# APPROXIMATION OF A PLANE WAVE BY SUPERPOSITIONS OF PLANE WAVES OF GIVEN DIRECTIONS

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Let $\mathbf{x} = (x_1, \cdots, x_n)$ be a point of the Euclidean space $R_n$, and let $D$ be a certain domain in this space. Further, let $\mathbf{a} = (a_1, \cdots, a_n)$ be a point of the $(n-1)$-dimensional real projective space $\Pi_{n-1}$, with given homogeneous coordinates $a_1, \cdots, a_n$. In the space $\Pi_{n-1}$, we choose a set of points $M$ and any point $\mathbf{a}$. Let $f(t)$ be an arbitrary function which is continuous in the interval

$$\inf_{\mathbf{x}\in D} (\mathbf{ax}) < t < \sup_{\mathbf{x}\in D} (\mathbf{ax}), \quad \mathbf{ax} = a_1x_1 + \ldots + a_nx_n. \tag{1}$$

We will derive a necessary and sufficient condition that every function $f(\mathbf{ax})$, $\mathbf{x} \in D$, can be approximated uniformly on the compact subsets of the domain $D$ by a summation of the form

$$\sum_{i=1}^{N} \varphi_i (\mathbf{a}_i\mathbf{x}), \tag{2}$$

where $N$ is an arbitrary natural number, $\mathbf{a}_i$ is a point of the set $M$ and $\phi_i(t_i)$ is a function continuous in the interval

$$\inf_{\mathbf{x}\in D} (\mathbf{a}_i\mathbf{x}) < t_i < \sup_{\mathbf{x}\in D} (\mathbf{a}_i\mathbf{x}), \quad \mathbf{a}_i\mathbf{x} = a_{i1}x_1 + \ldots + a_{in}x_n.$$

Toward that end, we will consider an arbitrary homogeneous polynomial $P(\mathbf{y}) = P(y_1, \cdots, y_n)$ in the variables $y_1, \cdots, y_n$ with real coefficients, and we will introduce the following definition.

**Definition.** We will say that the point $\mathbf{a}$, $\mathbf{a} \in \Pi_{n-1}$ is *algebraically related* to the set $M$, if every polynomial containing the set $M$ (i.e. vanishing at each of the points of $M$), contains also the point $\mathbf{a}$.

**Theorem.** *In order that an arbitrary continuous function $f(\mathbf{ax})$, $\mathbf{x} \in D$, can be uniformly approximated on the compact subsets of the domain $D$, by a summation of the form* (2), *it is necessary and sufficient that the point $\mathbf{a} = (a_1, \cdots, a_n)$ be algebraically related to the set $M$.*

**Proof of the necessity.** We assume that the arbitrary continuous function $f(\mathbf{ax})$ can be uniformly approximated on the compact subsets of the domain $D$ by a summation of the form (2). Let $P(y_1, \cdots, y_n)$ be an arbitrary polynomial of degree $m$ containing the set $M$. We choose any closed sphere $\overline{K}$ lying together with its boundary in the domain $D$. We denote by $v_{\overline{K}}$ the class of all functions $v(\mathbf{x})$ which are $m$ times continuously differentiable in the sphere $\overline{K}$, and which vanish together with their partial derivatives up to the $(m-1)$ st order inclusively, on the boundary of the sphere $\overline{K}$. For the class of functions $v_{\overline{K}}$, we define the functional

$$(u, v) = \int_{\overline{K}} u(\mathbf{x})\, P\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) v(\mathbf{x})\, d\mathbf{x},$$

where $u(\mathbf{x})$ is an arbitrary function which is continuous in the sphere $\overline{K}$. The functional $(u, v)$ is additive in $u$ and $v$ and, for a fixed function $v(\mathbf{x})$, is continuous in the sense of uniform convergence within the class of all functions $u(\mathbf{x})$ which are continuous in the sphere $\overline{K}$. We can easily see also that, if $P(a_{01}, \cdots, a_{0n}) = 0$, then $(\phi(\mathbf{a}_0\mathbf{x}), v) = 0$ for functions of the class $v_{\overline{K}}$, whatever be the continuous function $\phi(\mathbf{a}_0\mathbf{x})$, $\mathbf{x} \in \overline{K}$. From this, and because of the relatively suitable function $f(\mathbf{ax})$ assumed, it follows that $(f(\mathbf{ax}), v) = 0$ if $v(\mathbf{x}) \in v_{\overline{K}}$. Since the function $f(t)$ is arbitrary, we choose it such that, in the interval $\inf_{\mathbf{x}\in\overline{K}} (\mathbf{ax}) \leq t \leq \sup_{\mathbf{x}\in\overline{K}} (\mathbf{ax})$, it possesses a continuous arbitrary $f^{(m)}(t)$, where $f^{(m)}(t) \neq 0$ in this interval. Integrating by parts, we obtain

$$(f, v) = (-1)^m \int_D f^{(m)}(\mathrm{ax})\, P(a_1, \ldots, a_n)\, v(\mathrm{x})\, dx = 0,$$

from which, by virtue of the arbitrariness of the function $v(\mathbf{x}) \in v_{\overline{K}}$, it follows that

$$f^{(m)}(\mathrm{ax})\, P(a_1, \ldots, a_n) \equiv 0, \quad \mathrm{x} \in \overline{K},$$

i.e. $P(a_1, \cdots, a_n) = 0$ and the point $\mathbf{a} = (a_1, \cdots, a_n)$ is algebraically related to the set $M$.

Proof of the sufficiency. Since any continuous function in the interval (1) can be uniformly approximated by a polynomial on the compact subsets of this interval, it is enough for the proof of our statement to show, for any natural number $m$, the possibility of the representation

$$(\mathrm{ax})^m = \sum_{i=1}^{k} \lambda_i (\mathrm{a}_i\mathrm{x})^m, \quad \mathrm{a}_i \in M, \tag{3}$$

where $k$ is some natural number and $\lambda_1, \cdots, \lambda_k$ are real numbers.

It is obvious that the number $\lambda_i$ should satisfy the system of equations

$$\begin{gathered} a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n} = \sum_{i=1}^{k} a_{i1}^{m_1} a_{i2}^{m_2} \ldots a_{in}^{m_n} \lambda_i, \\ m_j \geqslant 0,\ m_1 + m_2 + \ldots + m_n = m. \end{gathered} \tag{4}$$

We choose $k = C_{m+n-1}^{n-1}$ and consider the determinant obtained from the system (4)

$$\Delta(\mathrm{a}_1, \ldots, \mathrm{a}_k) = | a_{i1}^{m_1} a_{i2}^{m_2} \ldots a_{in}^{m_n} |$$

for the totality of points belonging to the set $M$. If for a given $m$, there is no polynomial of degree $m$ containing the set $M$, other than the identically vanishing one, then it is easily seen that the totality of points of the set $M$ $\mathbf{a}_i^*$, $i = 1, \cdots, k$, for which $\Delta(\mathbf{a}_1^*, \cdots, \mathbf{a}_k^*) \neq 0$ are chosen and the possibility of the representation (3) for this $m$ is proved. If, however, some nontrivial polynomial does exist, then $\Delta(\mathbf{a}_1, \cdots, \mathbf{a}_k) = 0$, whatever be the totality of points $(\mathbf{a}_i, i = 1, \cdots, k)$, from the set $M$. In this case the choice is made from among the minors of the determinant $\Delta(\mathbf{a}_1, \cdots, \mathbf{a}_k)$ if only one minor $\Delta(\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_s})$, $s < k$ and only one system of points $\mathbf{a}_{i_1}^*, \cdots, \mathbf{a}_{i_s}^*$ exist such that $\Delta(\mathbf{a}_{i_1}^*, \cdots, \mathbf{a}_{i_s}^*) \neq 0$, while all the minors of the determinant $\Delta(\mathbf{a}_1, \cdots, \mathbf{a}_k)$ which are of higher order will be equal to zero at all the points of $M$.

If among the rows of the minor $\Delta(\mathbf{a}_{i_1}^*, \cdots, \mathbf{a}_{i_s}^*)$, there is no row of the form $a_{i_1}^{m_1^0} a_{i_2}^{m_2^0} \cdots a_{i_n}^{m_n^0}$, $m_1^0 + m_2^0 + \cdots + m_n^0 = m$, $m_j^0 \geq 0$, then we augment the minor $\Delta(\mathbf{a}_{i_1}^*, \cdots, \mathbf{a}_{i_s}^*)$ by the elements of that row and the elements of any column of the original determinant, taken at the arbitrary point $\mathbf{a}_i \in M$. We obtain some minor $\Delta(\mathbf{a}_{i_1}^*, \cdots, \mathbf{a}_{i_s}^*, \mathbf{a}_i)$. We fix the points $\mathbf{a}_{i_1}^*, \cdots, \mathbf{a}_{i_s}^*$ and let the point $\mathbf{a}_i$ run through all the points of the set $M$, thus tracing each time $\Delta(\mathbf{a}_{i_1}^*, \cdots, \mathbf{a}_{i_s}^*, \mathbf{a}_i) = 0$. Expanding this minor about the elements of the added column, we find that on the set $M$ there is a linear relationship between the power $a_{i_1}^{m_1^0} a_{i_2}^{m_2^0} \cdots a_{i_n}^{m_n^0}$ and the powers corresponding to the rows of the minor $\Delta(\mathbf{a}^*_{i_1}, \cdots, \mathbf{a}^*_{i_s})$, the power $a_{i_1}^{m_1^0} a_{i_2}^{m_2^0} \cdots a_{i_n}^{m_n^0}$ being linearly expressed in terms of the rest of the mentioned powers. We put the obtained linear combination into the corresponding homogeneous polynomial $P(y_1, \cdots, y_n)$, replacing, in the determinant $\Delta(\mathbf{a}^*_{i_1}, \cdots, \mathbf{a}^*_{i_s}, \mathbf{a}_i)$, the components $a_{i_1}, a_{i_2}, \cdots, a_{i_n}$ of the point $\mathbf{a}_i$ by the corresponding variables $y_1, y_2, \cdots, y_n$. Since the polynomial $P(y)$ contains the set $M$, it will necessarily contain the point $\mathbf{a}$ also, by the statement of the theorem. Hence it follows that the function $u = (\mathbf{a}x)^m$ satisfies the equation in the partial derivatives

$$P\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) u = 0.$$

Substituting the power $(\mathbf{a}x)^m$ in this equation, we find that exactly the same relationship that was found above for the point $\mathbf{a}_i \in M$, exists between the power $a_1^{m_1^0} a_2^{m_2^0} \cdots a_n^{m_n^0}$ and the powers corresponding to the rows of the minor $\Delta(\mathbf{a}^*_{i_1}, \cdots, \mathbf{a}^*_{i_s})$ taken at the point $\mathbf{a}$. Since the chosen row of the form $a_{i_1}^{m_1^0} a_{i_2}^{m_2^0} \cdots a_{i_n}^{m_n^0}$ does not enter in the composition of the rows of the determinant $\Delta(\mathbf{a}^*_{i_1}, \cdots, \mathbf{a}^*_{i_s})$, which is arbitrary, the solvability of the system (4) in terms of $\lambda_i$ follows from the proof and thus the possibility of the representation (3).

The mentioned theorem allows the notion of the algebraic relationship of a point to the set $M$ to be formulated as follows:

The point $\mathbf{a} \in \Pi_{n-1}$ is algebraically related to the set $M$, if it exists in the domain $D$ in such a manner that the arbitrary function $f(\mathbf{a}x)$, $x \in D$ ($f(t)$ continuous in the interval (1)), can be uniformly approximated on the compact subsets of $D$ by a summation of the form (2).

Received 20/JAN/62

Translated by:
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