55. ON THE REPRESENTATION OF CONTINUOUS FUNCTIONS OF
SEVERAL VARIABLES AS SUPERPOSITIONS OF CONTINUOUS
FUNCTIONS OF A SMALLER NUMBER OF VARIABLES *

Theorem 3, stated below, implies the following somewhat unexpected consequence: any continuous function of an arbitrarily large number of variables is representable as a finite superposition of continuous functions of at most three variables. For an arbitrary function of four variables the representation has the form

\[ f(x_1, x_2, x_3, x_4) = \sum_{r=1}^{4} h^r[x_4, g_1^r(x_1, x_2, x_3), g_2^r(x_1, x_2, x_3)]. \]

The question whether an arbitrary continuous function of three variables can be represented as a superposition of continuous functions of two variables remains open. The proof of the possibility of such a representation would give the complete solution to Hilbert’s 13th problem [1], in the sense of a refutation of the conjecture put forward by Hilbert. Theorem 2 only shows that the representation of an arbitrary continuous function of three variables in the form of a superposition of continuous functions of two variables is possible if we admit as auxiliary variables some variables running over a one-dimensional formation somewhat more complicated than a closed interval on the number line, namely a universal tree (by a tree is meant a locally connected continuum not containing a homeomorphic image of a circle; as was shown by Menger [2], there exists a universal tree \( \Xi \) containing homeomorphic images of all the trees).

In what follows \( k, m, n \) and \( r \) are natural numbers; \( a, b, c, C, d, M, R, x, y, u, v, f, F, g, h, \epsilon, \delta \) and \( \rho \) are real numbers; \( \xi, \phi \) and \( \psi \) are tree elements; \( E^n \) is the \( n \)-dimensional unit cube; \( 0 \leq x_i \leq 1; \ i = 1, \ldots, n. \)

**Theorem 1.** a) For any \( n \geq 2 \) there are continuous functions

\[ \phi^1, \ldots, \phi^{n+1} \]

on \( E^n \) with values belonging to the universal tree \( \Xi \) such that any continuous real function \( f \) on \( E^n \) can be represented as

\[ f(x_1, \ldots, x_n) = \sum_{r=1}^{n+1} h^r_f[\phi^r(x_1, \ldots, x_n)]. \]

where $h_j^r(\xi)$ are continuous real functions on $\Xi$.

b) The functions $h_j^r(\xi)$ can be chosen so that they depend continuously on $f$ in the sense of the topology of uniform convergence in the spaces of continuous functions on $E^n$ and $\Xi$.

Theorem 1 implies almost immediately

**Theorem 2.** For any $n \geq 3$ there are continuous functions

$$\phi^1, \ldots, \phi^n$$

on $E^n$ with values belonging to $\Xi$ such that any continuous function $f$ on $E^n$ can be represented in the form

$$f(x_1, \ldots, x_n) = \sum_{r=1}^{n} h^r[x_n, \phi^r(x_1, \ldots, x_{n-1})],$$

where $h^r(x, \xi)$ are continuous real functions on the product $E^1 \times \Xi$.

The universal tree $\Xi$ can be regarded (see [2]) as having a realization as a continuum in the unit square $E^2$. Denoting by $g^r_1$ and $g^r_2$ the coordinates of the point $\phi^r$, we obtain as an immediate consequence of Theorem 2 the following proposition:

**Theorem 3.** For any $n \geq 3$ there are continuous real functions

$$g^1_1, \ldots, g^n_1, \; g^1_2, \ldots, g^n_2$$

on $E^{n-1}$ such that any continuous function $f$ on $E^n$ can be represented in the form

$$f(x_1, \ldots, x_n) = \sum_{r=1}^{n} h^r[x_n, g^r_1(x_1, \ldots, x_{n-1}), g^r_2(x_1, \ldots, x_{n-1})],$$

where $h^r$ are continuous functions on $E^3$.

Theorem 3 is trivial for $n = 3$; it is of actual interest only for $n \geq 4$.

It remains to indicate briefly a way of proving Theorem 1. The proof proceeds from the following lemma.

**Main lemma.** For any $n \geq 2$ there is a system of functions

$$u^r_{km}(x_1, \ldots, x_n)$$
on $E^n$, with indices $r, k$ and $m$ such that

$$1 \leq r \leq n + 1, \quad 1 \leq k < \infty, \quad 1 \leq m \leq m_k,$$

possessing the following properties:

1) $u_{km}^r \geq 0$;
2) $u_{km}^r \neq 0$ only on a set $G_{km}^r$ of a diameter $d_k$, where $d_k \rightarrow 0$ as $k \rightarrow \infty$;
3) two sets $G_{km}^r$ and $G_{km'}^r$, with common indices $r$ and $k$ are disjoint for $m' \neq m$;
4) for any $k$ at each point $P \in E^n$,

$$c \leq \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} u_{km}^r \leq C,$$

where $c$ and $C$ do not depend on $k$;
5) the function $u_{km}^r$ is constant on each set $G_{km'}^r$, with the same superscript $r$ for $k' > k$ and arbitrary $m'$.

The construction of the system of functions $u_{km}^r$ cannot be presented within the framework of this paper. In what follows this system of functions is assumed to be given.

**Lemma 1.** a) Any continuous function $f$ on $E^n$ can be represented as

$$f(P) = \sum_{k=1}^{\infty} \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} a_{km}^r(f) u_{km}^r(P), \quad (1)$$

where the coefficients $a_{km}^r(f)$ do not depend on $P$.

b) The coefficients $a_{km}^r(f)$ can be chosen in the form of continuous functionals of $f$ and so that

$$|a_{km}^r(f)| \leq a(\mathcal{F}), \quad \sum_{k=1}^{\infty} a_k(\mathcal{F}) < \infty$$

on each family $\mathcal{F}$ of uniformly bounded and equicontinuous functions $f$.

The proof of Lemma 1 is based on properties 1), 2), and 4) of the system $u_{km}^r$ and begins with estimation of the remainder $R$ in the representation

$$f(P) = \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} b_{km}^r u_{km}^r(P) + R,$$
where
\[ b_m^r = \frac{1}{C} f(P_{km}) \]
and \( P_{km} \) are arbitrary points belonging to the corresponding sets \( G_{km}^r \). It can easily be shown that for an appropriate choice of the coefficients \( b_m^r \) we have
\[ |R| \leq (|1 - c/C| + \delta_k)M, \]
where
\[ M = \sup_{P \in E^n} |f(P)|, \quad \delta_k = \sup_{\rho(P,P') \leq d_k} |f(P) - f(P')|. \]

The complete proof of Lemma 1 falls outside the framework of this paper. We now write expansion (1) in the form
\[
f(P) = \sum_{r=1}^{n+1} f^r(P), \quad f^r(P) = \sum_{k=1}^{\infty} \sum_{m=1}^{m_k} a_{km}^r u_{km}(P). \tag{2}
\]

Properties 2), 3) and 5) of the system \( u_{km}^r \) readily imply the following property of the functions \( f^r \).

Lemma 2. The function \( f^r(P) \) is constant on each component of any level set of the function
\[
F^r(P) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{m_k} u_{km}^r(P).
\]

We now note that, as was shown by A.S. Kronrod [3], the components of the level sets of any continuous function form a tree in a certain natural topology. We denote the tree of the components of the level sets of the function \( F^r \) by \( \Xi^r \) and map the trees \( \Xi^1, \ldots, \Xi^{n+1} \) by means of the homeomorphisms
\[
\psi_r(\Xi^r) = \Xi^r_0 \subseteq \Xi
\]
ono into pairwise disjoint subsets of the universal tree \( \Xi \). We put
\[
\phi^r(P) = \psi_r(\xi^r)
\]
if \( P \in \xi^r \subseteq \Xi^r \) and define continuous functions \( h^r(\xi) \) on \( \Xi \) such that for \( \xi \in \Xi^r \)
\[
h^r(\xi) = y, \text{ if } f^r(P) = y \text{ for } P \in \psi_r^{-1}(\xi).
\]
It can easily be verified that
\[ f^r(P) = h^r[\phi^r(P)]. \] (3)

Formulas (2) and (3) lead to a proof of assertion a) of Theorem 1. Assertion b) of Theorem 1 is proved on the basis of assertion b) of Lemma 1.

In conclusion we also state without proof the following proposition.

**Theorem 4.** Given any \( n \geq 2 \) and \( \epsilon > 0 \), for each continuous function \( f \) on \( E^n \) there exist polynomials
\[ b(u_1, \ldots, u_{n-1}), \quad a_r(x), c_r(x); \quad r = 1, \ldots, n + 1, \]
such that
\[ |f(P) - \tilde{f}(P)| < \epsilon \]
at all the points \( P \in E^n \), where
\[ \tilde{f}(x_1, \ldots, x_n) = \sum_{r=1,2} a_r(x_n)b[c_r(x_n) + x_1, \ldots, c_r(x_n) + x_{n-1}]. \] (4)

For \( n = 3 \), by setting
\[ d(u, v) = u + v, \quad g_r(x, y) = a_r(x)y, \quad h_r(x, x') = c_r(x) + x', \]
we obtain from (4)
\[ \tilde{f}(x_1, x_2, x_3) = d(g_1\{x_3, b[h_1(x_3, x_1), h_1(x_3, x_2)]\}, \]
\[ g_2\{x_3, b[h_2(x_3, x_1), h_2(x_3, x_2)]\}). \] (5)

By virtue of Theorem 4, any continuous function of three variables can be approximated arbitrarily accurately by an expression of the form (5), where \( d, g_r, b \) and \( h_r \) are polynomials in two variables. This remark also illuminates from a new viewpoint the group of problems related to Hilbert's 13th problem.

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**References**