56. ON THE REPRESENTATION OF CONTINUOUS FUNCTIONS OF SEVERAL VARIABLES AS SUPERPOSITIONS OF CONTINUOUS FUNCTIONS OF ONE VARIABLE AND ADDITION*

The aim of this paper is to present a brief proof of the following theorem:

Theorem. *For any integer $n \geq 2$ there are continuous real functions $\psi^{p,q}(x)$ on the closed unit interval $E^1 = [0;1]$ such that each continuous real function $f(x_1, \dots, x_n)$ on the n-dimensional unit cube E^n is representable as*

$$f(x_1, \dots, x_n) = \sum_{q=1}^{q=2n+1} \chi_q \Big[\sum_{p=1}^{n} \psi^{pq}(x_p) \Big], \tag{1}$$

where $\chi_q(y)$ are continuous real functions.

For $n = 3$, by setting

$$\phi_q(x_1, x_2) = \psi^{1q}(x_1) + \psi^{2q}(x_2), \quad h_q(y, x_3) = \chi_q[y + \psi^{3q}(x_3)],$$

we obtain from (1)

$$f(x_1, x_2, x_3) = \sum_{q=1}^{7} h_q[\phi_q(x_1, x_2), x_3], \tag{2}$$

which is a slight strengthening of a result by V.I. Arnol'd [2], who showed that any continuous function of three variables can be represented as a sum of *nine* summands of the same form as the *seven* summands involved in formula (2). The results of my paper [1] do not follow from the new theorem presented here in their exact statements, but their essence (in the sense of the possibility of representing functions of several variables by means of superpositions of functions of a smaller number of variables and their approximation by superpositions of a fixed form involving polynomials in one variable and addition) is obviously contained in the new theorem. The method for proving the new theorem is more elementary than that in [1, 2] and reduces to direct constructions and calculations. In particular, it is no longer necessary to use trees of components of level lines. However, the constructions used in this paper were in fact found by analyzing those employed in [1, 2] and discarding some of their details unnecessary for the derivation of the final result.

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* *Dokl. Akad. Nauk SSSR* **114**:5 (1957), 953–956 (in Russian).

§1. Construction of the functions ψ^{pq}

Everywhere in what follows, the indices p, q, k and i run over the integer values:

$$1 \leq p \leq n, \ 1 \leq q \leq 2n+1, \ k = 1, 2, \dots, 1 \leq i \leq m_k = (9n)^k + 1.$$

When summing and multiplying within these limits we do not indicate the limits.

Consider the closed intervals

$$A^q_{k,i} = \left[\frac{1}{(9n)^k}\left(i - 1 - \frac{q}{3n}\right), \ \frac{1}{(9n)^k}\left(i - \frac{1}{3n} - \frac{q}{3n}\right)\right].$$

The intervals $A^q_{k,i}$ have lengths $\frac{1}{(9n)^k}\left(1 - \frac{1}{3n}\right)$ and are obtained from one another for fixed k and q by passing from i to $i' = i + 1$ using a shift to the right over a distance $1/(9n)^k$, that is, not only do they not overlap, but they do not even overlap intervals of lengths $1/3n(9n)^k$ and to within the presence of these intervals, they cover the whole closed unit interval E^1. Accordingly, for fixed k and q the cubes

$$S^q_{k,i_1\dots i_n} = \prod_n A^q_{k,i_p}$$

with edges of lengths $1/(9n)^k$ cover the unit cube E^n to within the separating slits of widths $1/3n(9n)^k$. It is easy to verify the following.

Lemma 1. *The system of all cubes $S^q_{k,i_1\dots i_n}$ with constant k and variable q and $i_1, \dots, i_n$ covers the unit cube E^n so that each point belonging to E^n is covered at least $n+1$ times.*

Using induction on k we can prove the following

Lemma 2. *The constants $\lambda^{pq}_{k,i}$ and ϵ_k can be chosen so that the following conditions hold:*

1) $\lambda^{pq}_{k,i} < \lambda^{pq}_{k,i+1} \leq \lambda^{pq}_{k,i} + 1/2^k$;

2) $\lambda^{pq}_{k,i} \leq \lambda^{pq}_{k+1,i'} \leq \lambda^{pq}_{k,i} + \epsilon_k - \epsilon_{k+1}$ *if the closed intervals $A^q_{k,i}$ and $A^q_{k+1,i'}$ do not intersect;*

3) *the closed intervals $\Delta^q_{k,i_1\dots i_n} = \left[\sum_p \lambda^{pq}_{k,i_p}; \sum_p \lambda^{pq}_{k,i_p} + n\epsilon_k\right]$ are pairwise disjoint for fixed k and q.*

It is easy to note that 1) and 3) imply

4) $\epsilon_k \leq 1/2^k$.

On the basis of the above-indicated properties of the closed intervals $A^q_{k,i}$ and properties 1), 2) and 4) of the constants $\lambda^{pq}_{k,i}$ and ϵ_k, one can easily prove the following

Lemma 3. *For fixed p and q the conditions*

5) $\lambda^{pq}_{k,i} \leq \psi^{pq}(x) \leq \lambda^{pq}_{k,i} + \epsilon_k$ *for* $x \in A^q_{k,i}$ *uniquely determine a continuous function* ψ^{pq} *on* E^1.

Remark. It can easily be seen that, by construction, the functions ψ^{pq} are monotonically increasing. This property could have been included in the statement of the theorem.

From 5) and 3) it follows that

6) $\sum_p \psi^{pq}(x_p) \in \Delta^q_{k,i_1\dots i_n}$ *for* $(x_1,\dots,x_n) \in S^q_{k,i_1\dots i_n}$.

§2. Construction of the functions χ^q

On establishing the existence of the functions ψ^{pq} and the constants $\lambda^{pq}_{k,i}$ and ϵ_i possessing properties 1)–6) we proceed to the proof of the main theorem. The desired functions $\chi^q(y)$ will be constructed in the form

$$\chi^q = \lim_{r\to\infty} \chi^q_r,$$

where $\chi^q_0 \equiv 0$, while for $r > 0$ χ^q_r will be defined by induction on r simultaneously with the natural numbers k_r.

We will use the notation

$$f_r(x_1,\dots,x_n) = \sum_q \chi^q_r[\sum_p \psi^{pq}(x_p)], \tag{3}$$

$$M_r = \sup_{E^n} |f - f_r|. \tag{4}$$

It is obvious that

$$f_0 \equiv 0, \quad M_0 = \sup_{E^n} |f|.$$

Assume that the continuous functions χ^q_{r-1} and the number k_{r-1} have already been determined. In this way a continuous function f_{r-1} on E^n has also been determined. Since the diameters of the cubes $S^q_{k,i_1\dots i_n}$ tend to zero as

$k \to \infty$, we can choose k_r so large that the oscillation of the difference $f - f_{r-1}$ does not exceed $(1/(2n+2))M_{r-1}$ on any $S^q_{k_r,i_1\dots i_n}$.

Let $\xi^q_{k,i}$ be arbitrary points belonging to the corresponding closed intervals $A^q_{k,i}$. For the closed interval $\Delta^q_{k,i_1\dots i_n}$ we put

$$\chi^q_r(y) = \chi^q_{r-1}(y) + \frac{1}{n+1}[f(\xi^q_{k,i_1}, \dots, \xi^q_{k,i_n}) - f_{r-1}(\xi^q_{k,i_1}, \dots, \xi^q_{k,i_n})]. \tag{5}$$

Obviously, the values of the function χ^q_r fixed in this way satisfy the inequality

$$|\chi^q_r(y) - \chi^q_{r-1}(y)| \leq \frac{1}{n+1} M_{r-1}. \tag{6}$$

Outside the closed intervals $\Delta^q_{k,i_1\dots i_n}$ the function χ^q_r is defined arbitrarily, with preservation of the same inequality (6) and continuity.

We now estimate $f - f_r$ at an arbitrary point $(x_1, \dots, x_n)$ belonging to E^n. It is obvious that

$$f(x_1, \dots, x_n) - f_r(x_1, \dots, x_n) = f(x_1, \dots, x_n) - f_{r-1}(x_1, \dots, x_n) - \\ - \sum_q \Big\{ \chi^q_r \Big[\sum_p \psi^{pq}(x_p)\Big] - \chi^q_{r-1}\Big[\sum_p \psi^{pq}(x_p)\Big]\Big\}. \tag{7}$$

We represent the sum $\sum_q$ in (7) in the form $\sum' + \sum''$, where the sum $\sum'$ extends over certain $n+1$ values of q for which the point $(x_1, \dots, x_n)$ is contained in one of the cubes $S^q_{k,i_1\dots i_n}$ (by Lemma 1, such cubes exist) and the sum $\sum''$ extends over the remaining n values of q.

By virtue of (5), for each term in $\sum'$ we have

$$\chi^q_r \Big[\sum_p \psi^{pq}(x_p)\Big] - \chi^q_{r-1}\Big[\sum_p \psi^{pq}(x_p)\Big] = \\ = \frac{1}{n+1}[f(\xi^q_{k,i_1}, \dots, \xi^q_{k,i_n}) - f_{r-1}(\xi^q_{k,i_1}, \dots, \xi^q_{k,i_n})] = \\ = \frac{1}{n+1}[f(x_1, \dots, x_n) - f_{r-1}(x_1, \dots, x_n)] + \frac{\omega^q}{n+1}, \tag{8}$$

where

$$|\omega^q| \leq \frac{1}{2n+2} M_{r-1}. \tag{9}$$

The terms in $\sum''$ are estimated using (6). Relation (5), together with (8), (9) and (6) implies

$$|f - f_r| = \Big|\frac{1}{n+1}\sum{}' \omega^q + \sum{}''(\chi^q_r - \chi^q_{r-1})\Big| \leq \\ \leq \frac{1}{2n+2} M_{r-1} + \frac{n}{n+1} M_{r-1} = \frac{2n+1}{2n+2} M_{r-1}. \tag{10}$$

Since inequality (10) holds at any point $(x_1, \dots, x_n) \in E^n$, we have

$$M_r \le \frac{2n+1}{2n+2} M_{r-1}, \quad M_r \le \left(\frac{2n+1}{2n+2}\right)^r M_0. \tag{11}$$

From (6) and (11) it follows that the absolute values of the differences $\chi_r^q - \chi_{r-1}^q$ do not exceed the corresponding terms of the absolutely convergent series

$$\sum_r \frac{1}{n+1} M_{r-1}.$$

Therefore the functions χ_r^q converge uniformly to continuous limit functions χ^q for $r \to \infty$.

From relations (3) and (4) and estimate (11), passing to the limit for $r \to \infty$, we obtain relation (1), which completes the proof of the theorem.

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References

1. A.N. Kolmogorov, 'On the representation of continuous functions of several variables as superpositions of continuous functions of a smaller number of variables', *Dokl. Akad. Nauk SSSR* **108**:2 (1956), 179–182 (in Russian). (See No. 55.)
2. V.I. Arnol'd, 'On the representation of continuous functions of three variables as superpositions of continuous functions of two variables', *Dokl. Akad. Nauk SSSR* **114**:4 (1957), 679–681 (in Russian).