APPROXIMATION OF A PLANE WAVE BY SUPERPOSITIONS
OF PLANE WAVES OF GIVEN DIRECTIONS

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Let \( x = (x_1, \cdots, x_n) \) be a point of the Euclidean space \( \mathbb{R}^n \), and let \( D \) be a certain domain in this space. Further, let \( a = (a_1, \cdots, a_n) \) be a point of the \((n - 1)\)-dimensional real projective space \( \Pi_{n-1} \), with given homogeneous coordinates \( a_1, \cdots, a_n \). In the space \( \Pi_{n-1} \), we choose a set of points \( M \) and any point \( a \). Let \( f(t) \) be an arbitrary function which is continuous in the interval

\[ \inf_{x \in D} (ax) < t < \sup_{x \in D} (ax), \quad ax = a_1x_1 + \cdots + a_nx_n. \]  

We will derive a necessary and sufficient condition that every function \( f(ax), x \in D \), can be approximated uniformly on the compact subsets of the domain \( D \) by a summation of the form

\[ \sum_{i=1}^{N} \phi_i(a_i, x), \]  

where \( N \) is an arbitrary natural number, \( a_i \) is a point of the set \( M \) and \( \phi_i(t) \) is a function continuous in the interval

\[ \inf_{x \in D} (a_i x) < t_i < \sup_{x \in D} (a_i x), \quad a_i x = a_1 x_1 + \cdots + a_n x_n. \]

Toward that end, we will consider an arbitrary homogeneous polynomial \( P(y) = P(y_1, \cdots, y_n) \) in the variables \( y_1, \cdots, y_n \) with real coefficients, and we will introduce the following definition.

Definition. We will say that the point \( a, a \in \Pi_{n-1} \) is algebraically related to the set \( M \), if every polynomial containing the set \( M \) (i.e., vanishing at each of the points of \( M \)), contains also the point \( a \).

Theorem. In order that an arbitrary continuous function \( f(ax), x \in D \), can be uniformly approximated on the compact subsets of the domain \( D \), by a summation of the form (2), it is necessary and sufficient that the point \( a = (a_1, \cdots, a_n) \) be algebraically related to the set \( M \).

Proof of the necessity. We assume that the arbitrary continuous function \( f(ax) \) can be uniformly approximated on the compact subsets of the domain \( D \) by a summation of the form (2). Let \( P(y_1, \cdots, y_n) \) be an arbitrary polynomial of degree \( m \) containing the set \( M \). We choose any closed sphere \( K \) lying together with its boundary in the domain \( D \). We denote by \( v_K \) the class of all functions \( v(x) \) which are \( m \) times continuously differentiable in the sphere \( K \), and which vanish together with their partial derivatives up to the \((m - 1)\) st order inclusively, on the boundary of the sphere \( K \). For the class of functions \( v_K \), we define the functional

\[ (u, v) = \int_{K} u(x) P \left( \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right) v(x) dx, \]

where \( u(x) \) is an arbitrary function which is continuous in the sphere \( K \). The functional \( (u, v) \) is additive in \( u \) and \( v \) and, for a fixed function \( v(x) \), is continuous in the sense of uniform convergence.
within the class of all functions \( u(x) \) which are continuous in the sphere \( \bar{K} \). We can easily see also that, if \( P(a_0, \ldots, a_n) = 0 \), then \( (\phi(a_0 x), v) = 0 \) for functions of the class \( v_K \), whatever be the continuous function \( \phi(a_0 x), x \in \bar{K} \). From this, and because of the relatively suitable function \( f(ax) \) assumed, it follows that \( (f(ax), v) = 0 \) if \( v(x) \in v_K \). Since the function \( f(t) \) is arbitrary, we choose it such that, in the interval \( \inf (ax) \leq t \leq \sup (ax) \), it possesses a continuous arbitrary \( f^{(m)}(t) \), where \( f^{(m)}(t) \neq 0 \) in this interval. Integrating by parts, we obtain

\[
(f, v) = (-1)^m \int_P f^{(m)}(ax) \, P(a_1, \ldots, a_n) \, v(x) \, dx = 0,
\]

from which, by virtue of the arbitrariness of the function \( v(x) \in v_K \), it follows that

\[
f^{(m)}(ax) \, P(a_1, \ldots, a_n) = 0, \quad x \in \bar{K},
\]

i.e. \( P(a_1, \ldots, a_n) = 0 \) and the point \( a = (a_1, \ldots, a_n) \) is algebraically related to the set \( M \).

Proof of the sufficiency. Since any continuous function in the interval \( (1) \) can be uniformly approximated by a polynomial on the compact subsets of this interval, it is enough for the proof of our statement to show, for any natural number \( m \), the possibility of the representation

\[
(ax)^m = \sum_{i=1}^k \lambda_i (a_i x)^m, \quad a_i \in M,
\]

where \( k \) is some natural number and \( \lambda_1, \ldots, \lambda_k \) are real numbers.

It is obvious that the number \( \lambda_i \) should satisfy the system of equations

\[
a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} = \sum_{i=1}^k a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} \lambda_i,
\]

\[
m_1 \geq 0, \quad m_1 + m_2 + \ldots + m_n = m.
\]

We choose \( k = \binom{n}{m+n-1} \) and consider the determinant obtained from the system (4)

\[
\Delta(a_1, \ldots, a_k) = |a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}|
\]

for the totality of points belonging to the set \( M \). If for a given \( m \), there is no polynomial of degree \( m \) containing the set \( M \), other than the identically vanishing one, then it is easily seen that the totality of points of the set \( M a_i^*, i = 1, \ldots, k \), for which \( \Delta(a_1^*, \ldots, a_k^*) \neq 0 \) are chosen and the possibility of the representation (3) for this \( m \) is proved. If, however, some nontrivial polynomial does exist, then \( \Delta(a_1, \ldots, a_k) = 0 \), whatever be the totality of points \( (a_i, i = 1, \ldots, k) \), from the set \( M \). In this case the choice is made from among the minors of the determinant \( \Delta(a_1, \ldots, a_k) \) if only one minor \( \Delta(a_1^*, \ldots, a_s^*) \neq 0 \), while all the minors of the determinant \( \Delta(a_1, \ldots, a_k) \) which are of higher order will be equal to zero at all the points of \( M \).

If among the rows of the minor \( \Delta(a_1^*, \ldots, a_k^*) \), there is no row of the form \( a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} \), \( m_1 + m_2 + \ldots + m_n = m \), \( m_j \geq 0 \), then we augment the minor \( \Delta(a_1^*, \ldots, a_k^*) \) by the elements of that row and the elements of any column of the original determinant, taken at the arbitrary point \( a_i \in M \). We obtain some minor \( \Delta(a_1^{*1}, \ldots, a_s^{*s}, a_i) \). We fix the points \( a_1^{*1}, \ldots, a_s^{*s} \) and let the point \( a_i \) run through all the points of the set \( M \), thus tracing each time \( \Delta(a_1^{*1}, \ldots, a_s^{*s}, a_i) = 0 \). Expanding this minor about the elements of the added column, we find that on the set \( M \) there is a linear relationship
between the powers \( a_{i1}^0 a_{i2}^0 \cdots a_{in}^0 \) and the powers corresponding to the rows of the minor \( \Lambda(a_{i1}^*, \ldots, a_{is}^*) \), the power \( a_{i1}^0 a_{i2}^0 \cdots a_{in}^0 \) being linearly expressed in terms of the rest of the mentioned powers. We put the obtained linear combination into the corresponding homogeneous polynomial \( P(y_1, \ldots, y_n) \), replacing, in the determinant \( \Lambda(a_{i1}^*, \ldots, a_{is}^*, a_{i}) \), the components \( a_{i1}, a_{i2}, \ldots, a_{in} \) of the point \( a_i \) by the corresponding variables \( y_1, y_2, \ldots, y_n \). Since the polynomial \( P(y) \) contains the set \( M \), it will necessarily contain the point \( a \) also, by the statement of the theorem. Hence it follows that the function \( u = (ax)^m \) satisfies the equation in the partial derivatives
\[
P \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) u = 0.
\]

Substituting the power \( (ax)^m \) in this equation, we find that exactly the same relationship that was found above for the point \( a_i \in M \), exists between the power \( a_{i1}^0 a_{i2}^0 \cdots a_{in}^0 \) and the powers corresponding to the rows of the minor \( \Lambda(a_{i1}^*, \ldots, a_{is}^*) \) taken at the point \( a \). Since the chosen row of the form \( a_{i1}^0 a_{i2}^0 \cdots a_{in}^0 \) does not enter in the composition of the rows of the determinant \( \Lambda(a_{i1}^*, \ldots, a_{is}^*) \), which is arbitrary, the solvability of the system (4) in terms of \( \lambda_i \) follows from the proof and thus the possibility of the representation (3).

The mentioned theorem allows the notion of the algebraic relationship of a point to the set \( M \) to be formulated as follows:

The point \( a \in \Pi_{n-1} \) is algebraically related to the set \( M \), if it exists in the domain \( D \) in such a manner that the arbitrary function \( f(ax), x \in D \) (\( f(t) \) continuous in the interval (1)), can be uniformly approximated on the compact subsets of \( D \) by a summation of the form (2).

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