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APPROXIMATION OF CONTINUOUS FUNCTIONS BY SUPERPOSITIONS OF PLANE WAVES

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A sequence of directions determined by the vectors $\mathbf{l}_i = (a_i) \neq 0$, $a_i = (a_{i1}, a_{i2}, \cdots, a_{in})$, where a_{ij} are real numbers and $i = 1, 2, \cdots$, will be called *basic*, if for a certain domain D of the n-dimensional euclidean space and an arbitrary function $f(x)$, $x = (x_1, x_2, \cdots, x_n)$, continuous in the domain there are functions $\phi_{ik}(t_i)$, $i = 1, 2, \cdots, k$, each continuous in the corresponding interval

$$\inf_{x\in D} (a_i x) < t_i < \sup_{x \in D} (a_i x); \quad a_i x = a_{i1}x_1 + \ldots + a_{in}x_n, \quad k = 1, 2, \cdots,$$

such that the sequence of sums

$$\Phi_k(x) = \sum_{i=1}^{k} \phi_{ik}(a_i x) \tag{1}$$

converges uniformly inside D to the function $f(x)$.

In this note we give necessary and sufficient conditions satisfied by any basic system of directions.

To formulate these necessary and sufficient conditions we shall consider the coordinates $a_1, a_2, \cdots, a_n$ of a vector $\mathbf{l} \neq 0$ as homogeneous coordinates of a point $A = (a) = (a_1, a_2, \cdots, a_n)$ of the $(n-1)$-dimensional projective space Π_{n-1}.

Theorem. *For a sequence of directions determined by the vectors $\mathbf{l}_i = (a_i) \neq 0$, $i = 1, 2, \cdots$ to be basic it is necessary and sufficient that the sequence of the points $A_i = (a_i)$ of the space Π_{n-1} does not belong to any $(n-2)$ dimensional algebraic surface of this space.*

From this theorem it follows in particular that a sequence of vectors $\mathbf{l}_i = (a_i)$ determining a basic sequence of directions cannot be entirely contained in any hyperplane of the n-dimensional vector space.

1. Proof of the necessity. We will show that if $(a_i) \in M$, $i = 1, 2, \cdots$, where M is some $(n-2)$-dimensional algebraic surface of the space Π_{n-1}, then in any domain D of the n-dimensional euclidean space there exist continuous functions which are not uniform limits of any convergent sequence of sums of the form (1).

Let

$$P(a_1, a_2, \cdots, a_n) = \sum_{m_1+m_2+\cdots+m_n=m} c_{m_1, m_2, \cdots, m_n} a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} = 0,$$

where $c_{m_1}, c_{m_2}, \cdots, c_{m_n}$ are constants, $m_j \geq 0$, $j = 1, 2, \cdots, n$, be the equation of the surface M in homogeneous coordinates. We take an arbitrary point $x_0 \in D$ and choose $\delta > 0$ so that the sphere $\bar{K}$, $\sum_{i=1}^{n} (x_i - x_{0i})^2 \leq \delta$, is in the domain D. Let us consider the operator

$$O_v(u) = \iint\limits_{\bar K} u(x)\, L\, [v(x)]\, dx, \quad dx = dx_1 \ldots dx_n,$$

where $L = P(\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_n)$ and $v(x)$ is a function m times continuously differentiable in the sphere $\bar K$ which vanishes on the boundary of the sphere $\bar K$ together with all its partial derivatives up to the order $(m-1)$ inclusive; we shall call such functions admissible.

Every function $u(x)$ continuous in the sphere $\bar K$ and of the form $u = \phi(ax)$, $(a) \in M$, annihilates the operator $O_v(u)$ for every admissible function $v(x)$. In fact, let us substitute an arbitrary admissible function $v(x)$ and the function $u = \phi(ax)$ in the operator $O_v(u)$ and, assuming that $a_1 \neq 0$ (which does not restrict generality) let us make under the integral sign in the operator L a change of variable setting

$$y_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n, \; y_2 = x_2, \cdots, y_n = x_n. \tag{2}$$

We shall have

$$\iint\limits_{\bar K} \varphi(ax)\, L\, [v(x)]\, dx = \frac{1}{|a_1|} \iint\limits_{\bar K'} \varphi(y_1)\, \bar L\, [\bar v(y)]\, dy, \tag{3}$$

where $\bar K'$ is the image of the sphere $\bar K$ and $\bar L$ results from transforming the operator L by means of the formulas (2).

Since

$$\bar L\, [\phi(y_1)] = L\, [\phi(ax)] = P(a_1, a_2, \cdots, a_n)\, \phi^{(m)}(ax) = 0$$

for any m times differentiable function ϕ, it is easy to see that the coefficient of $\partial^m/\partial y_1^m$ in the operator $\bar L$ is equal to 0. Furthermore let us observe that the function $\bar v(y) = v(x)$ vanishes on the boundary of the domain $\bar K'$ together with all its partial derivatives. Using this and applying in the right side of (3) a single termwise integration with respect to a variable not coinciding with y_1 we see the validity of the equation

$$O_v[\phi(ax)] = 0$$

for any admissible function $v(x)$.

From this and the additiveness of the integral and the possibility of passing to the limit in the operator $O_v(u)$ for a fixed function $v(x)$ it follows that if the functions $\phi_{ik}(a_ix)$, $i = 1, 2, \cdots, k$; $k = 1, 2, \cdots$, are continuous in the sphere $\bar K$, then every function $f(x)$ which is the limit of a sequence of sums of the form (1) uniformly converging in $\bar K$ also annihilates the operator $O_v(u)$ for every admissible function $v(x)$.

To complete the proof it is necessary to show the existence of a function $u = u_0(x)$ continuous in the domain D and not annihilating the operator $O_v(u)$ for any admissible function $v(x)$.

If $c_{m_1^0, m_2^0, \cdots, m_n^0}$ is a nonzero coefficient of the polynomial $P(a_1, a_2, \cdots, a_n)$ then such a function will be, for example, the function $u_0(x) = x_1^{m_1^0} x_2^{m_2^0} \ldots x_n^{m_n^0}$. For it and the function

$$v_0(x) = [\delta^2 - \sum_{i=1}^{n} (x_i - x_{0i})^2]^m$$

we shall have

$$O_v(u_0) \neq 0,$$

and thus the necessity of the conditions of the theorem is established.

2. **Proof of the necessity.** Let a sequence of directions determined by the vectors

$$\mathbf{l}_i = (a_i) \neq 0, \quad i = 1, 2, \cdots,$$

be such that the corresponding sequence of points

$$A_i = (a_i) \tag{4}$$

of the space Π_{n-1} does not belong to any $(n-2)$-dimensional algebraic surface. We may, obviously, consider the vectors $\mathbf{l}_i$, $i = 1, 2, \cdots$, as pairwise noncollinear.

For any natural number m and any aggregate of points $\{(ak_r)\} \subset \{(a_i)\}$, where k_r is a natural number, $r = 1, 2, \cdots, N$, $N = c^n_{m+n}$, $i = 1, 2, \cdots$, let us consider the identities

$$(a_{k_r1}x_1 + a_{k_r2}x_2 + \ldots + a_{k_rn}x_n)^m =$$
$$= \sum_{m_1+m_2+\ldots+m_n=m} \frac{m!}{m_1!m_2!\ldots m_n!} a^{m_1}_{k_r1} a^{m_2}_{k_r2} \ldots a^{m_n}_{k_rn} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}, \tag{5}$$
$$m_j \geqslant 0, \quad j = 1, 2, \ldots, n,$$

which we shall treat as a system of linear equations in the quantities

$$\frac{m!}{m_1!\, m_2! \ldots m_n!} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}. \tag{6}$$

The determinant of this system is equal to

$$\Delta(a_{k_1}, \ldots, a_{k_N}) = \left| a^{m_1}_{k_r1} a^{m_2}_{k_r2} \ldots a^{m_n}_{k_rn} \right|,$$
$$m_1 + m_2 + \ldots + m_n = m, \quad m_j \geqslant 0, \; j = 1, 2, \ldots, n.$$

We shall call a system $\{(a_{k_r})\}$, $r = 1, 2, \cdots, N$ for which $\Delta(a_{k_1}, \cdots, a_{k_N}) \neq 0$, a nondegenerate system of points of order m.

Let now $n = 2$. In this case for the sequence $\mathbf{l}_i$, $i = 1, 2, \cdots$, we may take any sequence of pairwise noncollinear vectors. It is also easy to see that when $n = 2$ any sequence of distinct points of the sequence (4) contaning $m + 1$ points is a nondegenerate system of order m, $m = 1, 2, \cdots$. Hence putting $k_r = r$, $r = 1, 2, \cdots, m + 1$ and solving the system (5) with respect to the unknowns (6) we find that every product $x_1^{m_1} x_2^{m_2}$, $m_1 + m_2 = m$, $m_1, m_2 \geq 0$, and consequently any homogeneous polynomial of degree m in the variables x_1, x_2, is representable as a linear combination of the powers $(a_ix)^m$, $i = 1, 2, \cdots, m + 1$. From this and the arbitrariness of m it follows that every polynomial $P_k(x)$ of degree k is representable in the form

$$P_k(x) = \sum_{i=1}^{k+1} \phi_i(a_ix), \tag{7}$$

where the functions $\phi_i(t)$ are also polynomials of degree k in t.

Hence taking a sequence of polynomials $\{P_k(x)\}$, $k = 1, 2, \cdots$, uniformly approximating the function $f(x)$ in D, and using the equation (7) we obtain the proof of sufficiency in the case $n = 2$.

For $n > 2$ an arbitrary system of points $\{(a_{k_r})\}$, $r = 1, 2, \cdots, N$, from the sequence (4) need not be nondegenerate. We shall show, however, that for any natural m there exist nondegenerate systems of points belonging to the indicated sequence. For this end assume the contrary and select from the sequence (4) any system of distinct points $\{(a^*_{k_r})\}$, $r = 1, 2, \cdots, N$. We fix any $N - 1$ among these points and make run the remaining point, say $(a^*_{k_j})$ through all the points of the sequence (a_i), $i = 1, 2, \cdots$.

By hypothesis, $\Delta(a^*_{k_1}, \cdots, a_{k_j}, \cdots, a_{k_N}) = 0$ each time. Since the initial sequence does not belong to any $(n-2)$-dimensional algebraic surface, all minors of the jth row of the determinant $\Delta(a^*_{k_1}, \cdots, a^*_{k_N})$ are equal to 0. In view of the arbitrariness of the choice of the system of the points $\{(a_{k_r})\}$ and the variable point $(a^*_{k_j})$ among the points of this system, it follows from the foregoing that, generally, any minor of order $(N-1)$ of the determinant $\Delta(a_{k_1}, \cdots, a_{k_N})$, taken for an arbitrary system of points of the sequence (4) is equal to 0. Repeating this argument for each minor of order $(N-1)$ we find that all minors of order $(N-2)$ of the determinant under consideration are also equal to 0 for all possible systems of points of our sequence. Continuing in the same way we arrive at the identical vanishing of all minors of order 1 of the determinant $\Delta(a_{k_1}, \cdots, a_{k_N})$ for all points of the sequence (a_i), $i = 1, 2, \cdots$, and this contradicts the hypothesis since $1_i \neq 0$, $i = 1, 2, \cdots$.

Hence the sequence (a_i), $i = 1, 2, \cdots$, contains nondegenerate systems of points of any order m. For each m we fix one such system and substitute the coordinates of the points of this system into the relations (5). If now $P_k(x)$ is an arbitrary polynomial of degree k, then solving the so obtained system (5) with respect to the unknowns (6) for $m = 1, 2, \cdots, k$, and replacing in the polynomial $P_k(x)$ the powers $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, $m_1 + m_2 + \cdots + m_n = m$, $m_j \geq 0$, by the solutions we arrive at the equation

$$P_k(x) = \sum_{i=1}^{N_k} \phi_i(a_i x),$$

where $\phi_i(t)$, $i = 1, 2, \cdots, N_k$, are certain polynomials in t of degree not exceeding k. The proof of the sufficiency of the conditions of the theorem then reduces to the possibility of a uniform approximation of the function $f(x)$ in the interior of the domain D by polynomials.

Using the preceding result we can, for example, assert the following.

Let t_i, $i = 1, 2, \cdots$, be a sequence consisting of infinitely many distinct real numbers and converging to some number t_0. Then the sequence of directions $(t_i, e^{t_i}, 1)$, $i = 1, 2, \ldots$, in the three-dimensional space is basic.

This assertion follows from the fact that the end points of the vectors taken in the plane Π_2 do not lie on any algebraic curve.

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