SCIENCE RECORD

New Ser. Vol. III, No. 6, 1959

MATHEMATICS

A REMARK ON THE FUNDAMENTAL THEOREM IN THE THEORY OF GAMES*

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§1. The fundamental minimax theorem of von Neumann in the theory of games[1,2] has been generalized in various manners, all of which require strong *algebraic* hypothesis on the strategy spaces or the payoff functions or both, namely, linearity or convexity in certain sense (cf. e. g. [13–14]). The present note furnishes another generalization which is, however, purely *topological* in character, in contrast to all the known generalizations, especially to that of H. Weyl, which is purely *algebraic* in character. In our proof of the theorem, only elementary facts about point sets, neither fixed point theorems or the like nor theorems about convex sets, are used.

§2. Let R be a closed interval which, as is well known, may be characterized topologically as a continuum with all but two non-cut points, which are the end points of R. On R an order relation may then be established in just two manners so that for any subset E of R, the g. l. b. inf E and the l. u. b. sup E may be well-defined with respect to a chosen order relation $<$ and $>$ on R. For any point $\lambda \in R$, we shall denote by $\bar{R}_\lambda^+$, R_λ^+, R_λ^- and $\bar{R}_\lambda^-$ the subsets of R consisting of points z for which $z \geqslant \lambda$, $z > \lambda$, $z < \lambda$ and $z \leqslant \lambda$ respectively with regard to the above chosen order which will be fixed henceforth. For any spaces X, Y and any mapping f, continuous or not, of $X \times Y$ in R, we shall write $f(x, y) = f_x(y) = f_y(x)$, $x \in X$, $y \in Y$. We shall say that the mapping f is *strongly connected in* X, if it possesses the following two properties:

(P_1) For any a, $b \in X$, there exists a continuous mapping h of the closed interval I with end points $\bar{a}$, $\bar{b}$ in X such that $h(\bar{a}) = a$, $h(\bar{b}) = b$, and for any $y \in Y$ and any $\lambda \in R$, $h^{-1} f_y^{-1}(\bar{R}_\lambda^+)$ is a connected set, if not empty. (This implies that X should be arcwise connected.)

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* Received May 6, 1959.

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(P_2) For any finite number of points $x_1, \cdots, x_k \in X$ and any $\lambda \in R$, the set $f_{x_1}^{-1}(R_\lambda^-) \cap \cdots \cap f_{x_k}^{-1}(R_\lambda^-)$ is connected, if not empty.

The generalized von Neumann's theorem in question is then the following

Theorem. *Let* Y *be a compact separable space while* X *is arcwise connected.* If f: X×Y→R *is strongly connected in* X *and* f_x, f_y *are all continuous for any* $x \in X$, $y \in Y$, *then*

$$\operatorname{Inf}\{\operatorname{Sup} f_y(X)/y \in Y\} = \operatorname{Sup}\{\operatorname{Inf} f_x(Y)/x \in X\}. \tag{1}$$

§3. The proof of the theorem depends on the following lemmas:

Lemma 1. Let X be a closed interval with end points a, b, and Y be a separable space. Let λ be a fixed point of R and f be a mapping of $X \times Y$ in R such that for any $x \in X$ and $y \in Y$, $f_y^{-1}(\bar{R}_\lambda^+)$ and $f_x^{-1}(R_\lambda^-)$ are connected if not empty, f_x, f_y are all continuous, and $f_y^{-1}(\bar{R}_\lambda^+)$ contains either a or b. Then there exists a point $\xi \in X$ such that $f_\xi(Y) \subset \bar{R}_\lambda^+$.

Proof. Let $y_1, y_2, \cdots, y_n, \cdots$ be a countable dense set in Y. Put $I_i = f_{y_i}^{-1}(\bar{R}_\lambda^+)$, then I_i contains either a or b and is connected so that it is an interval containing, or a point reducing to, a or b. Take n fixed and suppose that $I_1 \cap \cdots \cap I_n$ is empty. Then for some integers α, β of the set $\{1, 2, \cdots, n\}$, we should have $I_\alpha \cap I_\beta = \phi$ so that $x_n \in X$ exists with $f_{y_\alpha}(x_n) \in R_\lambda^-$, $f_{y_\beta}(x_n) \in R_\lambda^-$. Hence $J = f_{x_n}^{-1}(R_\lambda^-)$ contains both y_α and y_β, and is by hypothesis connected. Say $a \in I_\alpha$. Then $a \notin I_\beta$, $b \in I_\beta$, $b \notin I_\alpha$, and $J_\alpha = J \cap f_a^{-1}(\bar{R}_\lambda^+)$ contains y_α while $J_\beta = J \cap f_b^{-1}(\bar{R}_\lambda^+)$ contains y_β. By hypothesis J_α, J_β are closed in J and $J_\alpha \cup J_\beta = J$. It follows that $J_\alpha \cap J_\beta \neq \phi$. Take $\eta \in J_\alpha \cap J_\beta \subset J$. Then $I = f_\eta^{-1}(\bar{R}_\lambda^+)$ contains both a and b. As I is connected, it coincides with X. Whence $x_n \in I = f_\eta^{-1}(\bar{R}_\lambda^+)$, or $\eta \in f_{x_n}^{-1}(\bar{R}_\lambda^+)$, contrary to $\eta \in J = f_{x_n}^{-1}(R_\lambda^-)$. Therefore $I_1 \cap \cdots \cap I_n \neq \phi$, and we may take $\xi_n \in I_1 \cap \cdots \cap I_n$ so that $f_{y_i}(\xi_n) \in \bar{R}_\lambda^+$, $i = 1, 2, \cdots, n$. Let ξ be a limiting point of $\xi_n, n = 1, 2, \cdots$, then we should have $f_{y_i}(\xi) = f_\xi(y_i) \in \bar{R}_\lambda^+$, for all i. As the set $\{y_i\}$ is dense in Y we have $f_\xi(y) \in \bar{R}_\lambda^+$ for all $y \in Y$, q. e. d.

Lemma 2_n. Let Y be a separable space, while X is arcwise connected. Let λ be a fixed point of R and $f: X \times Y \to R$ be a mapping strongly connected in X. If f_x, f_y are continuous for all $x \in X$, $y \in Y$ and if there exist n points $a_1, \cdots, a_n \in X$ such that $f_{a_1}^{-1}(\bar{R}_\lambda^+) \cup \cdots \cup f_{a_n}^{-1}(\bar{R}_\lambda^+) = Y$, then there exists a point $\xi \in X$ with $f_\xi(Y) \subset \bar{R}_\lambda^+$.

Proof. We shall use induction on n. The lemma is trivial for $n = 1$. Suppose the lemma is true for $n-1$, $n \geqslant 2$. Put $Y' = f_{a_n}^{-1}(R_\lambda^-)$. Then the hypotheses in Lemma 2_{n-1} for the pair of spaces X, Y' the point $\lambda \in R$, the mapping $f/X \times Y'$, and the set of points $a_1, \cdots, a_{n-1} \in X$ are verified and hence there exists a point $\xi' \in X$ with

$f_{\xi'}(Y') \subset \bar{R}_\lambda^+$. By ($P_1$) of the strong connectedness of the mapping f, there exists a continuous mapping h of the closed interval I with end points a, b in X such that $h(a) = a_n$, $h(b) = \xi'$ and $(f_y h)^{-1}(\bar{R}_\lambda^+)$ is a connected set for any $y \in Y$, if not empty. It follows that the hypotheses of Lemma 1 are verified for the pair of spaces (I, Y), the point $\lambda \in R$, and the mapping $\bar{f}$: $I \times Y \to R$ defined by $\bar{f}(x, y) = f_y h(x)$, $x \in I$, $y \in Y$. Hence there exists a point $\bar{\xi} \in I$ with $\bar{f}_{\bar{\xi}}(Y) \subset \bar{R}_\lambda^+$. For $\xi = h(\bar{\xi})$, we have then $f_\xi(Y) \subset \bar{R}_\lambda^+$, as required.

§4. Our theorem follows now easily by usual arguments. Suppose in fact that (1) is not true so that

$$\operatorname{Inf}\{\operatorname{Sup} f_y(X)/y \in Y\} > \operatorname{Sup}\{\operatorname{Inf} f_x(Y)/x \in X\}.$$

Then $\lambda \in R$ exists with

$$\operatorname{Inf}\{\operatorname{Sup} f_y(X)/y \in Y\} > \lambda > \operatorname{Sup}\{\operatorname{Inf} f_x(Y)/x \in X\}. \tag{2}$$

For any $y \in Y$ there exists then a point $x_y \in X$ with $f(x_y, y) \in R_\lambda^+$. As each set $U_y = f_{x_y}^{-1}(R_\lambda^+) \subset Y$ is open, the compactness of Y implies the existence of a finite number of points $y_1, \cdots, y_n \in Y$ with $a_i = x_{y_i}$ such that $U_{y_i} = f_{a_i}^{-1}(R_\lambda^+)$, *à fortiori* $f_{a_i}^{-1}(\bar{R}_\lambda^+)$, $i = 1, 2, \cdots, n$, cover Y. The hypotheses of Lemma 2_n are then satisfied with respect to the pair of spaces X, Y, the point $\lambda \in R$, the mapping f, and the system of points $a_1, \cdots, a_n \in X$. Hence there exists a point $\xi \in X$ with $f_\xi(Y) \subset \bar{R}_\lambda^+$. It would follow that $\operatorname{Sup}\{\operatorname{Inf} f_x(Y)/x \in X\} \geqslant \lambda$ in contradiction to (2). Hence (1) must be true and our theorem is proved.

§5. *Examples and Remarks.*

(A) Let X', Y' be spaces and f': $X' \times Y' \to R$ be a mapping verifying the conditions of our theorem. Let X, Y be any spaces homeomorphic to X', Y' under the homeomorphisms φ and ψ respectively and h any order-preserving topological transformation of R. Define f: $X \times Y \to R$ by $f(x, y) = hf'(\varphi(x), \psi(y))$, $x \in X$, $y \in Y$. Then (X, Y, f) verify also the conditions of our theorem. This shows the pure topological character of the above generalized von Neumann's theorem.

(B) If X, Y are convex subsets of linear topological spaces and f a real-valued function on $X \times Y$ which is quasi-concave in X and quasi-convex in Y in the sense of Nikaidô[12], then f is strongly connected in X. Hence our theorem contains the generalizations of von Neumann's theorem by Nikaidô on the further hypothesis about separability of the space Y and also those of Ville, Wald, Kneser, etc. On the other hand, it is independent of all other known generalizations, since no algebraic hypotheses are assumed in our theorem while all others do make them.

(C) Almost all generalizations suppose that the strategy spaces X and Y are convex so that they are topologically contractible to a point. The following is an example which is not so but meets the requirements of our theorem. Let X and Y be circles and let us imbed the torus $X \times Y$ in an ordinary manner in the 3-space with rectangular coordinates (x, y, z) such that the axis of the imbedded torus is the x–axis and the parallel circles correspond to the sets $X \times (y)$, $y \in Y$. The z–coordinate then defines a real-valued function f on $X \times Y$ which satisfies the conditions of our theorem. The unique optimal strategy may be seen to correspond to one of the two saddle points of the function f. If we define the function by means of the x–coordinate, then the set of optimal strategies is seen to correspond to a circle on the torus.

(D) That certain connectivity hypotheses about the spaces X, Y and the function f should be assumed in order to ensure the equality (1) without imposing any algebraic conditions, is quite natural. However, the following example shows that our conditions on connectivity are rather too strong. Let X, Y be circles as before and let us represent the torus $X \times Y$ as a square $ABCD$ with its opposite sides identified. Take a real-valued continuous function f on $X \times Y$ such that $f = 0$ on the four sides as well as its diagonal AC of the square $ABCD$, and $f > 0$ respectively $f < 0$ in the interior of the triangle ABC respectively ACD, with a single maximum respectively a single minimum in their interior, the level lines $f = c$ being triangles with sides parallel to those of ABC or ACD, according as $c > 0$ or $c < 0$. Then the condition (P_2) of the strong connectedness is satisfied but (P_1) is not. However, the equality (1) still holds with the set of optimal strategies reduced to a single point corresponding to the four vertices of the square and the value of the game is equal to zero. The same is true, if in the first example of (c) the roles of X and Y are interchanged.

References

[1] Von Neumann, J. 1928 *Math. Ann.* **100**, 295—320.

[2] Von Neumann, J. and Morgenstern, O. 1953 *Theory of games and economic behaviour*, 3rd. ed., Princeton.

[3] Ville, J. 1938 *Sur la théorie générale des jeux où intervient l'habileté des joueurs, Traité du calcul des probabilités et de ses applications*, **2**, 5, 105—113, Paris.

[4] Kakutani, S. 1941 *Duke Math. J.* **8**, 457—459.

[5] Wald, A. 1945 *Annals of Math.* **46**, 281—286.

[6] Karlin, S. 1950 *Operator treatment of minimax principle, Contributions to the theory of games*, **1**, Princeton.

[7] Weyl, H. 1950 *Elementary proof of a minimax theorem due to von Neumann*, *Contributions to the theory of games*, Vol. 1, Princeton.

[8] Kneser, H. 1952 *C. R. Paris.* **234**, 2418—2420.

[9] Fan, K. 1952 *Proc. Nat. Acad. Sci.* **38**, 121—126.

[10] Fan, K. 1953 *Proc. Nat. Acad. Sci.* **39**, 42—47.

[11] Berge, C. 1954 *Bull. Soc. Math.* France, **82**, 301—319.

[12] Nikaidô, H. 1954 *Pacific J. Math.* **4**, 65—72.

[13] Peck, J. E. L. and Dulmage, A. L. 1957 *Canadian J. Math.* **9**, 450—458.

[14] Sion, M. 1958 *Pacific J. Math.* **8**, 171—176.