10

ITERATIVE METHODS FOR CONCAVE PROGRAMMING

HIROFUMI UZAWA

1. Introduction

In order to approach saddle-points of the Lagrangian of the concave programming problem, we have, in Chapters 6, 7, and 8, considered a system of differential equations—the so-called gradient method. In the present chapter we shall formulate the gradient method in a system of difference equations and investigate the stability of the solution. It will first be shown that the solution monotonically converges to any given neighborhood of a saddle-point provided the rate of change is sufficiently small. Then the method will be slightly modified so that the system consists of the Lagrangian multipliers, and the approximate stability of the modified system will be shown. Especially, for the case where the restrictions are linear, the system is proved to be stable provided the rate of change is small. Finally, linear programming will be reduced to strictly concave quadratic programming and the above iterative methods will be applied to solve linear programming.

2. Concave Programming and Saddle-Point Problem

Let $f(x)$ and $g(x) = (g_1(x), \cdots, g_m(x))$ be functions defined for $x = (x_1, \cdots, x_n) \geqq 0$, and consider

PROBLEM A. *Find a vector $\bar{x}$ that maximizes $f(x)$ subject to the restrictions*

$$(1) \qquad x \geqq 0\,, \qquad g(x) \geqq 0\,.$$

It will be assumed that

(a) *$f(x)$ and $g_1(x), \cdots, g_m(x)$ are concave functions in $x \geqq 0$ and have continuous partial derivatives*, and

(b) *there exists a vector x^0 such that*

$$(2) \qquad x^0 \geqq 0\,, \qquad g(x^0) > 0\,.$$

Then the Kuhn-Tucker Theorem on concave programming (Chapter 3, Theorem 2) may be applied: a vector $\bar{x}$ is an optimum solution to the problem if, and only if, there is an m-vector $\bar{u}$ such that $(\bar{x}, \bar{u})$ is a saddle-point of the Lagrangian

$$(3) \qquad \varphi(x, u) = f(x) + u \cdot g(x)$$

in $x \geqq 0$ and $u \geqq 0$; i.e.,

$$(4) \qquad \varphi(\bar{x}, \bar{u}) = \max_{x \geqq 0} \varphi(x, \bar{u}) = \min_{u \geqq 0} \varphi(\bar{x}, u)\,.$$

Therefore, solving concave programming is reduced to finding a saddle-point of the Lagrangian $\varphi(x, u)$.

We shall denote by $\bar{X}$ and $\bar{U}$ the sets of the x-components and the u-components of saddle-points of $\varphi(x, u)$; i.e.,

$$\overline{X} = \{\bar{x} \mid (\bar{x}, \bar{u}) \text{ is a saddle-point of } \varphi(x, u) \text{ for some } \bar{u}\} ,$$
$$\overline{U} = \{\bar{u} \mid (\bar{x}, \bar{u}) \text{ is a saddle-point of } \varphi(x, u) \text{ for some } \bar{x}\} .$$

It will be noted that *the set* $\overline{U}$ *is compact* (i.e., closed and bounded).

Let $\bar{x} \in \overline{X}$, $\bar{u} \in \overline{U}$. Then

(5) $$\bar{u} \cdot g(\bar{x}) = 0 ,$$

(6) $$f(x) + \bar{u} \cdot g(x) \leqq f(\bar{x}) , \qquad \text{for all } x \geqq 0.$$

Take the vector x^0 for which (2) is satisfied. Then (6) implies that

$$0 \leqq \bar{u}_k \leqq \frac{f(\bar{x}) - f(x^0)}{g_k(x^0)} \qquad (k = 1, \cdots, m) ,$$

which shows that $\overline{U}$ is bounded. Since $\bar{u} \in \overline{U}$ is characterized by (5) and (6), $\overline{U}$ is closed.

In the following sections it will be assumed that

(c) $f(x)$ *is strictly concave in* x.

The optimum solution $\bar{x}$ is then uniquely determined.

3. The Arrow-Hurwicz Gradient Method

Consider a system of difference equations defined by

(I) $$\begin{cases} x(t+1) = \max\,\{0, x(t) + \rho\varphi_x(x(t), u(t))\} , \\ u(t+1) = \max\,\{0, u(t) - \rho\varphi_u(x(t), u(t))\} , \end{cases}$$

with an initial position $(x(0), u(0))$ such that

$$x(0) \geqq 0 , \qquad u(0) \geqq 0 ,$$

where ρ is a given positive number, φ_x and φ_u stand for the partial derivatives of φ with respect to x and u, respectively:

$$\varphi_x(x, u) = f_x(x) + u \cdot g_x(x) ,$$
$$\varphi_u(x, u) = g(x) .$$

We shall define the system (I) as **-stable* if the following condition is satisfied:

(*) *For any initial position* $(x(0), u(0)) \geqq 0$ *and any positive number* $\varepsilon > 0$, *there exists a positive number* $\rho_0 > 0$ *such that, for the solution* $(x(t), u(t))$ *of the system* (I) *with* $\rho \leqq \rho_0$, *there is an integer* t_0 *with the properties*

(7) $$V[x(t+1), u(t+1)] \leqq V[x(t), u(t)] , \qquad \textit{for } 0 \leqq t < t_0 ,$$

and

(8) $$V[x(t), u(t)] \leqq \varepsilon , \qquad \textit{for } t \geqq t_0 ,$$

where

$$V(x, u) = \min_{\bar{u} \in \overline{U}} \{|x - \bar{x}|^2 + |u - \bar{u}|^2\} .$$

Now we prove

THEOREM 1. *Let Problem A satisfy* (a), (b), *and* (c). *Then the system* (I) *is *-stable.*

PROOF. It is first noted that, by concavity of $\varphi(x, u)$ in x,

(9) $$(\bar{x} - x) \cdot \varphi_x - (\bar{u} - u) \cdot \varphi_u > 0\,, \qquad \text{for } x \neq \bar{x} \text{ ~~or~~ } u \notin \bar{U}\,.$$

Now from the first equation of (I) we have

$$|x(t+1)|^2 \leqq |x(t)|^2 + 2\rho x(t) \cdot \varphi_x(x(t), u(t)) + \rho^2 |\varphi_x(x(t), u(t))|^2\,,$$
$$-2\bar{x} \cdot x(t+1) \leqq -2\bar{x} \cdot x(t) - 2\rho\bar{x} \cdot \varphi_x(x(t), u(t))\,.$$

Hence, we get

(10) $$|x(t+1) - \bar{x}|^2 \leqq |x(t) - \bar{x}|^2 - 2\rho(\bar{x} - x(t)) \cdot \varphi_x(x(t), u(t)) + \rho^2 |\varphi_x(x(t), u(t))|^2\,.$$

Similarly, from the second equation of (I) we have

$$|u(t+1)|^2 \leqq |u(t)|^2 - 2\rho u(t) \cdot \varphi_u(x(t), u(t)) + \rho^2 |\varphi_u(x(t), u(t))|^2\,,$$
$$-2\bar{u} \cdot u(t+1) \leqq -2\bar{u} \cdot u(t) + 2\rho\bar{u} \cdot \varphi_u(x(t), u(t))\,.$$

Hence, we get

(11) $$|u(t+1) - \bar{u}|^2 \leqq |u(t) - \bar{u}|^2 + 2\rho(\bar{u} - u(t)) \cdot \varphi_u(x(t), u(t)) + \rho^2 |\varphi_u(x(t), u(t))|^2\,.$$

From (10) and (11),

(12) $$[|x(t+1) - \bar{x}|^2 + |u(t+1) - \bar{u}|^2] \leqq [|x(t) - \bar{x}|^2 + |u(t) - \bar{u}|^2] - \rho\{2[(\bar{x} - x(t)) \cdot \varphi_x(x(t), u(t)) - (\bar{u} - u(t)) \cdot \varphi_u(x(t), u(t))] - \rho[|\varphi_x(x(t), u(t))|^2 + |\varphi_u(x(t), u(t))|^2]\}\,.$$

Let ε be a given positive number. We define ρ_0 as the minimum of the following two numbers :

$$\min\left\{\sqrt{\frac{(\varepsilon/2)}{|\varphi_x|^2 + |\varphi_u|^2}}\;\middle|\; V(x, u) \leqq \frac{\varepsilon}{2}\right\}$$

and

$$\min\left\{\frac{(\bar{x} - x) \cdot \varphi_x - (\bar{u} - u) \cdot \varphi_u}{|\varphi_x|^2 + |\varphi_u|^2}\;\middle|\; \frac{\varepsilon}{2} \leqq V(x, u) \leqq K,\, \bar{u} \in \bar{U}\right\},$$

where

$$K = \max\{\varepsilon, V(x(0), u(0))\} > 0\,.$$

~~By (9) and compactness of $\bar{U}$, $\{(x, u) \mid V(x, u) \leqq \varepsilon/2\}$ and $\{(x, u) \mid \varepsilon/2 \leqq V(x, u) \leqq K\}$, ρ_0 is *positive*.~~

Let $(x(t), u(t))$ be the solution of (I) with $\rho \leqq \rho_0$. Then (12) and the definition of ρ_0 imply that, for any $\bar{u} \in \bar{U}$,

(13) $$|x(t+1) - \bar{x}|^2 + |u(t+1) - \bar{u}|^2 < |x(t) - \bar{x}|^2 + |u(t) - \bar{u}|^2, \qquad \text{if } \varepsilon/2 \leqq V(x(t), u(t)) \leqq K,$$

and

(14) $$|x(t+1) - \bar{x}|^2 + |u(t+1) - \bar{u}|^2 \leqq \varepsilon\,, \quad \text{if } V(x(t), u(t)) \leqq \varepsilon/2.$$

Since $V(x(0), u(0)) \leqq K$, we have

$$V(x(t), u(t)) \leqq K \qquad (t = 0, 1, 2, \cdots)\,.$$

Hence, the sequence $\{(x(t), u(t))\}$ is bounded. Let (x^*, u^*) be a limiting point of $\{(x(t), u(t))\}$ such that $V(x^*, u^*)$ is the minimum among the limiting points, i.e.,

$$V(x^*, u^*) \leqq V(x^{**}, u^{**}) , \tag{15}$$

for any limiting point (x^{**}, u^{**}) of $\{(x(t), u(t))\}$.

Then we have

$$V(x^*, u^*) \leqq \varepsilon/2 . \tag{16}$$

In order to show (16), we may without loss of generality assume that

$$(x^*, u^*) = \lim_{\nu\to\infty} (x(t_\nu), u(t_\nu))$$

such that $(x(t_\nu + 1), u(t_\nu + 1))$ will converge, say to (x^{**}, u^{**}). Then

$$x^{**} = \max\{0, x^* + \rho\varphi_x(x^*, u^*)\} ,$$
$$u^{**} = \max\{0, u^* - \rho\varphi_u(x^*, u^*)\} .$$

If we had assumed $V(x^*, u^*) > \varepsilon/2$, then, by a formula similar to (13),

$$V(x^{**}, u^{**}) < V(x^*, u^*) ,$$

which would contradict (15). By (13), (14), and (16), there is an integer t_0 for which (7) and (8) are satisfied, q.e.d.

4. A Modified Arrow-Hurwicz Gradient Method

In this section we consider an iterative method, which is a modification of the one described in Section 3.

Here the maximum problem may be formulated as follows:

PROBLEM B. *To find a vector x that maximizes $f(x)$ subject to the restriction*

$$g(x) \geqq 0 . \tag{17}$$

The non-negativity restriction on x, if there is any, may be included in (17), so that Problem A can be reduced to the problem in this section.

It will be assumed that Problem B satisfies (a), (b), (c), and

(d) *for any $u \geqq 0$, $\varphi(x, u)$ has a finite maximum with respect to x.*

In this case, a vector $\bar{x}$ is an optimum solution to the problem if and only if there is a vector $\bar{u}$ such that $(\bar{x}, \bar{u})$ is a saddle-point of the Lagrangian $\varphi(x, u)$ in x unrestricted and $u \geqq 0$; i.e.,

$$\varphi(x, \bar{u}) \leqq \varphi(\bar{x}, \bar{u}) \leqq \varphi(\bar{x}, u) \qquad \text{for all } x \text{ unrestricted and } u \geqq 0. \tag{18}$$

Now, for any given $u \geqq 0$, the vector that maximizes $\varphi(x, u)$ with respect to unrestricted x is uniquely determined by u. We shall denote it by $x(u)$:

$$\varphi(x(u), u) = \max_x \varphi(x, u) . \tag{19}$$

The vector $x(u)$ is characterized as the solution of $\varphi_x = 0$, i.e.,

$$f_x(x(u), u) + u \cdot g_x(x(u)) = 0 . \tag{20}$$

We may consider the kth component u_k of u as an imputed price of the kth factor, and $x(u)$ as the optimal level of production that maximizes the net profit

$$f(x) + u \cdot g(x) ,$$

supposing there be no factor limitations. Then $g_k(x(u))$ represents the excess of supply over demand of the kth factor for the price system u. Therefore in setting the u_k in the next stage, it may be reasonable to determine u_k higher if there is an excess demand, i.e., $g_k(x(u)) < 0$, and lower if there is an excess supply, i.e., $g_k(x(u)) > 0$; the rates of increase and decrease are proportional to the amounts of the excess demand and excess supply, respectively. Furthermore, we have to take into consideration that the imputed price u_k should not be negative.

The above consideration leads us to the following formulation of an iterative method:

$$\text{(II)} \qquad u(t+1) = \max\{0, u(t) - \rho g(x(t))\} \qquad (t = 0, 1, 2, \cdots) ,$$

with an initial position $u(0) \geqq 0$, and a given rate of change $\rho > 0$, where

$$\text{(21)} \qquad x(t) = x(u(t)) \qquad (t = 0, 1, 2, \cdots) .$$

We define the system (II) as **-stable with respect to* $u(t)$ if, for any initial position $u(0) \geqq 0$ and any positive number $\varepsilon > 0$, there exists a positive number $\rho_0 > 0$ such that, for the solution $u(t)$ of the system (II) with $\rho \leqq \rho_0$, there exists an integer t_0 with the properties:

$$\text{(22)} \qquad V(u(t+1)) < V(u(t)) , \qquad \text{for } 0 \leqq t < t_0 ,$$

and

$$\text{(23)} \qquad V(u(t)) \leqq \varepsilon , \qquad \text{for } t \geqq t_0 ,$$

where

$$V(u) = \min_{\bar{u} \in \bar{U}} |u - \bar{u}|^2 .$$

THEOREM 2. *Let Problem* B *satisfy* (a), (b), (c), *and* (d). *Then the system* (II) *is *-stable with respect to* $u(t)$.

Consequently, $x(t)$ *converges to an arbitrary small neighborhood of* $\bar{x}$ *provided the rate of change* ρ *is sufficiently small.*

PROOF. Since $x(u)$ uniquely maximizes $\varphi(x, u) = f(x) + u \cdot g(x)$ with respect to x, we have

$$\text{(24)} \qquad f(x) + u \cdot g(x) < f(x(u)) + u \cdot g(x(u)) \qquad \text{for } x \neq x(u) .$$

Let $u \notin \bar{U}$ and $\bar{u} \in \bar{U}$. If $x(u) \neq \bar{x} = x(\bar{u})$, then

$$\text{(25)} \qquad f(\bar{x}) + u \cdot g(\bar{x}) < f(x(u)) + u \cdot g(x(u)) ,$$

$$\text{(26)} \qquad f(x(u)) + \bar{u} \cdot g(x(u)) < f(\bar{x}) + \bar{u} \cdot g(\bar{x}) .$$

Summing (25) and (26), and noting that $g(\bar{x}) \geqq 0$, $\bar{u} \cdot g(\bar{x}) = 0$, we have

$$\text{(27)} \qquad (u - \bar{u}) \cdot g(x(u)) > 0 , \qquad \text{for any } u \notin \bar{U} \text{ and } \bar{u} \in \bar{U}.$$

If $x(u) = \bar{x}$, then $u \cdot g(x(u)) > 0$ and $\bar{u} \cdot g(x(u)) = 0$. Therefore, the relation (27) is also valid.

Now, from (II),

$$|u(t+1)|^2 \leqq |u(t)|^2 - 2\rho u(t)\cdot g(x(t)) + \rho^2 |g(x(t))|^2 ,$$
$$-2\bar{u}\cdot u(t+1) \leqq -2\bar{u}\cdot u(t) + 2\rho\bar{u}\cdot g(x(t)) .$$

Then we have

(28) $$|u(t+1)-\bar{u}|^2 \leqq |u(t)-\bar{u}|^2 - \rho\{2(u(t)-\bar{u})\cdot g(x(t)) - \rho|g(x(t))|^2\} .$$

Let ρ_0 be a number defined by

(29) $$\rho_0 = \min\left\{\min_{V(u)\leqq \varepsilon/2} \frac{\sqrt{\varepsilon/2}}{|g(x(u))|},\ \min_{\substack{\varepsilon/2\leqq V(u)\leqq K\\ \bar{u}\in\bar{U}}} \frac{(u-\bar{u})\cdot g(x(u))}{|g(x(u))|^2}\right\}$$

where

$$K = \max\{\varepsilon,\ V(u(0))\} .$$

By (27), ρ_0 is *positive.*

Then, (28) and the definition of ρ_0 imply that, for the solution $u(t)$ of (II) with $\rho \leqq \rho_0$,

(30) $$|u(t+1)-\bar{u}|^2 < |u(t)-\bar{u}|^2, \qquad \text{if } \varepsilon/2 \leqq V(u(t)) \leqq K,\ \bar{u}\in\bar{U} ,$$

and

(31) $$|u(t+1)-\bar{u}|^2 \leqq \varepsilon , \qquad \text{if } V(u(t)) \leqq \varepsilon/2,\ \bar{u}\in\bar{U}.$$

Similar to the proof of Theorem 1, (30) and (31) imply the monotonic convergence of $u(t)$ to the ε-neighborhood of $\bar{U}$, q.e.d.

A careful examination of the proof of Theorem 2 shows that the system (II) is *-stable for a broader class of the problems; namely, we can easily prove the following theorem.

THEOREM 3. *Let the Lagrangian*

$$\varphi(x, u) = f(x) + u\cdot g(x)$$

satisfy the following conditions:

(i) *There exists a closed set A of n-vectors such that, for any $u \geqq 0$, $\varphi(x, u)$ has a finite maximum with respect to $x \in A$ and the vector $x_A(u)$ maximizing $\varphi(x, u)$ in A is uniquely determined.*

(ii) *There is a saddle-point $(\bar{x}_A, \bar{u}_A)$ of $\varphi(x, u)$ in $x \in A$ and $u \geqq 0$.*

Then the system

(II)′ $$u_A(t+1) = \max\{0,\ u_A(t) - \rho g(x_A(t))\} \qquad (t = 0, 1, 2, \cdots) ,$$

*with $x_A(0) \in A$, is *-stable.*

It will be noted that, in Theorem 3, we do not assume concavity of functions $f(x)$ and $g(x)$.

5. Concave Programming with Linear Restrictions

We shall now consider the case where the restrictions are linear, i.e.,

$$g(x) = b - Bx ,$$

and show that the method explained in Section 4 converges to $\bar{U}$, provided ρ is sufficiently small.

It is again assumed that

(b) *there is* x^0 *such that* $b - Bx^0 > 0$,
(c) $f(x)$ *is strictly concave and has continuous partial derivatives* f_{xx},
(d) *for any* $u \geqq 0$, $\max_x \varphi(x, u)$ *is finite.*

The Lagrangian $\varphi(x, u)$ in the present case becomes

(32) $$\varphi(x, u) = f(x) + u \cdot (b - Bx) .$$

For any $u \geqq 0$, the vector $x(u)$ that maximizes $\varphi(x, u)$ in x is characterized by the solution of the following equation:

(33) $$f_x(x(u)) - B'u = 0 .$$

The system (II) in this case will be written as follows:

(III) $$u(t+1) = \max\{0, u(t) - \rho(b - Bx(t))\} \qquad (t = 0, 1, 2, \cdots) .$$

THEOREM 4. *Suppose* $f(x)$ *and* $g(x) = b - Bx$ *satisfy the conditions* (b), (c), *and* (d). *Then there exists a positive number* $\rho_0 > 0$ *such that the solution* $u(t)$ *of* (III) *with* $\rho \leqq \rho_0$ *monotonically converges to a vector* $\bar{u} \in \bar{U}$. *Consequently the corresponding* $x(t)$ *converges to the optimum vector* $\bar{x}$.

PROOF. Let $\bar{x}$ be the unique optimum vector for the problem and II be the set of corner indices; i.e., $II = \{k \mid g_k(\bar{x}) > 0\}$, and $I = \{1, \cdots, m\} - II$.

Then, for any $\bar{u} \in \bar{U}$, we have

(34) $$\bar{u}_{II} = 0 .$$

By (33) and (c), $x(u)$ is a continuous function of u, so that *there exists a positive number* $\varepsilon > 0$ *such that*

(35) $$V(u) \leqq \varepsilon \qquad \textit{implies } g_{II}(x(u)) > 0 .$$

For this ε, let $\rho_0(u(0), \varepsilon)$ be the positive number defined by (29).

Since $f(x)$ is *strictly* concave, the matrix (f_{xx}) is negative definite. Let us denote by $\lambda(x)$ the maximum value of characteristic roots of $B(-f_{xx})^{-1}B'$ at x. Then

(36) $$\lambda(x) \geqq 0 .$$

Let

(37) $$\rho_0 = \min\left\{\rho_0(u(0), \varepsilon),\ \min_{V(u) \leqq \varepsilon} \frac{1}{\lambda(x(u))}\right\} .$$

By (36) and compactness of the set $\{x(u) \mid V(u) \leqq \varepsilon\}$, ρ_0 is *positive.*

We shall now show that, for any $\rho \leqq \rho_0$, the solution $u(t)$ of (III) converges to a $\bar{u} \in \bar{U}$. We may, by Theorem 3, suppose that

(38) $$V(t) \leqq \varepsilon , \qquad \text{for } t = 0, 1, 2, \cdots .$$

It will be first noted that *there is an integer* $\bar{t}$ *such that*

(39) $$u_{II}(t) = 0 , \qquad \text{for } t \geqq \bar{t} .$$

In fact, (35), (38), and compactness of the set $\{u \mid V(u) \leqq \varepsilon\}$ imply that

$$\min g_k(x(t)) > 0 , \qquad \text{for } k \in II \qquad (t = 0, 1, 2, \cdots) .$$

Therefore there is an integer $\bar{t}$ such that

$$u_{II}(\bar{t}-1) - \rho g_{II}(x(\bar{t}-1)) \leqq 0 .$$

Then

$$u_{II}(\bar{t}) = 0 ,$$

and (39) is satisfied.

Now, from (III), we have

$$|u_I(t+1)|^2 \leqq |u_I(t)|^2 - 2\rho u_I(t)\cdot(b_I - B_I x_I(t)) + \rho^2 |b_I - B_I x_I(t)|^2 ,$$
$$-2\bar{u}_I \cdot u_I(t+1) \leqq -2\bar{u}_I \cdot u_I(t) + 2\rho\bar{u}_I \cdot (b_I - B_I x_I(t)) ,$$

where

$$u(t) = \begin{pmatrix} u_I(t) \\ \\ u_{II}(t) \end{pmatrix}, \quad b = \begin{pmatrix} b_I \\ \\ b_{II} \end{pmatrix}, \quad B = \begin{pmatrix} B_I \\ \\ B_{II} \end{pmatrix}.$$

Then we have

(40) $$|u_I(t+1) - \bar{u}_I|^2 \leqq |u_I(t) - \bar{u}_I|^2 - \rho\{2(u_I(t) - \bar{u}_I)\cdot(b_I - B_I x(t)) - \rho |b_I - B_I x(t)|^2\} .$$

Now, by the definition of I, we have

(41) $$b_I - B_I \bar{x} = 0 .$$

Hence,

(42) $$b_I - B_I \cdot x(t) = B_I(\bar{x} - x(t)) .$$

Now the relations (33) and (39) imply, for $t \geqq \overline{t},$

(43) $$\left(\frac{df}{dx}\right)_{x(t)} - B'_I u_I(t) = 0 ,$$

(44) $$\left(\frac{df}{dx}\right)_{\bar{x}} - B'_I \bar{u}_I = 0 .$$

On the other hand, we have

(45) $$\left(\frac{df}{dx}\right)_{x(t)} = \left(\frac{df}{dx}\right)_{\bar{x}} + (f^{\theta}_{xx})(x(t) - \bar{x}) ,$$

where

$$f^{\theta}_{xx} = (f_{xx})_{x^\theta} , \quad x^\theta = \bar{x} + \theta(x(t) - \bar{x}) , \quad 0 \leqq \theta \leqq 1 .$$

Equations (43), (44), and (45) imply

(46) $$B'_I(u_I(t) - \bar{u}_I) = (-f^{\theta}_{xx}) \cdot (\bar{x} - x(t)) .$$

Since (f^{θ}_{xx}) is non-singular,

(47) $$\bar{x} - x(t) = (-f^{\theta}_{xx})^{-1} B'_I \cdot (u_I(t) - \bar{u}_I) .$$

Substituting (47) into (42), we get

(48) $$b_I - B_I \cdot x(t) = B_I(-f^{\theta}_{xx})^{-1} B'_I \cdot (u_I(t) - \bar{u}_I) , \qquad t \geqq t_1 .$$

Therefore, by (37), (47), and (48), we have

(49)
$$\begin{aligned} &2(u_I(t) - \bar{u}_I) \cdot (b_I - B_I x(t)) - \rho \,| b_I - B_I x(t) |^2 \\ &= (\bar{x} - x(t)) \cdot (-f^0_{xx}) \cdot (\bar{x} - x(t)) \\ &\quad + (u_I(t) - \bar{u}_I) \cdot B_I(-f^0_{xx})^{-1}B'_I \cdot (u_I(t) - \bar{u}_I) \\ &\qquad - \rho(u_I(t) - \bar{u}_I) \cdot (B_I(-f^0_{xx})^{-1}B'_I)^2 \cdot (u_I(t) - \bar{u}_I) \\ &\geqq (\bar{x} - x(t)) \cdot (-f^0_{xx}) \cdot (\bar{x} - x(t)) \begin{cases} > 0 , & \text{if } x(t) \neq \bar{x} , \\ \geqq 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, by (40), we have, for any $\bar{u} \in \overline{U}$,

(50)
$$| u_I(t + 1) - \bar{u}_I | \leqq | u_I(t) - \bar{u}_I | ,$$

with the strict inequality for $x(t) \neq \bar{x}$.

For $\bar{u} \in \overline{U}$, let u^* be a limiting point of the sequence $u(t)$ such that

(51)
$$| u^* - \bar{u} | \leqq | u^{**} - \bar{u} |$$

for any limiting point u^{**} of $u(t)$. Take a sub-sequence $\{u(t_\nu)\}$ such that

$$\lim_\nu u(t_\nu) = u^* .$$

We may, without loss of generality, assume that $u(t_\nu + 1)$ also converges, say to u^{**}.

Then

$$u^{**} = \max \{0, u^* - \rho g(x(u^*))\} .$$

By a formula similar to (50), we have

$$| u^{**} - \bar{u} | \leqq | u^* - \bar{u} | ,$$

which, by (51), implies

$$| u^{**} - \bar{u} | = | u^* - \bar{u} | .$$

Hence, by (50),

$$x(u^*) = \bar{x} \qquad \text{and} \qquad u^* \in \overline{U} .$$

Since the inequality (50) holds for any $\bar{u} \in \overline{U}$, we may put $\bar{u} = u^*$ in (50). Then we know that the sequence $\{u(t)\}$ itself converges to u^*. Consequently,

$$\lim_{t \to \infty} x(t) = \bar{x}, \text{ q.e.d.}$$

The modified gradient method will be applied to solve concave *quadratic programming: find a vector* $\bar{x}$ *that maximizes* $a'x - x'Ax/2$ *subject to* $Bx \leqq b$, where A is positive definite.

The Lagrangian is given by

(52)
$$\varphi(x, u) = a'x - \frac{1}{2}x'Ax + u'(b - Bx) .$$

For $u \geqq 0$, the vector $x(u)$ that maximizes $\varphi(x, u)$ with respect to unrestricted x is characterized by

$$a - Ax(u) - B'u = 0 .$$

Therefore,

(53)
$$x(u) = A^{-1}(a - B'u) .$$

The modified gradient method may be written as follows:

$$\text{(IV)} \qquad \begin{cases} x(t) = A^{-1}a - A^{-1}B'u(t) \\ u(t+1) = \max\{0, (I - \rho BA^{-1}B')u(t) - \rho(b - BA^{-1}a)\} \end{cases}$$
$$(t = 0, 1, 2, \cdots) .$$

Since we can beforehand compute $I - \rho BA^{-1}B'$ and $\rho(b - BA^{-1}a)$, the computation of $u(t)$ by (IV) will be easily performed.

THEOREM 5. *The solution $x(t)$ of the system* (IV) *converges to the optimum solution provided ρ is a sufficiently small positive number.*

6. Linear Programming

In the maximum problem with which we have been concerned so far, the maximand has been assumed to be strictly concave. We shall in this section treat linear programming problems and show how the above method can be applied.

Linear programming is formulated as follows:

PROBLEM C. *To find a vector $\bar{x}$ that maximizes $a'x$ subject to $Bx \leqq b$.*

The following conditions will be assumed to be satisfied:

(b) *There is a vector x^0 such that $Bx^0 < b$.*

(e) *The feasible set is bounded.*

LEMMA. *Consider Problem* C′: *Maximize $c'x$ subject to $Bx \leqq b$. Then there exists a positive number $\delta > 0$ such that, if $|c - a| \leqq \delta$, then every optimum vector $\bar{x}$ for Problem* C′ *is also optimum to Problem* C.

PROOF. Since the feasible set is a bounded convex polyhedral set, there exists a matrix

$$K = (k^1 \cdots k^N) = \begin{pmatrix} k_{11} & \cdots & k_{1N} \\ & \cdots & \\ k_{n1} & \cdots & k_{nN} \end{pmatrix}$$

such that a vector x satisfies $Bx \leqq b$ if and only if

$$x = Kw , \qquad w = \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix} \geqq 0 , \qquad \sum_{\nu=1}^{N} w_\nu = 1 .$$

We may without loss of generality assume that

$$a \cdot k^1 = \cdots = a \cdot k^r > a \cdot k^{r+1} \geqq \cdots \geqq a \cdot k^N .$$

Let

$$\delta = \min\left\{ \frac{a \cdot (k^\nu - k^\mu)}{2\,|k^\nu - k^\mu|} \,\middle|\, 1 \leqq \nu \leqq r < \mu \leqq N \right\} > 0 .$$

and $|a - c| \leqq \delta$. Then

$$|a \cdot (k^\nu - k^\mu) - c \cdot (k^\nu - k^\mu)| \leqq |a - c|\,|k^\nu - k^\mu| < a \cdot (k^\nu - k^\mu) ,$$
$$\text{for any } 1 \leqq \nu \leqq r < \mu \leqq N .$$

Therefore,

$$c \cdot (k^\nu - k^\mu) > 0 , \qquad \text{for } 1 \leqq \nu \leqq r < \mu \leqq N .$$

Hence, if a vector $\bar{x}$ is optimum for Problem C′, then

$$\bar{x} = \sum_{\nu=1}^{r} w_\nu k^\nu , \qquad w_\nu \geqq 0 , \qquad \sum_{\nu=1}^{r} w_\nu = 1 .$$

Therefore, $\bar{x}$ is optimum for Problem C, q.e.d.

Now we shall consider the following strictly concave quadratic programming problem :

PROBLEM C_ε. *Find a vector x that maximizes $a'x - \varepsilon x'x/2$ subject to $Bx \leqq b$, where ε is a given positive number.*

Since the optimum vector for Problem C_ε is unique, we denote it by x_ε.

We shall prove the following :

THEOREM 6. *There exists a positive number ε_0 such that the optimum solution x_ε of Problem C_ε with $\varepsilon \leqq \varepsilon_0$ is optimum for Problem C.*

PROOF. According to the Kuhn-Tucker Theorem, a vector $\bar{x}$ is optimum for Problem C_ε if and only if there exists a vector $\bar{u}$ such that $(\bar{x}, \bar{u})$ is a saddle-point of the Lagrangian form $\varphi_\varepsilon(x, u) = (a'x - \varepsilon x'x/2) + u \cdot (b - Bx)$ with x unrestricted and $u \geqq 0$. Any saddle-point $(x_\varepsilon, \bar{u})$ is characterized as the solution of

$$\text{(54)} \qquad \begin{cases} a - \varepsilon x_\varepsilon - B'\bar{u} = 0 \\ b - Bx_\varepsilon \geqq 0 \\ \bar{u} \geqq 0, \ \bar{u} \cdot (b - Bx_\varepsilon) = 0 . \end{cases}$$

But (54) shows that $(x_\varepsilon, \bar{u})$ is also a saddle-point of

$$\varphi(x, u) = (a - \varepsilon x_\varepsilon)'x + u'(b - Bx)$$

with x unrestricted, $u \geqq 0$.

Therefore x_ε maximizes $(a - \varepsilon x_\varepsilon)'x$ subject to $Bx \leqq b$.

Let

$$\varepsilon_0 = \frac{\delta}{K}$$

where δ is the positive number in the Lemma, and

$$K = \max_{x:\ \text{feasible}} |x| .$$

Since the feasible set is compact, K is finite, and ε_0 is *positive*. Then, by the Lemma, if $0 < \varepsilon \leqq \varepsilon_0$, x_ε is optimum for Problem C, q.e.d.

By Theorem 6, solving linear programming Problem C is reduced to solving the strictly concave programming Problem C_ε with $0 < \varepsilon \leqq \varepsilon_0$, to which the modified gradient method will be applied. The modified gradient method for Problem C_ε is now written as follows :

$$\text{(V)} \qquad \begin{cases} u(t+1) = \max\left\{0, \left(I - \frac{\rho}{\varepsilon}BB'\right)u(t) - \left(\rho b - \frac{\rho}{\varepsilon}Ba\right)\right\} \\ x(t) = \frac{1}{\varepsilon}a - \frac{1}{\varepsilon}B'u(t) \end{cases}$$

$$(t = 0, 1, 2, \cdots , \ u(0) \geqq 0) .$$

The above method can be applied to the case where A is only positive semi-definite. In this case, the iterative method (IV) will be modified as follows:

$$
\text{(IV)}' \quad \begin{cases} x(t) = (A + \varepsilon I)^{-1}(a - B'u(t)) , \\ u(t+1) = \max\{0, [I - B(A + \varepsilon I)^{-1}B']u(t) - \rho[b - B(A + \varepsilon I)^{-1}a]\} \end{cases}
$$
$$
(t = 0, 1, 2, \cdots) .
$$

The system is stable provided ρ and ε are sufficiently small positive numbers.