the projections $\tilde{\varphi}, \tilde{\psi}$: $\tilde{X} \to \tilde{Y}$, φ_0, ψ_0: $X_0 \to Y_0 = Y \times (0)$, and φ_1, ψ_1: $X_1 \to Y_1 = Y \times (1)$ by $\tilde{\varphi}(\tilde{y}, \tilde{y}') = \tilde{y}$, $\tilde{\psi}(\tilde{y}, \tilde{y}') = \tilde{y}'$,

$\varphi_k(y_k, y_k') = y_k$, $\psi_k(y_k, y_k') = y_k'$, where $(\tilde y, \tilde y') \in \tilde X$, $(y_k, y_k') \in X_k$, $k=0, 1$. Denote the natural injection of Y into $Y_k = Y \times (k)$ of $\tilde Y$ by λ_k where $\lambda_k(y) = (y, k)$, $y \in Y$, $k=0, 1$; similarly denote the natural injection of X_k into $\tilde X$ by θ_k, where $k=0, 1$. Take a basis $\{\tilde Z_i^p\}$ of $H(\tilde Y)$, then $\{\lambda_k^* \tilde Z_i^p\}$ is a basis of $H(Y)$ for each $k=0$ or 1. Now we have

$$\Lambda(\tilde T) = \Lambda(\tilde\varphi, \tilde\psi) = \sum_p (-1)^p S_p(b_{ij}^p),$$

where $\tilde\psi^*(\tilde Z_i^p) = \sum_j b_{ij}^p \tilde\varphi^*(\tilde Z_j^p)$. As $\tilde\varphi\theta_k = \lambda_k \varphi_k$, $\tilde\psi\theta_k = \lambda_k \psi_k$, we get by applying θ_k^* to the last equation,

$$\psi_k^*(\lambda_k^* \tilde Z_i^p) = \sum_j b_{ij}^p \varphi_k^*(\lambda_k^* \tilde Z_j^p), \qquad k = 0, 1.$$

It follows that

$$\Lambda(T_k) = \Lambda(\varphi_k, \psi_k) = \sum_p (-1)^p S_p(b_{ij}^p).$$

Therefore $\Lambda(T_0) = \Lambda(T_1) = \Lambda(\tilde T)$ and the theorem is proved.

Remark. For simplicity we shall say that the two mappings verifying the conditions in our theorem are "homotopic in a simple manner".

VIII. Definition of the Game and the Main Theorem

Let us consider an n-person game with strategy space S_i and payoff function $H_i(x_1, \cdots, x_n)$, $x_i \in S_i$, $i=1, \cdots, n$, for the player "i". We shall suppose that S_i are all bicompact Hausdorff spaces and H_i are all continuous over $S = S_1 \times \cdots \times S_n$. For each S_i let $\mathfrak{F}_i = \{F_1^{(i)}, \cdots, F_{m_i}^{(i)}\}$ be a given finite closed covering and B_i the σ-field of all Borel sets of S_i and $\{c_i\} = \{c_{i1}, \cdots, c_{im_i}\}$ a set of number $\geqslant 0$ and $\leqslant 1$. As defined in V, let $S_i^* = m_{c_i}(\mathfrak{F}_i)$ be the set of all regular probability measures μ_i over B_i with $\mu_i(F_j^{(i)}) \geqslant c_{ij}$ for at least one of the indices j, $1 \leqslant j \leqslant c_i$, this set having a topology as induced by that of the topological space $R^\omega(S_i) = R_i^\omega$.

Consider now for each $i=1, \cdots, n$, a multiple-valued mapping τ_i of S_i^* into itself verifying the following conditions:

(i) $\mu_i \in \tau_i(\mu_i)$, $\mu_i \in S_i^*$,

(ii) τ_i is closed and uniformly closed,

(iii) for each $\mu_i \in S_i^*$, the set $\tau_i(\mu_i)$ is convex with respect to the linear structure of the Banach space $R_i = R(S_i)$.

Definition. The system $\Gamma = \langle I, \{S_i\}, \{H_i\}, \{\mathfrak{F}_i\}, \{c_i\}, \{\tau_i\}\rangle$ in which $I = \{1, \cdots, n\}$ is the set of players will be called a *game with restricted domains of activities.* The closed sets $F_j^{(i)}$ of the covering $\mathfrak{F}_i$ will be called the *domain of activities,* τ_i *the domain of alternations* and c_{ij} *the factors of concentration* of the player "i" in the game Γ. The game $\Gamma^* = \langle I, \{S_i^*\}, \{H_i^*\}, \{\tau_i\}\rangle$ with the same set of players I, strategy spaces $S_i^* = m_{c_i}(\mathfrak{F}_i) = \sum_j m_{c_{ij}}(F_j^{(i)}) \subset R^\omega(S_i)$, and payoff functions[1] $H_i^*(\mu_1, \cdots, \mu_n) = \int_S H_i(x_1, \cdots, x_n)\, \mu(dx)$ where μ is the product measure over the product space $S = S_1 \times \cdots \times S_n$ of the regular probability measures $\mu_i \in S_i^*$, will be called the *natural extension* of the game Γ. We call $(\mu_1^*, \cdots, \mu_n^*) \in S_1^* \times \cdots \times S_n^*$ an *equilibrium point* of Γ or Γ^* if

$$H_i^*(\mu_1^*, \cdots, \mu_i^*, \cdots, \mu_n^*) \geqslant H_i^*(\mu_1^*, \cdots, \mu_i, \cdots, \mu_n^*)$$

for any

$$\mu_i \in \tau_i(\mu_i^*), \qquad i = 1, \cdots, n.$$

Denote the nerve complex of the covering $\{m_{c_{ij}}(F_{ij})\}$ of $m_{c_i}(\mathfrak{F}_i)$ by $K_i = K_{c_i}(\mathfrak{F}_i)$ and its Euler-Poincaré characteristic by χ_i. Then the number $\chi(\Gamma) = \chi_1, \cdots, \chi_n$ will be called the *characteristic* of the game Γ.

Main Theorem. *The game with restricted domains of activities $\Gamma = \langle I, \{S_i\}, \{H_i\}, \{\mathfrak{F}_i\}, \{c_i\}, \{\tau_i\}\rangle$ has equilibrium points if all the nerve complexes K_i are connected and $\chi(\Gamma) \neq 0$.*

Proof. For any $\mu = (\mu_1, \cdots, \mu_n) \in S^* = S_1^* \times \cdots \times S_n^*$ let $\Phi^{(i)}(\mu)$ be the set of all $\mu_i' \in \tau_i(\mu_i) \subset S_i^*$ such that $H_i^*(\mu_1, \cdots, \mu_i', \cdots, \mu_n) = \sup_{\nu_i \in \tau_i(\mu_i)} H_i^*(\mu_1, \cdots, \nu_i, \cdots, \mu_n)$ and let $\Phi(\mu) = \Phi^{(1)}(\mu) \times \cdots \times \Phi^{(n)}(\mu) \subset S^*$. As τ_i is closed, $\Phi(\mu)$ is non-empty. As τ_i is also uniformly closed, Φ is closed by the Lemmas of VI. As $\tau_i(\mu_i)$ is convex, $\Phi(\mu)$ is also convex with respect to the linear structure of the Banach space $R = R(S_1) \times \cdots \times R(S_n)$. Moreover, Φ is "homotopic in a simple manner" to the identical mapping J of S^* into S^* since $\tau_i(\mu_i)$ is convex and contains μ_i. It follows from Theorems E, D of VII that

$$\Lambda(\Phi) = \Lambda(J) = \chi(S^*) = \prod_{i=1}^{n} \chi(S_i^*).$$

Again by Lemma 1 of VII as well as Lemma 3 of V we have

$$\chi(S_i^*) = \chi(K_i) = \chi_i.$$

1) Sometimes $H_i^*(\mu_1, \cdots, \mu_n)$ will also be written simply $H_i(\mu_1, \cdots, \mu_n)$.

Hence

$$\Lambda(\Phi) = \chi_1 \cdots \chi_n = \chi(\Gamma) \neq 0.$$

By Theorem C of VII, there exists a point $\mu^* \in S^*$ with $\mu^* \in \Phi(\mu^*)$. This point μ^* is then an equilibrium point of our game and the theorem is proved.

Corollary 1. *If $c_{ij_1} + \cdots + c_{ij_s} > s-1$ for each i and each set of indices $j_1, \cdots, j_s$ among $1, \cdots, m_i$, then the game of restricted domains of activities $\Gamma = \langle I, \{S_i\}, \{H_i\}, \{\mathfrak{F}_i\}, \{c_i\}, \{\tau_i\}\rangle$ has always equilibrium points if none of the Euler-Poincaré characteristic $\chi(N_i)$ is 0, $i=1, \cdots, n$, where N_i is the nerve complex of the covering $\mathfrak{F}$, supposed to be connected.*

Proof. This follows from $\chi(N_i) = \chi_i = \chi(K_{c_i}(\mathfrak{F}_i))$ by (v) of the Theorem of V.

Corollary 2. *If each $\mathfrak{F}_i$ consists of a single set, namely S_i itself, then the game with restricted domains of activities $\Gamma = \langle I, \{S_i\}, \{H_i\}, \{\mathfrak{F}_i\}, \{c_i\}, \{\tau_i\}\rangle$ which may be simply written $\Gamma = \langle I, \{S_i\}, \{H_i\}, \{\tau_i\}\rangle$ has always equilibrium points.*

Proof. For in that case $K_{c_i}(\mathfrak{F}_i)$ is simply a point so that $\chi_i = 1 \neq 0$.

Corollary 3. (Nash-Glicksberg)[3,5]. *The game $\Gamma = \langle I, \{S_i\}, \{H_i\}\rangle$ in which S_i are all bicompact Hausdorff spaces and H_i are all continuous over $S = S_1 \times \cdots \times S_n$ has always equilibrium points.*

Proof. For the game Γ may be considered as a game with restricted domains of activities for which $\mathfrak{F}_i$ consists of a single set, namely S_i itself, $c_i = 1$ and $\tau_i(\mu_i) = S_i^*$ for any $\mu_i \in S_i^*$, $1 \leqslant i \leqslant n$.

Conclusions. For a finite closed covering $\mathfrak{F}_i = \{F_j^{(i)}\}$, $1 \leqslant j \leqslant m_i$, of a space S_i with nerve complex N_i, the Euler-Poincaré characteristic $\chi_i = \chi(N_i)$ is equal to

$$\chi_i = \sum_{s=0}^{m_i-1} (-1)^s a_s(\mathfrak{F}_i),$$

where $a_s(\mathfrak{F}_i)$ denotes the number of $(s+1)$-tuples of closed sets among $F_j^{(i)}$ which have non-empty intersections. Thus χ_i is a number determined by the mutual interrelations between the various closed sets of the covering $\mathfrak{F}_i$. The Corollary 1 of our theorem assures therefore the existence of equilibrium points whenever the choice of strategies is to be sufficiently concentrated and the mutual interrelations of the restricted domains of activities are such that $\chi(\Gamma) \neq 0$. The Corollary 2 of our theorem shows that if the choice of strategies is entirely unrestricted, then equilibrium points exist always irrespective of the structure of the strategy spaces and the domains of alternations. This becomes the theorem of Nash-Glicksberg if the alternations of strategies are further unrestricted (Corollary 3 of our theorem). On the other hand, simple examples (see the Example in IX) show that if $\chi(\Gamma)=0$, then equilibrium points may not exist even in the case of simple strategy spaces consisting of finite number of points only. Thus, our theorem shows that:

The main factors which determine the existence of equilibrium points of a game with restricted domains of activities are rather the mutual interrelations of the domains of activities than the strategy spaces themselves.

IX. An Example

Let us define a 2-person game with restricted domains of activities $\Gamma=\langle I, \{S_i\}, \{H_i\}, \{\mathfrak{F}_i\}, \{c_i\}, \{\tau_i\}\rangle$ as follows.

Let Player I possess 4 (pure) strategies a_i, $1 \leqslant i \leqslant 4$, and Player II possess 4 (pure) strategies b_j, $1 \leqslant j \leqslant 4$. The payoff functions H_1 and H_2 are given in the following tables:

H_1	a_1	a_2	a_3	a_4
b_1	γ	β	α	δ
b_2	β	α	δ	γ
b_3	α	δ	γ	β
b_4	δ	γ	β	α

H_2	a_1	a_2	a_3	a_4
b_1	β	γ	δ	α
b_2	γ	δ	α	β
b_3	δ	α	β	γ
b_4	α	β	γ	δ

The numbers α, β, γ, δ in the tables will be chosen to satisfy the inequalities

$$\delta < \alpha < \beta < \gamma, \tag{1}$$

$$\alpha < 2\delta, \tag{2}$$

$$\gamma + \delta < 2\alpha, \tag{3}$$

and

$$\alpha + \gamma < 2\beta. \tag{4}$$

The covering $\mathfrak{F}_i$, $i = 1, 2$, will each consist of 4 closed sets $F_j^{(i)}$, $1 \leqslant j \leqslant 4$, where

$$F'_j = \{a_j, a_{j+1}\},$$

$$F''_j = \{b_j, b_{j+1}\},$$

(with the convention $a_5=a_1$, $b_5=b_1$).

The numbers $\{c_{ij}\}$, $i=1,2$, will be taken to be all equal to $c>0$ and <1 which is sufficiently near to 1. The spaces S_i^*, $i=1,2$, may then be considered as spaces of points $\sum_{j=1}^{4} x_j a_j$ and $\sum_{j=1}^{4} y_j b_j$ with x, y satisfying the following sets of inequalities respectively.

For x:

$$x_j \geqslant 0, \qquad 1 \leqslant j \leqslant 4,$$

$$\sum_{j=1}^{4} x_j = 1,$$

and $x_1 + x_2 \geqslant c$, or $x_2 + x_3 \geqslant c$, or $x_3 + x_4 \geqslant c$ or $x_4 + x_1 \geqslant c$.

For y:

$$y_j \geqslant 0, \qquad 1 \leqslant j \leqslant 4,$$

$$\sum_{j=1}^{4} y_j = 1,$$

and $y_1 + y_2 \geqslant c$, or $y_2 + y_3 \geqslant c$, or $y_3 + y_4 \geqslant c$ or $y_4 + y_1 \geqslant c$.

Let a_j', a_j'', b_j', b_j'', $1 \leqslant j \leqslant 4$, be respectively the points defined by

$$\begin{aligned}
a_1' &= c a_1 + (1-c) a_3, \\
a_2' &= (2c-1) a_2 + (1-c) a_1 + (1-c) a_3, \\
a_3' &= c a_3 + (1-c) a_1, \\
a_4' &= (2c-1) a_4 + (1-c) a_1 + (1-c) a_3, \\
a_1'' &= (2c-1) a_1 + (1-c) a_2 + (1-c) a_4, \\
a_2'' &= c a_2 + (1-c) a_4, \\
a_3'' &= (2c-1) a_3 + (1-c) a_2 + (1-c) a_4, \\
a_4'' &= c a_4 + (1-c) a_2.
\end{aligned}$$

Similarly for b_j' and b_j'' defined by equations as above with all a_j replaced by b_j. The spaces of all mixed strategies of players I and II will be considered as tetrahedrons T_1 and T_2 with vertices a_j, $1 \leqslant j \leqslant 4$, and b_j, $1 \leqslant j \leqslant 4$, respectively. Then S_1^* is part of T_1 surrounding the 4 edges a_1a_2, a_2a_3, a_3a_4, a_4a_1. The boundary of this part consists of 4 parallelograms $a_1'a_1''a_2'a_2''$, $a_2'a_2''a_3'a_3''$, $a_3'a_3''a_4'a_4''$, $a_4'a_4''a_1'a_1''$, as well as other 8 trapezoids with two lying on each the four faces of the tetrahedron T_1. We shall denote by C_j, $1 \leqslant j \leqslant 4$, the four corners of S_1^* about a_j for which C_1 is given by

$$\begin{cases}
x_1 + x_2 \geqslant c, \ x_1 + x_4 \geqslant c, \\
x_1 \geqslant 0, \ x_2 \geqslant 0, \ x_3 \geqslant 0, \ x_4 \geqslant 0, \\
x_1 + x_2 + x_3 + x_4 = 1.
\end{cases}$$

Similarly for the other corners C_j, $j = 2, 3, 4$. We shall also denote by $C_{j,j+1}$ the prisms defined by

$$\begin{cases} x_j + x_{j+1} \geqslant c, \ x_{j-1} + x_j < c, \ x_{j+1} + x_{j+2} < c, \\ x_1 \geqslant 0, \ x_2 \geqslant 0, \ x_3 \geqslant 0, \ x_4 \geqslant 0, \\ x_1 + x_2 + x_3 + x_4 = 1, \end{cases}$$

where $1 \leqslant j \leqslant 4$ and by convention $x_{k+4} = x_k$, $C_{4,5} = C_{4,1}$. It is now easy to define the domains of alternations $\tau_1(\mu)$, $\mu \in S_1^*$, such that τ_1 should, besides being convex, closed and uniformly closed and containing μ themselves, also satisfy the following conditions:

(i) $\tau_1(\mu) = C_j$, for μ on the segment $a_j' a_j''$, $1 \leqslant j \leqslant 4$.

(ii) $\tau_1(\mu) \supset C_j$, for $\mu \in C_j$, $1 \leqslant j \leqslant 4$.

(iii) $\tau_1(\mu) \subset C_{j,j+1} \cup C_j \cup C_{j+1}$, for $\mu \in C_{j,j+1}$, $1 \leqslant j \leqslant 4$.

(iv) $\mu \in \operatorname{Int} \tau_1(\mu)$ if μ is not on the segments $a_j' a_j''$, $1 \leqslant j \leqslant 4$.

Similarly for τ_2.

For numbers (u, u', u'') and (v, v', v'') with $u \geqslant 0$, $u' \geqslant 0$, $u'' \geqslant 0$, $u + u' + u'' = 1$, $v \geqslant 0$, $v' \geqslant 0$, $v'' \geqslant 0$, and $v + v' + v'' = 1$, let

$$\begin{aligned} \bar{a}_j &= u a_j + u' a_j' + u'' a_j'', \\ \bar{b}_j &= v b_j + v' b_j' + v'' b_j'', \end{aligned} \qquad (1 \leqslant j \leqslant 4).$$

The values of $H_1(\bar{a}_i, \bar{b}_j)$ and $H_2(\bar{a}_i, \bar{b}_j)$ will be tabulated as follows:

H_1	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$
$\bar{b}_1$	$\bar{\gamma}_{11}^1$	$\bar{\beta}_{21}^1$	$\bar{\gamma}_{31}^1$	$\bar{\delta}_{41}^1$
$\bar{b}_2$	$\bar{\beta}_{12}^1$	$\bar{\alpha}_{22}^1$	$\bar{\delta}_{32}^1$	$\bar{\gamma}_{42}^1$
$\bar{b}_3$	$\bar{\alpha}_{13}^1$	$\bar{\delta}_{23}^1$	$\bar{\gamma}_{33}^1$	$\bar{\beta}_{43}^1$
$\bar{b}_4$	$\bar{\delta}_{14}^1$	$\bar{\gamma}_{24}^1$	$\bar{\beta}_{34}^1$	$\bar{\alpha}_{44}^1$

H_2	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$
$\bar{b}_1$	$\bar{\beta}_{11}^2$	$\bar{\gamma}_{21}^2$	$\bar{\delta}_{31}^2$	$\bar{\alpha}_{41}^2$
$\bar{b}_2$	$\bar{\gamma}_{12}^2$	$\bar{\delta}_{22}^2$	$\bar{\alpha}_{32}^2$	$\bar{\beta}_{42}^2$
$\bar{b}_3$	$\bar{\delta}_{13}^2$	$\bar{\alpha}_{23}^2$	$\bar{\beta}_{33}^2$	$\bar{\gamma}_{43}^2$
$\bar{b}_4$	$\bar{\alpha}_{14}^2$	$\bar{\beta}_{24}^2$	$\bar{\gamma}_{34}^2$	$\bar{\delta}_{44}^2$

Now for $c \to 1$ it is evident that $H_1(\bar{a}_i, \bar{b}_j) \to H_1(a_i, b_j)$, $H_2(\bar{a}_i, \bar{b}_j) \to H_2(a_i, b_j)$ for the arbitrary systems (u, u', u'') and (v, v', v'') chosen above. It follows that we can choose such a $c > 0$ sufficiently near to 1 so that values $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$ have the same relative magnitudes as exhibited in the inequalities (1)—(4), e. g.,

$$\bar{\delta}_{i_1, j_1}^k < \bar{\alpha}_{i_2, j_2}^k < \bar{\beta}_{i_3, j_3}^k < \bar{\gamma}_{i_4, j}^k \qquad (\bar{1})$$

$$2\bar{\delta}_{i_1, j_1}^k < \bar{\alpha}_{i_2, j_2}^k, \qquad (\bar{2})$$

etc.,

for any $k = 1, 2$ and any $i_r, j_r = 1, 2, 3, 4$. Now let P_1 and P_2 be the closed polygons $\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4\bar{a}_1$ and $\bar{b}_1\bar{b}_2\bar{b}_3\bar{b}_4\bar{b}_1$ respectively. The space $P_1 \times P_2$ is topologically a torus which we shall represent as a square with opposite sides identified. Suppose that equilibrium point exists, say (μ_1^*, μ_2^*), lying on $P_1 \times P_2$. We shall prove the impossibility by distinguishing the following cases according to the values of u, u', etc.

Case I. $u > 0$, $v > 0$.

Owing to our choice of the domains of alternations it is evident in that case that

$$H_1(\mu_1^*, \mu_2^*) \geqslant H_1(\mu_1, \mu_2^*) \tag{5}$$

for any μ_1 in a certain neighbourhood about μ_1^* on P_1 and

$$H_2(\mu_1^*, \mu_2^*) \geqslant H_2(\mu_1^*, \mu_2) \tag{6}$$

for any μ_2 in a certain neighbourhood about μ_2^* on P_2. As a consequence of inequalities $(\bar{1})-(\bar{4})$ we see that (μ_1^*, μ_2^*) should lie on the dark lines in the following diagrams in order to satisfy (5) and (6) respectively.

Fig. 1

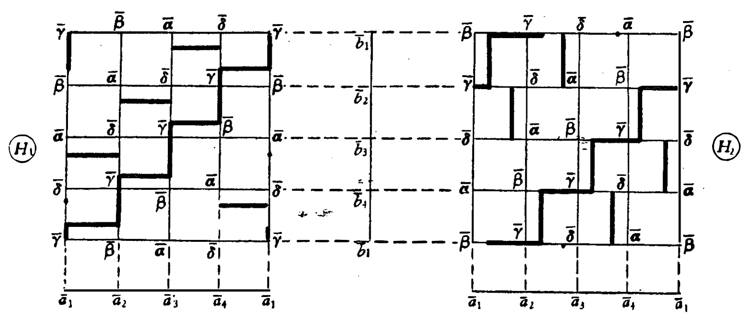

Fig. 2

(The numbers $\bar{\alpha}, \cdots, \bar{\delta}$ indicate the corresponding values $H_1(\bar{a}_i, \bar{b}_j)$ or $H_2(\bar{a}_i, \bar{b}_j)$ and are abbreviations for $\bar{\alpha}_{ij}^k$, etc.) As these dark lines are disjoint, it follows that no such equilibrium points can exist.

Case II. $u > 0$, $v = 0$.

In that case (5) should still be satisfied for μ_1 in certain neighbourhood about μ_1^* on P_1 as before. As for (6), it should still be satisfied for μ_2 in certain neighbourhood about μ_2^* on P_2 if $\mu_2^* \neq \bar{b}_1, \bar{b}_2, b_3$ or $\bar{b}_4$. It follows that an equilibrium point on $P_1 \times P_2$, if such one exists, should lie on the one hand on the dark lines in Fig. 1, and on the other hand should lie on the dark lines or on the horizontal lines in Fig. 2. The only possible equilibrium points are thus

$$(\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_4), (\bar{a}_3, \bar{b}_3) \text{ or } (\bar{a}_4, \bar{b}_2).$$

However, we have

$$H_2(a_1, b_1) - H_2(a_1, b_1') = (1-c)(\beta - \delta) > 0,$$
$$H_2(a_1, b_1) - H_2(a_1, b_1'') = (1-c)(2\beta - \gamma - \alpha) > 0,$$
$$H_2(a_2, b_1) - H_2(a_2, b_1') = (1-c)(\gamma - \alpha),$$
$$H_2(a_2, b_1) - H_2(a_2, b_1'') = (1-c)(2\gamma - \beta - \delta),$$
$$H_2(a_3, b_1) - H_2(a_3, b_1') = (1-c)(\delta - \beta),$$
$$H_2(a_3, b_1) - H_2(a_3, b_1'') = (1-c)(2\delta - \alpha - \gamma),$$
$$H_2(a_4, b_1) - H_2(a_4, b_1') = (1-c)(\alpha - \gamma),$$
$$H_2(a_4, b_1) - H_2(a_4, b_1'') = (1-c)(2\alpha - \beta - \delta).$$

It follows that $\frac{1}{1-c}[H_2(\bar{a}_1, b_1) - H_2(\bar{a}_1, \bar{b}_1)] \rightarrow v'(\beta - \delta) + v''(2\beta - \gamma - \alpha) > 0$ as $c \rightarrow 1$. Hence

$$H_2(\bar{a}_1, b_1) > H_2(\bar{a}_1, \bar{b}_1)$$

inasmuch as c is sufficiently near to 1. As $b_1 \in \tau_1(\bar{b}_1)$, the above inequality shows that $(\bar{a}_1, \bar{b}_1)$ cannot be an equilibrium point. Similarly for $(\bar{a}_2, \bar{b}_4)$, etc.. Hence there exist no equilibrium points in the present case.

Case III. $u = 0$, $v > 0$.

In this case an equilibrium point (μ_1^*, μ_2^*) lying on $P_1 \times P_2$ should lie on the dark lines in Fig. 2 and also on the dark lines or the vertical lines in Fig. 1. The only possibilities are then

$$(\bar{a}_1, \bar{b}_2), (\bar{a}_2, \bar{b}_1), (\bar{a}_3, \bar{b}_4) \text{ or } (\bar{a}_4, \bar{b}_3).$$

As in Case II, all these are impossible.

Case IV. $u = 0$, $v = 0$.

As before, the only possibilities are the 16 points

$$(\bar{a}_i, \bar{b}_j), \quad i, j = 1, 2, 3, 4.$$

Now the points

$$(\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_4), (\bar{a}_3, \bar{b}_3), (\bar{a}_4, \bar{b}_2)$$

are impossible as in Case II, and the points

$$(\bar{a}_1, \bar{b}_2), (\bar{a}_2, \bar{b}_1), (\bar{a}_3, \bar{b}_4), (\bar{a}_4, \bar{b}_3)$$

are also impossible as in Case III. The only points remaining to be tested are then

$$(\bar{a}_1, \bar{b}_3), (\bar{a}_2, \bar{b}_2), (\bar{a}_3, \bar{b}_1), (\bar{a}_4, \bar{b}_4),$$
$$(\bar{a}_1, \bar{b}_4), (\bar{a}_2, \bar{b}_3), (\bar{a}_3, \bar{b}_2), (\bar{a}_4, \bar{b}_1).$$

For the point $(\bar{a}_1, \bar{b}_3)$ let us put

$$b_3^* = cb_3 + (1-c)b_2.$$

Then

$$H_2(a_1, b_3^*) - H_2(a_1, b_3') = (1-c)(\gamma - \beta) > 0,$$
$$H_2(a_1, b_3^*) - H_2(a_1, b_3'') = (1-c)(2\delta - \alpha) > 0, \text{ etc.}\,.$$

It follows that

$$H_2(\bar{a}_1, b_3^*) - H_2(\bar{a}_1, \bar{b}_3) > 0$$

inasmuch as c is sufficiently near to 1. As

$$b_3^* \in \tau_2(\bar{b}_3)$$

we see that $(\bar{a}_1, \bar{b}_3)$ cannot be an equilibrium point. Similarly for the points $(\bar{a}_2, \bar{b}_2)$, $(\bar{a}_3, \bar{b}_1)$ and $(\bar{a}_4, \bar{b}_4)$.

For the point $(\bar{a}_1, \bar{b}_4)$ let us put

$$a_1^* = ca_1 + (1-c)a_2.$$

Then

$$H_1(a_1^*, b_4) - H_1(a_1', b_4) = (1-c)(\gamma - \beta) > 0,$$
$$H_1(a_1^*, b_4) - H_1(a_1'', b_4) = (1-c)(2\delta - \alpha) > 0, \text{ etc.}\,.$$

It follows that

$$H_1(a_1^*, \bar{b}_4) - H_1(\bar{a}_1, \bar{b}_4) > 0$$

inasmuch as c is sufficiently near to 1. As $a_1^* \in \tau_1(\bar{a}_1)$ we see that $(\bar{a}_1, \bar{b}_4)$ cannot be an equilibrium point. Similarly for the points $(\bar{a}_2, \bar{b}_3)$, $(\bar{a}_3, \bar{b}_2)$ and $(\bar{a}_4, \bar{b}_1)$.

From the above we see that no equilibrium points can exist for our game with restricted domains of activities, inasmuch as c is sufficiently near to 1, though each player possesses only a finite number of pure strategies.

References

[1] Begle, E. G. 1950 The Vietoris mapping theorem for bicompact spaces, *Annals of Math.*, **51**, 534—543.

[2] Dunford-Schwartz 1958 *Linear operators*, Part I. General theory, New York.

[3] Glicksberg, I. L. 1952 A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points, *Proc. Amer. Math. Soc.*, **3**, 170—174.

[4] Leray, J. 1945 Sur la forme des espaces topologiques et sur les points fixés des représentations, *J. de Math.*, **24**, 95—167.

[5] Nash, J. 1951 Non-cooperative games, *Annals of Math.*, **54**, 286—295.