55. ON THE REPRESENTATION OF CONTINUOUS FUNCTIONS OF SEVERAL VARIABLES AS SUPERPOSITIONS OF CONTINUOUS FUNCTIONS OF A SMALLER NUMBER OF VARIABLES *

Theorem 3, stated below, implies the following somewhat unexpected consequence: *any continuous function of an arbitrarily large number of variables is representable as a finite superposition of continuous functions of at most three variables.* For an arbitrary function of four variables the representation has the form

$$f(x_1, x_2, x_3, x_4) = \sum_{r=1}^{4} h^r[x_4, g_1^r(x_1, x_2, x_3), g_2^r(x_1, x_2, x_3)].$$

The question whether an arbitrary continuous function of *three* variables can be represented as a superposition of continuous functions of *two* variables remains open. The proof of the possibility of such a representation would give the complete solution to Hilbert's 13th problem [1], in the sense of a refutation of the conjecture put forward by Hilbert. Theorem 2 only shows that the representation of an arbitrary continuous function of three variables in the form of a superposition of continuous functions of two variables is possible if we admit as auxiliary variables some variables running over a *one-dimensional formation* somewhat more complicated than a closed interval on the number line, namely *a universal tree* (by a tree is meant a locally connected continuum not containing a homeomorphic image of a circle; as was shown by Menger [2], there exists a universal tree Ξ containing homeomorphic images of all the trees).

In what follows k, m, n and r are natural numbers; $a, b, c, C, d, M, R, x, y, u, v, f, F, g, h, \epsilon, \delta$ and ρ are real numbers; ξ, ϕ and ψ are tree elements; E^n is the n-dimensional unit cube; $0 \le x_i \le 1$; $i = 1, \dots, n$.

Theorem 1. a) *For any $n \ge 2$ there are continuous functions*

$$\phi^1, \dots, \phi^{n+1}$$

on E^n with values belonging to the universal tree Ξ such that any continuous real function f on E^n can be represented as

$$f(x_1, \dots, x_n) = \sum_{r=1}^{n+1} h_f^s[\phi^r(x_1, \dots, x_n)],$$

* *Dokl. Akad. Nauk SSSR* **108**:2 (1956), 179–182 (in Russian).

where $h_j^r(\xi)$ are continuous real functions on Ξ.

b) *The functions $h_j^r(\xi)$ can be chosen so that they depend continuously on f in the sense of the topology of uniform convergence in the spaces of continuous functions on E^n and Ξ.*

Theorem 1 implies almost immediately

Theorem 2. *For any $n \geq 3$ there are continuous functions*

$$\phi^1, \ldots, \phi^n$$

on E^n with values belonging to Ξ such that any continuous function f on E^n can be represented in the form

$$f(x_1, \ldots, x_n) = \sum_{r=1}^{n} h^r[x_n, \phi^r(x_1, \ldots, x_{n-1})],$$

where $h^r(x, \xi)$ are continuous real functions on the product $E^1 \times \Xi$.

The universal tree Ξ can be regarded (see [2]) as having a realization as a continuum in the unit square E^2. Denoting by g_1^r and g_2^r the coordinates of the point ϕ^r, we obtain as an immediate consequence of Theorem 2 the following proposition:

Theorem 3. *For any $n \geq 3$ there are continuous real functions*

$$g_1^1, \ldots, g_1^n;\ g_2^1, \ldots, g_2^n$$

on E^{n-1} such that any continuous function f on E^n can be represented in the form

$$f(x_1, \ldots, x_n) = \sum_{r=1}^{n} h^r[x_n, g_1^r(x_1, \ldots, x_{n-1}), g_2^r(x_1, \ldots, x_{n-1})],$$

where h^r are continuous functions on E^3.

Theorem 3 is trivial for $n = 3$; it is of actual interest only for $n \geq 4$.

It remains to indicate briefly a way of proving Theorem 1. The proof proceeds from the following lemma.

Main lemma. *For any $n \geq 2$ there is a system of functions*

$$u_{km}^r(x_1, \ldots, x_n)$$

on E^n, with indices r, k and m such that

$$1 \leq r \leq n+1, \quad 1 \leq k < \infty, \quad 1 \leq m \leq m_k,$$

possessing the following properties:

1) $u^r_{km} \geq 0$;

2) $u^r_{km} \neq 0$ *only on a set* G^r_{km} *of a diameter* d_k, *where* $d_k \to 0$ *as* $k \to \infty$;

3) *two sets* G^r_{km} *and* $G^r_{km'}$ *with common indices* r *and* k *are disjoint for* $m' \neq m$;

4) *for any* k *at each point* $P \in E^n$,

$$c \leq \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} u^r_{km} \leq C,$$

where c *and* C *do not depend on* k;

5) *the function* u^r_{km} *is constant on each set* $G^r_{k'm'}$ *with the same superscript* r *for* $k' > k$ *and arbitrary* m'.

The construction of the system of functions u^r_{km} cannot be presented within the framework of this paper. In what follows this system of functions is assumed to be given.

Lemma 1. a) *Any continuous function* f *on* E^n *can be represented as*

$$f(P) = \sum_{k=1}^{\infty} \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} a^r_{km}(f) u^r_{km}(P), \tag{1}$$

where the coefficients $a^r_{km}(f)$ *do not depend on* P.

b) *The coefficients* $a^r_{km}(f)$ *can be chosen in the form of continuous functionals of* f *and so that*

$$|a^r_{km}(f)| \leq a(\mathfrak{F}), \ \sum_{k=1}^{\infty} a_k(\mathfrak{F}) < \infty$$

on each family $\mathfrak{F}$ *of uniformly bounded and equicontinuous functions* f.

The proof of Lemma 1 is based on properties 1), 2), and 4) of the system u^r_{km} and begins with estimation of the remainder R in the representation

$$f(P) = \sum_{r=1}^{n+1} \sum_{m=1}^{m_k} b^r_m u^r_{km}(P) + R,$$

where

$$b^r_m = \frac{1}{C} f(P^r_{km})$$

and P^r_{km} are arbitrary points belonging to the corresponding sets G^r_{km}. It can easily be shown that for an appropriate choice of the coefficients b^r_m we have

$$|R| \leq (|1 - c/C| + \delta_k)M,$$

where

$$M = \sup_{P \in E^n} |f(P)|, \quad \delta_k = \sup_{\rho(P,P') \leq d_k} |f(P) - f(P')|.$$

The complete proof of Lemma 1 falls outside the framework of this paper.

We now write expansion (1) in the form

$$f(P) = \sum_{r=1}^{n+1} f^r(P), \quad f^r(P) = \sum_{k=1}^{\infty} \sum_{m=1}^{m_k} a^r_{km} u^r_{km}(P). \tag{2}$$

Properties 2), 3) and 5) of the system u^r_{km} readily imply the following property of the functions f^r.

Lemma 2. *The function $f^r(P)$ is constant on each component of any level set of the function*

$$F^r(P) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{m_k} u^r_{km}(P).$$

We now note that, as was shown by A.S. Kronrod [3], the components of the level sets of any continuous function form a tree in a certain natural topology. We denote the tree of the components of the level sets of the function F^r by Ξ^r and map the trees $\Xi^1, \dots, \Xi^{n+1}$ by means of the homeomorphisms

$$\psi_r(\Xi^r) = \Xi^r_0 \subseteq \Xi$$

onto pairwise disjoint subsets of the universal tree Ξ. We put

$$\phi^r(P) = \psi_r(\xi^r)$$

if $P \in \xi^r \in \Xi^r$ and define continuous functions $h^r(\xi)$ on Ξ such that for $\xi \in \Xi^r$

$$h^r(\xi) = y, \text{ if } f^r(P) = y \text{ for } P \in \psi_r^{-1}(\xi).$$

It can easily be verified that

$$f^r(P) = h^r[\phi^r(P)]. \tag{3}$$

Formulas (2) and (3) lead to a proof of assertion a) of Theorem 1. Assertion b) of Theorem 1 is proved on the basis of assertion b) of Lemma 1.

In conclusion we also state without proof the following proposition.

Theorem 4. *Given any $n \geq 2$ and $\epsilon > 0$, for each continuous function f on E^n there exist polynomials*

$$b(u_1, \dots, u_{n-1}), \quad a_r(x), c_r(x); \quad r = 1, \dots, n+1,$$

such that

$$|f(P) - \tilde{f}(P)| < \epsilon$$

at all the points $P \in E^n$, where

$$\tilde{f}(x_1, \dots, x_n) = \sum_{r=1,2} a_r(x_n) b[c_r(x_n) + x_1, \dots, c_r(x_n) + x_{n-1}]. \tag{4}$$

For $n = 3$, by setting

$$d(u, v) = u + v, \quad g_r(x, y) = a_r(x) y, \quad h_r(x, x') = c_r(x) + x',$$

we obtain from (4)

$$\tilde{f}(x_1, x_2, x_3) = d(g_1\{x_3, b[h_1(x_3, x_1),\ h_1(x_3, x_2)]\},$$
$$g_2\{x_3, b[h_2(x_3, x_1),\ h_2(x_3, x_2)]\}). \tag{5}$$

By virtue of Theorem 4, *any continuous function of three variables can be approximated arbitrarily accurately by an expression of the form* (5), *where d, g_r, b and h_r are polynomials in two variables.* This remark also illuminates from a new viewpoint the group of problems related to Hilbert's 13th problem.

5 May 1956

References

1. D. Hilbert, *Gesammelte Abhandlungen* Vol. 3, No. 17, Berlin, 1935.
2. K. Menger, *Kurventheorie*, Chapter 10, §6, Berlin, 1932.
3. A.S. Kronrod, *Uspekhi Mat. Nauk* 5:1 (1950), 24–134 (in Russian).