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ON THE NEWTON METHOD *

One of the most effective methods for finding the roots of an algebraic equation in the case when approximations to the roots are known is the Newton method, sometimes called the method of tangents. In this method the successive approximations are determined by formulas of the form

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The convergence of this method was studied by Cauchy and later by Ostrowski [1]-[3]. This method was generalized to systems of algebraic equations. The convergence conditions in this case were given by Willers [4], Stenin [5], and Ostrowski [6]. However, it turns out that this method is applicable to the case of arbitrary non-linear equations. In particular, at my suggestion, it was applied by D. N. Zagadsky [7, 8] to the case of non-linear integral equations.

In order to consider all the cases simultaneously, it is most profitable to develop and study the method in the general case of arbitrary functional equations. In order to simplify the unterstanding of this paper, readers who are not familiar with operator theory can find the necessary information for the sequel in Sections 1–3.

In Section 4 we establish the main theorem on the convergence of the Newton process. In it we give conditions which are in a certain sense conpletetive for the convergence of the method and establish the rate of convergence. This theorem is simultaneously a certain existence and uniqueness theorem for non-linear equations and therefore has purely theoretic significance.

In Section 5 we describe a modified method which has certain advantages in specific cases.

**L. V. Kantorovich, O metode N'yutona,* Trudy MIAN SSSR, 1949, 28, pp. 104–144. In the original text the author uses the terms 'operation' and 'functional operation'. For the present collection, the terminology of this paper was modernized throughout without further comments. *–Ed.note.*

In Section 6 we present applications of the method to algebraic equations and systems and in Section 7, to non-linear integral equations; Section 8 contains the development of a certain method for finding the eigenvalues and eigenvectors obtained by applying the Newton process to this method.

1. Linear Functional Operators

A normed linear space[1] is any linear or vector set, i.e., a set $X = \{x\}$ of elements of any nature for which the operations of addition $x + y$ and multiplication of an element by a real number λx are defined and satisfy the ordinary laws of algebra, and which also possesses a norm of elements $\|x\|$ which is a number obeying the ordinary properties of the length of a vector.

To be more precise, the following conditions for the norm must be met:

1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
2) $\|x + y\| \leq \|x\| + \|y\|$;
3) $\|\lambda x\| = |\lambda| \, \|x\|$.

Since the norm is defined, it is possible to introduce the notion of convergence: we say that $x_n \to x$ or $\lim_{n\to\infty} = x$ if $\|x_n - x\| \to 0$ as $n \to \infty$. The space is called *complete* or of *type* B (from S. Banach's name) if the Cauchy test for convergence holds, i.e., if $\lim_{n\to\infty} \|x_n - x_{n+p}\| = 0$ uniformly in p implies the convergence of the sequence to some element x, i.e., $x_n \to x$. Let us present the examples of spaces which we shall use in the sequel.

1. The set of real or complex numbers is obviously a space of type B, if the operations $x + y$ and λx are defined in the ordinary way for numbers and the norm is simply the absolute value of a number, i.e., $\|x\| = |x|$.

2. The Euclidian space R^n, or which is the same, the space of n-dimensional vectors

$$x = (\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(n)})$$

is a space of type B, if the operations $x+y$ and λx are carried out for vectors conventionally (componentwise) and the norm is defined as the length of a vector, i.e.,

$$\|x\| = \sqrt{(\xi^{(1)})^2 + (\xi^{(2)})^2 + \cdots + (\xi^{(n)})^2}.$$

3. Using the same elements as in the previous example, we can constitute a different space m_n by defining the norm in a different way, namely, by taking as the norm the greatest absolute value of the components $\|x\| = \max |\xi^{(i)}|$. Obviously, in this case all the conditions will hold. In particular, the space will be complete and the convergence of a sequence $x_m \to x$ means here, as in the previous case, that we have componentwise convergence: $\xi_m^{(i)} \to \xi^{(i)}$ as $m \to \infty$ $(i = 1, 2, \ldots, n)$.

[1] Concerning the main notions of functional analysis see, for example, [9].

4. The space C of continuous functions defined in some interval (a, b) with the norm

$$\|x\| = \max_{a \leq t \leq b} |x(t)|.$$

This is also a complete space and convergence in it is uniform convergence: $x_n \to x$ means that the sequence of functions $x_n(t)$ converges uniformly to $x(t)$.

5. The space L^2 of square integrable functions defined on the interval (a, b), for which the norm is

$$\|x\| = \left[\int_a^b x^2(t)\,dt\right]^{1/2},$$

while convergence is that in mean.

A mapping which transforms one such space X into another Y is a function $y = U(x)$ which assigns an element y of the space Y to every element x of the space X. In particular, if Y is the space of real numbers, then the operator is known as a functional. An operator $y = U(x)$ is called linear if it is additive

$$U(x_1 + x_2) = U(x_1) + U(x_2)$$

and continuous, i.e., if $U(x_n) \to U(x)$ as $x_n \to x$. A linear operator is always homogeneous. This means that $U(\lambda x) = \lambda U(x)$. Instead of continuity, we could have required the equivalent condition that there exists a constant C such that $\|U(x)\| \leq C\|x\|$. The least constant C ensuring that this inequality holds is called the norm of the given linear operator and is denoted by $\|U\|$.

Let us note the main inequality

$$\|U(x)\| \leq \|U\| \cdot \|x\|. \tag{1}$$

The set of linear operators from X to Y will be denoted by $(X \to Y)$; it is also a linear set and a complete normed space under the norm $\|U\|$ defined as above.

An operator inverse to the given operator $y = U(x)$ is any operator $x = U^{(-1)}(y)$ which takes Y to the X and satisfies $U(U^{(-1)}(y)) = y$. Such an inverse operator does not always exist. Note the following case in which its existence is guaranteed [10].

If U is a linear operator from X into X with norm $\|U\| < 1$ and I is the identity operator in X, i.e., $Ix = x$, then the operator $I - U$ possesses an inverse operator and we have

$$\|(I - U)^{-1}\| \leq (1 - \|U\|)^{-1}. \tag{2}$$

To prove this, consider the operator

$$Hx = x + Ux + U^2x + \cdots.$$

This series converges and defines a linear operator so that we have

$$\|Hx\| \leq \|x\| + \|U\|\|x\| + \cdots \leq \|x\|(1 - \|U\|)^{-1}, \qquad \|H\| \leq (1 - \|U\|)^{-1}.$$

Further, it is obvious that

$$(I - U)Hx = (x + Ux + U^2x + \cdots) - (Ux + U^2x + \cdots) = x,$$

which shows that H is an operator inverse to $I - U$. Therefore,

$$(I - U)^{-1} = H, \qquad \|(I - U)^{-1}\| = \|H\| \leq (1 - \|U\|)^{-1}.$$

Let us give some examples of linear operators.

Example 1. Consider the linear operators $y = U(x)$ from the space $X = m_n$ to $Y = m_\nu$. Introduce the elements

$$x_1 = (1, 0, \ldots, 0), \quad x_2 = (0, 1, \ldots, 0), \quad \ldots \quad x_n = (0, 0, \ldots, 1)$$

and suppose

$$y_i = U(x_i) = (a_{i1}, a_{i2}, \ldots, a_{i\nu}), \qquad i = 1, 2, \ldots, n;$$

then for any

$$x = (\xi^{(1)}, \ldots, \xi^{(n)}) = \sum_{i=1}^{n} \xi^{(i)}$$

we have

$$y = U(x) = (\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(\nu)}), \tag{3}$$

where

$$\eta^{(j)} = \sum_{i=1}^{n} a_{ij}\xi^{(i)}, \qquad j = 1, \ldots, \nu.$$

Thus, the operator U is a linear transformation determined by the matrix

$$A = \|a_{ij}\|, \qquad i = 1, \ldots, \quad j = 1, \ldots, \nu.$$

Let us show now that the norm of the operator is expressed as

$$\|U\| = \max_j \sum_i |a_{ij}|. \tag{4}$$

Indeed, we have

$$\|y\| = \|U(x)\| = \max_j |\eta^{(j)}| = \max_j \sum\nolimits_i \left|\sum\nolimits_{i=1}^{n} \xi^{(i)} a_{ij}\right|$$

$$= \max_i |x_i^{(i)}| \max_j \sum_{i=1}^n |a_{ij}| = \left(\max_j \sum_{i=1}^n |a_{ij}|\right) \|x\|.$$

Hence, it is clear that $\|U\|$ is not greater then the right-hand side of (4), but it is easy to check that we actually have an equality.

The relation (4) implies the obvious estimate

$$\|U\| \leq nL; \qquad L = \max |a_{ij}|. \tag{4a}$$

Example 2. Consider the case in which $X = R^n, Y = R^\nu$ are Euclidean spaces. Then linear operators are presented in the same form as linear transformations (3) except that their norm is defined in a different way. It can be shown that in the case when the matrix A is symmetric

$$\|U\| = |\Lambda_n|, \tag{5}$$

where Λ_n is the greatest eigennumber in absolute value of this matrix. In the case when the matrix A is not symmetric,

$$\|U\| = \sqrt{\Lambda}_n, \tag{6}$$

where Λ_n is the greatest eigenvalue of the matrix AA^*. These facts are easily obtained from theorems on the extremal properties of eigenvalues of matrices known from the course of algebra, or by using considerations similar to the ones given above in our study of Example 5.

Besides the exact expression for the norm, the following estimate for it is often useful:

$$\|U\| \leq \left(\sum_{k=1}^\nu \sum_{i=1}^n a_{ij}^2\right)^{-1}. \tag{7}$$

The latter is established as follows:

$$\|y\|^2 = \sum_{j=1}^\nu \eta_j^2 = \sum_{k=1}^\nu \left(\sum_{i=1}^n a_{ij}\xi_i\right)^2 \leq \left(\sum_{k=1}^\nu \sum_{i=1}^n a_{ij}^2\right) \sum_{i=1}^n \xi_i^2$$

$$\leq \left(\sum_{k=1}^\nu \sum_{i=1}^n a_{ij}^2\right) \|x\|^2.$$

Example 3. Consider linear operators in the space of complex numbers $X = \{x\}$; $x = x_1 + ix_2$. Such an operator in general is determined by a certain matrix, but we shall only consider operators of particular form

$$y = U(x) = wx, \tag{8}$$

where w is a complex number. Then we obviously have $|y| = |w||x|$. Since in the given case the norm of an element equals its modulus, we obviously have

$$\|U\| = |w|. \tag{9}$$

Example 4. Consider operators from C to C. Here we only consider operators of integral type

$$y = U(x); \qquad y(s) = \int_0^1 K(s,t)\,x(t)\,dt \tag{10}$$

and assume that $K(s,t)$ is continuous. This operator is obviously additive. Let us estimate its norm

$$\|y\| = \max_s |y(s)| = \max_s \left| \int_0^1 K(s,t)\,x(t)\,dt \right| \le \|x\| \max_{0\le s\le 1} \int_0^1 |K(s,t)|\,dt.$$

We see that the norm of U is no greater than the second factor, but it is easy to show that the former is actually equal to the latter:

$$\|U\| = \max_{0\le s\le 1} \int_0^1 |K(s,t)|\,dt. \tag{11}$$

In particular,

$$\|U\| \le M \quad \text{if} \quad |K(s,t)| \le M, \tag{11a}$$

Example 5. Now suppose that $X = Y = L^2$ is the space of square integrable functions. We confine ourselves to considering only operators of the special form

$$y(s) = x(s) - \int_0^1 K(s,t)\,x(t)\,dt, \tag{12}$$

first in the case when the kernel $K(s,t)$ is symmetric.

Denote by λ_i the eigenvalues of the kernel $K(s,t)$ and by $\phi_i(t)$ its eigenfunctions. Supplement the system of functions $\phi_i(t)$ by the functions $\phi_0(t), \phi_{-1}(t), \ldots$ so that the resulting system becomes a complete system in L^2. Here we assume that $\lambda_0 = \lambda_{-1} = \lambda_{-2} = \cdots = \infty$. Further, denote by ξ_i and η_i the Fourier coefficients of $x(t)$ and $Y(t)$ with respect to the system ϕ_i. Then substituting the bilinear expansion of K and the expansions for x and y

$$K(s,t) = \sum_i \frac{\phi_i(t)\phi_i(s)}{\lambda_i}, \qquad x(t) = \sum_i \xi_i \phi_i(t), \qquad y(t) = \sum_i \eta_i \phi_i(t)$$

in the expression for the operator (12), we find

$$\eta_i = \xi_i - \frac{1}{\lambda_i}\xi_i.$$

This mean that if we take L equal to the least upper bound of the numbers $|1 - 1/\lambda_i|$, we get

$$\|U\|^2 = \|y\|^2 = \sum_i \eta_i^2 = \sum_i \left(1 - \frac{1}{\lambda_i}\right)^2 \xi_i^2 = L^2 \sum_i \xi_i^2 = L^2 \|y\|^2.$$

Therefore, in the given case

$$\|U\| \le L = \sup |1 - 1/\lambda_i| \tag{13}$$

if $|K(s,t)| \leq M$ then $|\lambda_i| \geq 1/M$ implies the inequality

$$\|U\| \leq 1 + M. \tag{13a}$$

From this it is also easy to estimate the norm of the inverse operator. Indeed, we have

$$\|x\|^2 = \sum_i \xi_i^2 = \sum \left(1 - \frac{1}{\lambda_i}\right)^{-1} \eta_i^2 \leq L_i^2 \|y\|^2,$$

where $L_1 = \sup |1 - 1/\lambda_i|^{-1}$. Therefore,

$$\|U\| \, L_1 = \sup |1 - 1/\lambda_i|^{-1} \tag{14}$$

In the case of a non-symmetric kernel $K(s,t)$ it is easy to check that

$$\int y^2(s)\, ds = \int y^2(s)\, ds + \int\int \overline{K}(u,t) x(u) x(t)\, du\, dt,$$

where

$$\overline{K}(u,t) = K(u,t) + K(t,u) - \int K(u,s) K(s,u)\, ds. \tag{15}$$

Therefore, if Λ_i denotes the eigenvalues of the symmetric kernel $\overline{K}(u,t)$; then, arguing as above, we find

$$\|U\| \leq \sup_i \left|1 - \frac{1}{\Lambda_i}\right|^{-1}, \tag{16}$$

$$\|U^{-1}\| \leq \sup_i \left|1 - \frac{1}{\Lambda_i}\right|^{-1}. \tag{17}$$

2. Bilinear Operators

Consider the linear operator $u = B(x)$ which acts from the space X to the space of linear operators mapping X into Y, i.e., $u \in (X \to Y)$. Let us compute its value for an arbitrary element $x' \in X$. Then putting

$$B(x, x') = u(x") = B(x)(x'), \tag{1}$$

we obviously obtain an operator which is defined on pairs of elements x and x', is additive with respect to each of its arguments and satisfies

$$\|B(x, x')\| \leq \|u\| \, \|x'\| \leq \|b\| \, \|X\| \, \|X'\|. \tag{2}$$

An operator meeting the last conditions is called bilinear, and the least acceptable constant factor in the equality of the form (2) is its norm.

Conversely, suppose we are given a bilinear operator $B(x, x')$ which is additive in both of its arguments and satisfies the condition

$$\|B(x, x')\| \leq C\|x\|\,\|x'\|.$$

Then it is clear that for fixed x $B(x, x')$ is a certain linear operator $u(x')$ from X to Y. Putting $B(x) = u$ we get

$$\|B(x)\| = \|u\| \leq C\,\|x\|.$$

Hence, $B(x)$ is an additive operator in x and it follows that $u = B(x)$ is a linear operator taking X to $(X \to Y)$ with norm $\|B\| \leq C$. Thus it follows from the above that it is essentially equivalent to consider B as an operator from X to $(X \to Y)$ or as a bilinear operator. The values of the norm in both cases are the same. Let us present some examples of bilinear operators.

Example 1. Consider a bilinear operator acting from the space $X = m_n$ to the space $Y = m_\nu$. It is easy to see that it is of the form

$$y = B(x, x') = \left\{\sum\nolimits_{i,j=1}^{n} a_{ijk}\xi_i\xi'_j\right\}_{k=1,\dots,\nu}, \tag{3}$$

i.e., its value is the vector y whose components are quadratic forms. Clearly, we have

$$\|y\| = \|B\| = \max_k \left|\sum\nolimits_{i,j=1}^{n} a_{ijk}\xi_i\xi'_j\right| \leq \max_k \left(\sum\nolimits_{i,j=1}^{n} |a_{ijk}|\right) \|x\|\,\|x'\|,$$

which implies that

$$\|B\| \leq \max_k \sum_{i,j=1}^{n} |a_{ijk}| \leq n^2 M, \tag{4}$$

where $|a_{ijk}| \leq M$. However, these estimates do not give the exact value of the norm.

Example 2. A bilinear operator from R^n to R^ν has the same form as in the previous example. However, its norm is defined differently and has a different estimate.

Namely,

$$\left|\sum_{i,j=1}^{n} a_{ijk}\xi_i\xi'_j\right|^2 \leq \left(\sum_{i,j=1}^{n} a_{ijk}^2\right)\left(\sum_{i,j=1}^{n} \xi_i^2\xi_j'^2\right) = \left(\sum_{i,j=1}^{n} a_{ijk}^2\right)\left(\sum_{i=1}^{n} \xi_i^2\right)\left(\sum_{j=1}^{n} \xi_j'^2\right).$$

Hence,

$$\|y\| = \|B\| = \sum_{k=1}^{\nu}\left(\sum_{i,j=1}^{n} a_{ijk}\xi_i\xi'_j\right)^2 \leq \sum_{k=1}^{\nu}\sum_{i,j=1}^{n} a_{ijk}^2\|x\|^2\|x'\|^2;$$

and, therefore,

$$\|B\| \leq \left(\sum_{k=1}^{\nu} \sum_{i,j=1}^{n} a_{ijk}^2 \right)^{1/2} \leq n\sqrt{\nu} L \tag{5}$$

if all the a_{ijk} satisfy $|a_{ijk}| \leq L$.

Example 3. If X and Y are spaces of complex numbers, then an example of a bilinear operator will be an operator of the form

$$y = B(x, x') = wxx', \tag{6}$$

where w is a complex number. It is easy to check that for this operator

$$\|B\| = |w|. \tag{7}$$

Example 4. An example of a bilinear operator mapping C into C is the integral operator of the form

$$y = B(x, x'), \qquad y(s) = \int_0^1 \int_0^1 K(s, t, u) x(t) x'(u)\, dt\, du; \tag{8}$$

its norm is estimated as

$$\|B\| \leq \sup_s \int_0^1 \int_0^1 |K(s, t, u)|\, dt\, du \tag{9}$$

if $|K(s, t, u)| \leq M$.

Example 5. The same operator (8) may be viewed as an operator mapping L^2 into L^2. In this case its norm can be estimated as

$$\|B\| \leq \left[\int_0^1 \int_0^1 \int_0^1 K^2(s, t, u)\, ds\, dt\, du \right]^{1/2}. \tag{10}$$

3. Differentiation of Non-linear Operators

Suppose that $y = P(x)$ is a non-linear operator acting from the space$X$ to Y. We say that it is Fréchet differentiable [11] if for each fixed value x there is a linear operator $U \in (X \to Y)$ satisfying

$$\|[P(x + \Delta x) - P(x)] - U(\Delta x)\| \leq \|\Delta x\|\, \varepsilon(\|\Delta x\|); \tag{1}$$

here $\varepsilon(\delta)$ is a function which tends to zero as $\delta \to 0$. This operator U is said to be the derivative of the operator $P(x)$ at the given x:

$$P'(x) = U. \tag{2}$$

As we mentioned before, $P'(x)$ is an element of the space $(X \to Y)$.

In turn, $U = P'(x)$ is a non-linear operator which takes the space X to the space $(X \to Y)$. It can also turn out to be differentiable. Its derivative is called the second derivative of the non-linear operator $P(x)$

$$V = [P'(x)]' = P''(x). \tag{3}$$

This second derivative V is an element of the space $[X \to (X \to Y)]$, i.e., of the space of linear operators from X to $(X \to Y)$. As we saw before, the consideration of this operator is equivalent to that of a bilinear operator from the space X to Y so that $P''(x)$ may be viewed as such a bilinear operator. In this connection by $\|P'(x)\|$ and $\|P''(x)\|$ we may mean the norms that are calculated respectively in the spaces $(X \to Y)$ and $[X \to (X \to Y)]$ (see [12]).

Let us note some propositions concerning derivatives which we shall use in the sequel.

1. If $y = \phi(x)$, then $z = F(y) = F[\phi(x)]$, where the functions ϕ and F are differentiable and

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = F'[\phi(x)]\phi'(x). \tag{4}$$

Here we mean that the linear operators written side by side dz/dy and dy/dx must be applied successively (the chain rule). This rule is established in the same way as for the conventional derivatives of functions of one or several variables.

2. If $y = P(x)$ is a linear operator from X to Y, then obviously

$$P'(x) = P; \quad P''(x) = 0, \tag{5}$$

i.e., the derivative of a linear operator coincides with the operator itself.

3. If U is a linear operator from Y to Z, then

$$[U(P(x))]' = U(P'(x)), \tag{6}$$

i.e., a constant operator may be factored out of the bracket being differentiated. This immediately follows from rules 1 and 2. Here

$$U(P'(x)) = V \in (X \to Z) \quad \text{and} \quad V(x') = U[(P'(x))(x')].$$

4. If $P(x)$ is a differentiable operator, then we have the inequality [14]

$$\|P(\overline{x} + \Delta x) - P(x)\| \leq \sup \{\|P'(\overline{x})\|, \|\Delta x\|\}, \tag{7}$$

$$\overline{x} = x + \theta \Delta x, \quad 0 \leq \theta \leq 1,$$

which is an estimate of the increment similar to that for functions of one variable known as the finite increment formula.

For the proof we put $P(x+\Delta x)-P(x) = y$ and choose a linear functional T on the space Y such that $\|T\| = 1$, $T(y) = \|y\|$, which is always possible [20]. Consider the real-valued function $f(t) = T[P(x + t\Delta x)]$ of the real parameter t. For its derivative, using the rules 1 and 3, we obtain the expression $f'(t) = TP'(x + t\Delta x)\Delta x$.

Further, using the definition of $f(x)$ and applying the ordinary formula of finite increments, we get

$$T(y) = T[P(x + \Delta x) - P(x)] = f(1) - f(0) = f'(\theta) = TP'(x + \theta\Delta x)\Delta x.$$

This implies

$$\begin{aligned} \|P(x + \Delta x) - P(x)\| &= \|y\| = T(y)) \leq \|T\|\,\|P'(x + \theta\Delta x)\|\,\|\Delta x\| \\ &\leq \sup \|P'(\overline{x})\|\,\|\Delta x\|, \qquad 0 \leq \theta \leq 1, \quad \overline{x} = x + \theta\Delta x. \end{aligned}$$

5. If $P(x)$ is a twice differentiable function, then we have the following inequality

$$\|P(x + \Delta x) - P(x) - P'(x)\Delta x\| \leq \frac{1}{2} \sup \|P''(\overline{x})\|\,\|\Delta x\|^2, \tag{8}$$
$$\overline{x} = x + \theta\Delta x$$

related to the Taylor formula just as the previous statement is related to the finite increment formula.

The proof follows along the lines of the previous one. Denoting by y an element on the left-hand side of (8) whose norm must be estimated, we introduce a linear functional T satisfying $\|T\| = 1$ and $T(y) = \|y\|$. Further, we construct the auxiliary function $f(t) = T[P(x + \Delta x)]$ for which

$$f'(t) = T[P'(x + t\Delta x)\Delta x], \qquad f''(t) = T[P''(x + t\Delta x)\Delta x\Delta x],$$

where the expression in the square brackets means that the bilinear operator $P''(x+t\Delta x)$ must be computed for the pair of arguments equal to Δx. Then, applying the ordinary Taylor formula, we obtain

$$\begin{aligned} \|P(x + \Delta x) - P(x) - P'(x)\Delta x\| &= \|y\| = T(y) = f(1) - f(0) - f'(0) \\ &= \frac{1}{2}f''(\theta) \leq \frac{1}{2}\|T\| \sup \|P''(x + t\Delta x)\|\,\|\Delta x\|^2. \end{aligned}$$

Let us give some examples of the derivation of operators.

Example 1. Consider a non-linear operator mapping n-dimensional space to a ν-dimensional one. It is defined by a set of ν functions of n variables:

$$y = P(x), \qquad \eta_k = f_k(\xi_1, \ldots, \xi_n), \qquad k = 1, \ldots, \nu. \tag{9}$$

We assume that the function f_k has continuous partial derivative of the second order. Then for the differentials we have

$$d\eta_k = \sum_{i=1}^{n} \frac{\partial f_k}{\partial \xi_i} d\xi_i, \qquad k = 1, \ldots, \nu, \tag{10}$$

while the increment $\Delta y = \{\Delta \eta_k\}_{k=1,2,\ldots,\nu}$ is expressed by the same system of forms of differentials up to infinitesimals of higher orders. This implies, in the case in question, that $P'(x)$ is given by the matrix of partial derivatives

$$P'(x) = \left\| \frac{\partial f_k}{\partial \xi_i} \right\|_{\substack{i = 1, \ldots, n \\ k = 1, \ldots, \nu}} \tag{11}$$

or, to be more precise, $P'(x)$ is the linear transformation corresponding to this matrix (see (2) Section 1).

In a similar way, considering the increment of $P'(x)$ that corresponds to the increment of the argument $\Delta x' = (\Delta \xi_1', \ldots, \Delta \xi_n')$, we see that the second derivative is determined by a matrix depending on three indices

$$P''(x) = \left\| \frac{\partial_2 f_k}{\partial \xi_i \partial \xi_j} \right\|_{\substack{i, j = 1, \ldots, n \\ k = 1, \ldots, \nu}} . \tag{12}$$

If we consider it as bilinear operator, then it will be determined by a system of ν bilinear forms

$$P''(x)xx' = \left\{ \sum_{i,j=1}^{n} \frac{\partial_2 f_k}{\partial \xi_i \partial \xi_j} \xi_i \xi_j' \right\}_{k=1,\ldots,\nu} . \tag{13}$$

If we choose a specific norm in n-dimensional spaces corresponding to the norms of R^n or m_n, we can find estimates for $\|P'(x)\|$ and $\|P''(x)\|$ by using the results of sections 1 and 2.

Example 2. If in the space of complex numbers we consider an analytic function $y = P(x)$, then in the given case $\Delta y = P'(x)\Delta x$ up to infinitesimals of higher order; hence, the operator $P'(x)$ is multiplication by the complex number $P'(x)$ and its norm is

$$\|P'(x)\| = |P'(x)|. \tag{14}$$

In the same way, in the given case, the second derivative coincides with the ordinary second derivative if we consider the latter as a bilinear operator over pairs of complex numbers $P''(x) \cdot x \cdot x'$.

Example 3. Consider a non-linear integral operator in the form

$$y = P(x), \qquad y(s) = \int_0^1 K(s,t)\Delta x(t)\, dt, \tag{15}$$

where $K(s,t,u)$ is a twice continuously differentiable function of its arguments. Then up to infinitesimals of higher order, we have

$$\Delta y(s) = \int_0^1 K_x'(s,t,x(t))\Delta x(t\, dt)$$

whence it follows that $P'(x)$ in the given case is a linear integral operator with kernel $K(s,t) = K'_x(s,t,x(t))$:

$$P'(x)]\Delta x = \int_0^1 K(s,t)\Delta x(t)\,dt = \int_0^1 K'_x(s,t,x(t))\Delta x(t)\,dt. \tag{16}$$

If we now give $x(t)$ the increment $\Delta' x(t)$, it is easy to check that up to infinitesimal expression of higher order we have

$$[P'(x+\Delta' x) - P'(x)]\Delta x = [\Delta P'(x)]\Delta x = \int_0^1 K''_{x^2}(s,t,x(t))\Delta' x(t)\Delta x(t)\,dt,$$

which implies that the second derivative in the given case is a bilinear integral operator of special form

$$P''(x)\Delta x\Delta' x = \int_0^1 K_2(s,t)\Delta x(t)\Delta' x(t)\,dt,$$

where

$$K_2(s,t) = K''_{x^2}(s,t,x(t)). \tag{17}$$

4. Convergence of the Newton Process

Consider the application of the Newton process to non-linear functional equations of the form

$$P(x) = 0 \tag{1}$$

where $P(x)$ is an operator mapping the space X into Y; $P(x)$ is assumed twice differentiable. Formulas relating successive approximations are constructed on the basis of ideas similar to the ones applied for real equations.

Suppose that x_0 is an initial approximation to a solution. Replacing the increment $P(x) - P(x_0)$ by the differential at the point x_0, we can replace the given equation approximately by a linear one

$$P(x) \approx P(x_0) + P'(x_0)(x - x_0) = 0. \tag{2}$$

A solution x_1 of this equation gives a new approximation for the root. If the operator $P'(x_0)$ possesses an inverse operator $[P'(x_0)]^{-1} \in (Y \to X)$, then, using it, the expression x_1 may be obtained in explicit form. Indeed, applying this operator $[P'(xc_0)]^{-1}$ to both parts of relation (2), we find

$$x_1 = x_0 - [P'(x_0)]^{-1}P(x_0). \tag{3}$$

In a similar way, we can successively express the subsequent linear approximations in terms of those foregoing:

$$x_{n+1} = x_n - [P'(x_n)]^{-1}P(x_n). \tag{4}$$

Conditions for convergence of the sequence x_n to an exact solution of equation (1), which are at the same time sufficient conditions for the existence of such a solution, are given by the following theorem.

Theorem 1. (On existence of solutions and on convergence of the Newton process). Suppose the following conditions are met:

1) for each initial approximation x_0, the operator $P'(x_0)$ from X to Y possesses an inverse operator $\Gamma_0 = [P'(x_0)]^{-1}$ whose norm has the following estimate:

$$\|\Gamma_0\| \leq B_0 \tag{5),}$$

2) the element x_0 meets equation (1) approximately, an estimate of the expression $\Gamma_0 P(x_0)$ known:

$$\|\Gamma_0 P(x_0)\| \leq \eta_0, \tag{6}$$

3) the second derivative $P''(x)$ is bounded in the domain that interests us and is determined by inequality (9) below; namely,

$$\|P''(x)\| \leq K, \tag{7}$$

4) the constants B_0, η_0, K satisfy the relations

$$h_0 = B_0 \eta_0 K \leq 1/2. \tag{8}$$

Then equation (1) has a solution x^*, which lies in the domain about x_0 determined by the inequality

$$\|x - x^*\| \leq N(h_0)\eta_0 = \frac{1 - \sqrt{1 - 2h_0}}{h_0}\eta_0 \tag{9}$$

and the successive approximations of the Newton process x_n converge to the solution, the rate of convergence being characterized by the estimate

$$\|x_n - x^*\| \leq 2^{1-n}(2h_0)^{2^n - 1}\eta_0. \tag{10}$$

Proof. Introduce the notation

$$F_0(x) = x - \Gamma_0 P(x). \tag{11}$$

Using it, formula (3) (which relates x_1 and x_0) may be written as

$$x_1 = x_0 - [P'(x_0)]^{-1} P'(x_0) = F_0(x_0). \tag{12}$$

Let us show that when we pass from the value x_0 to x_1, all the conditions 1–4 will still hold. First of all, we have

$$\|x_1 - x_0\| = \|\Gamma_0 P(x_0)\| \leq \eta_0. \tag{13}$$

Further, applying an analogue of the Lagrange formula ((7) in Section 3) to $P'(x)$, we obtain an estimate for the norm of the following operator which maps X into X:

$$\|\Gamma_0(P'(x_0) - P'(x_1))\| \le B_0 \sup \|P''(\overline{x})\| \, \|x_1 - x_0\| \le B_0 K \eta_0 = h_0 < 1,$$

$$\overline{x} = x_1 + \theta(x_0 - x_1). \tag{14}$$

Hence, using the theorem of S. Banach ((2) of Section 1), we see that the inverse operator

$$H = [I - \Gamma_0(P'(x_0) - P'(x_1))]^{-1}$$

exists, where I is the identity operator taking X to X, and that for its norm we have the following estimate

$$\|H\| \le \frac{1}{1 - h_0}. \tag{15}$$

Hence, putting $\Gamma_1 = H\Gamma_0$ and using the rule for operator inverses $[AB]^{-1} = B^{-1}A^{-1}$, we find

$$\Gamma_1 = H\Gamma_0 = \{I - \Gamma_0[P'(x_0) - P'(x_1)]\}^{-1}[P'(x_0)]^{-1}$$

$$= \{P'(x_0)(I - \Gamma_0[P'(x_0) - P'(x_1)])\}^{-1} = [P'(x_1)]^{-1},$$

which proves the existence of the sought inverse operator. Hence, on the basis of inequalities (15), we obtain an estimate for it

$$\|\Gamma_1\| = \|[P'(x_1)]^{-1}\| \le \frac{B_0}{1 - h_0} = B_1. \tag{16}$$

Thus condition (1) has been checked.

Now, using the fact that

$$F_0'(x_0) = I - \Gamma_0 P'(x_0) = 0 \tag{17}$$

(it follows from rules 2 and 3 of Section 3) and using (11) for $x = x_1$, we get

$$-\Gamma_0 P(x_1) = F_0(x_1) - F_0(x_0) - F_0'(x_0)(x_1 - x_0).$$

Then applying an analogue of the Taylor formula (8) of Section 3 to $P = F_0$ and $\Delta x = x_1 - x_0$, we get

$$\|\Gamma_0 P(x_1)\| \le \frac{1}{2} \sup \|F_0''(\overline{x})\| \|x_1 - x_0\|^2$$

$$\le \frac{1}{2} \sup \|\Gamma_0 P''(\overline{x})\| \|x_1 - x_0\|^2 \le \frac{1}{2} B_0 K \eta_1^2 \le \frac{1}{2} h_0 \eta_0, \tag{18}$$

$$\bar{x} = x_0 + \theta(x_1 - x_0).$$

Finally, using (15) and (18), we get

$$\|\Gamma_1 P(x_1)\| = \|H\Gamma_0 P(x_1)\| \leq \|H\| \cdot \|\Gamma_0 P(x_1)\| \leq \frac{1}{2}\frac{h_0\eta_0}{1-h_0} = \eta_1. \tag{19}$$

Condition 3 holds for the point x_1, since the corresponding ball, as we check below, is contained inside the ball defined by inequality (9).

Finally, condition 4 can be checked directly by using (16) and (19). Indeed,

$$h_1 = B_1\eta_1 K = \frac{B_0}{1-h_0}\frac{1}{2}\frac{h_0\eta_0}{1-h_0}K = \frac{h_0^2}{2(1-h_0)^2} \leq 2h_0^2 \leq \frac{1}{2} \tag{20}$$

since $h_0 \leq 1/2$.

Thus for $x = x_1$ conditions of the form 1–4 with the numbers B_0, η_0 and h_0 replaced by B_1, η_1 and h_1 hold. This allows us to continue the successive determination of elements x_n and the related numbers B_n, η_n and h_n that satisfy formulas similar to (13), (16), (19) and (20)

$$\|x_n - x_{n+1}\| \leq \eta_n, \tag{13a}$$

$$B_n = \frac{B_{n-1}}{1-h_{n-1}}, \tag{16a}$$

$$\eta_n = \frac{1}{2}\frac{h_{n-1}\eta_{n-1}}{1-h_{n-1}}, \tag{19a}$$

$$h_n = \frac{1}{2}\frac{h_{n-1}^2}{(1-h_{n-1})^2}. \tag{20a}$$

Further, we have the following estimates for them

$$h_2 \leq 2h_1^2 \leq 8h_0^4, \quad \dots \quad h_n \leq \frac{1}{2}(2h_0)^{2^n}, \tag{21}$$

$$\begin{aligned}\eta_n &= \frac{1}{2}\frac{h_{n-1}}{1-h_{n-1}}\eta_{n-1} \leq h_{n-1}\eta_{n-1} \leq \cdots \leq h_{n-1}h_{n-2}\cdots h_0\eta_0 \\ &\leq 2^{-n}(2h_0)^{2^{n-1}}(2h_0)^{2^{n-2}}\dots(2h_0)\eta_0 = 2^{-n}(2h_0)^{2^n-1}\eta_0.\end{aligned} \tag{22}$$

Finally, note the following identity

$$N(h_n)\eta_n - N(h_{n+1})\eta_{n+1} = \eta_n \tag{23}$$

which can be checked by direct computation

$$N(h_{n+1})\eta_{n+1} = \eta_{n+1}\frac{1-\sqrt{1-2h_{n+1}}}{h_{n+1}} = \frac{1}{2}\frac{h_n\eta_n}{1-h_n}\frac{1-\sqrt{1-\frac{h_n^2}{(1-h_n)^2}}}{\frac{1}{2}\frac{h_n^2}{(1-h_n)^2}}$$

$$= \eta_n \frac{1 - h_n - \sqrt{1 - 2h_n}}{h_n} = \eta_n N(h_n) - \eta_n.$$

Using (13a), (23), and (22), we find

$$\|x_{n+p} - x_n\| \leq \|x_{n+1} - x_n\| + \cdots + \|x_{n+p} - x_{n+p-1}\|$$

$$\leq \eta_n + \cdots + \eta_{n+p-1}$$

$$= N(h_n)\eta_n - N(h_{n+p})\eta_{n+p} \leq N(h_n)\eta_n \leq 2\eta_n \leq 2^{1-n}(2h_0)^{2^n-1}\eta_0. \tag{24}$$

This proves the existence of the limit

$$\lim_{n\to\infty} x_n = x^*. \tag{25}$$

This also implies that as $p \to \infty$ the inequality (10) holds and if we then take $n = 0$, inequality (9) is also verified.

The fact that x^* is a root of equation (1) follows by passing to the limit in the relation

$$P'(x_n)(x_{n+1} - x_n) + P'(x_n) = 0.$$

Indeed, since $\|x_{n+1} - x_n\| \to 0$ and $\|P'(x_n)\|$ is bounded since

$$\|P'(x_n)\| \leq \|P'(x_0)\| + \|P'(x_n) - P'(x_0)\|$$

$$\leq \|P'(x_0)\| + K\|x_n - x_0\| \leq \|P'(x_0)\| + KN(h_0)\eta_0,$$

we see that $\|P(x_n)\| \to 0$ and since $x_n \to x^*$ the continuity of $P(x)$ implies $P(x^*) = 0$.

Note also that the statement used in the proof that the ball

$$\|x - x_n\| \leq N(h_n)\eta_n \tag{26}$$

is contained inside the ball (9) is now completely obvious. Indeed, if x is inside the ball (26), then

$$\|x - x_0\| \leq \|x_n - x_0\| + \|x - x_n\|$$

$$\leq [N(h_0)\eta_0 - N(h_n)\eta_n] + N(h_n)\eta_n = N(h_0)\eta_0,$$

i.e., x is also inside the ball (9). Thus the theorem is proved entirely.

Theorem 2. (On uniqueness of solution and convergence of the process for other initial conditions). Suppose conditions 1–4 of the previous theorem hold with the only difference that inequality (7)

$$\|P''(x)\| \leq K$$

now be assumed to hold inside the ball determined by inequality (27). Then the solution of equation (1) in the domain

$$\|x - x_0\| < L(h_0)\eta_0 = \frac{1 + \sqrt{1 - 2h_0}}{h_0}\eta_0 \tag{27}$$

is unique (in the case when $h_0 = 1/2$ the strict inequality $<$ in this relation must be replaced by the sign $\leq$). The process of successive approximations converges to the solution if we begin with any value of x which satisfies the inequality

$$\|x - x_0\| \leq \Delta = \frac{1 - 2h_0}{4h_0}\eta_0. \tag{28}$$

Proof. To establish uniqueness, let us assume that we have another solution $\tilde{x}$ of the equation $P(x) = 0$ which satisfies the conditions

$$\|\tilde{x} - x_0\| \leq \theta L(h_0)\eta_0, \qquad 0 \leq \theta < 1 \tag{29}$$

(we first confine ourselves to the case when $h_0 < 1/2$). Since for the given value $x = \tilde{x}$ we have $P(\tilde{x}) = 0$ it follows that (see (11)) $F_0(\tilde{x}) = \tilde{x}$. As above in the proof of Theorem 1 (compare (17) and (18)) we have:

$$\begin{aligned} \|\tilde{x} - x_1\| &= \|F_0(\tilde{x}) - F_0(x_0)\| = \|F_0(\tilde{x}) - F_0(x_0) - F_0(x_0)(\tilde{x} - x_0)\| \\ &\leq \frac{1}{2}B_0 K\|\tilde{x} - x_0\| \leq \frac{1}{2}B_0 K\theta^2 L^2(h_0)\eta_0^2 = \theta^2 L(h_1)\eta_1. \end{aligned} \tag{30}$$

The validity of the last relation can be checked by direct calculations

$$L(h_1)\eta_1 = \frac{1 + \sqrt{1 - 2h_1}}{h_1}\eta_1 = \frac{1 + \sqrt{1 - \frac{h_0^2}{(1-2h_0)^2}}}{\frac{1}{2}\frac{h_0^2}{(1-2h_0)^2}}\frac{1}{2}\frac{h_0}{1 - 2h_0}\eta_0$$

$$= \frac{1 - h_0 + \sqrt{1 - 2h_0}}{h_0}\eta_0 = (L(h_0) - 1)\eta_0,$$

$$\frac{1}{2}B_0 K h_0 \eta_0 = \frac{1}{2}h_0\left[\frac{2 - 2h_0 + 2\sqrt{1 - 2h_0}}{h_0}\right]\eta_0 = (L(h_0) - 1)\eta_0.$$

Inequality (30) that we have proved shows that for the elements $\tilde{x}$ and x_1 we have a condition similar to (29) except that θ is replaced by θ^2. Hence, we can continue the estimates, taking x_1 instead of the element x_0, then x_2, and so on. As the result, we get

$$\|\tilde{x} - x_n\| \leq \theta^{2^n} L(h_n)\eta_n \leq \theta^{2^n}\frac{2}{B_n K}, \tag{31}$$

the last inequality following from the fact that $L(h_n) < 2/h_n$.

Since $B - n \geq B_0$ and $\theta < 1$, the estimate (31) that we have obtained shows that $\|x_n - \tilde{x}\| \to 0$. Hence,

$$\tilde{x} = \lim_{n \to \infty} x_n = x^* \tag{32}$$

which proves uniqueness.

The case in which $h_0 = 1/2$ must be considered separately. Here, due to the fact that in condition (27) the equality sign is possible, in relations (29) and (31) we must take $\theta = 1$, but, in the case $h_0 = 1/2$, as is clear from (16), we have $B - {}_1 = 2B_0$ and generally $B_n = 2^n B_0$, so that as before on the basis of (31) we conclude that $\|x_n - \tilde{x}\| \to 0$, which implies uniqueness.

Now let us assume that for the initial value we have taken some x' contained in the ball (28):

$$\|x' - x_0\| \leq \Delta = \frac{1 - 2h_0}{4h_0}\eta_0, \tag{33}$$

and let us prove that the sequence of approximations beginning with this value also converges to x^*.

Let us show first that for $x = x'$ we have the same conditions as 1–4 of Theorem 1 for $x = x_0$. Partly repeating the argument in Theorem 1, we obtain the relation

$$\|[\Gamma_0[P'(x_0) - P'(x')]]\| \leq B_0 K \Delta < 1. \tag{34}$$

Hence, (compare (15)),

$$\|[I - \Gamma_0[P'(x_0) - P'(x')]]^{-1}\| \leq \frac{1}{1 - B_0 K \Delta}, \tag{35}$$

$$\|[P'(x')]^{-1}\| \leq \|\Gamma'\| = \|H'\Gamma_0\| \leq \frac{B_0}{1 - B_0 K \Delta} = B'; \tag{36}$$

thus, condition (1) has been verified.

Further, we have (compare (6) and (34)):

$$\|\Gamma_0 P(x')\| \leq \|\Gamma_0[P(x') - P(x_0) - P'(x_0)(x' - x_0)]\|$$
$$+\|\Gamma_0[P(x_0) + P'(x_0)(x' - x_0)]\| \leq \frac{1}{2}B_0 K \Delta^2 + \eta_0 + \Delta. \tag{37}$$

Then

$$\|\Gamma' P(x')\| = \|H'\Gamma_0 P(x')\| \leq \frac{1}{1 - B_0 K \Delta}(\frac{1}{2}B_0 K \Delta^2 + \eta_0 + \Delta) = \eta'; \tag{38}$$

i.e., condition 2 has also been checked. The validity of condition 3 will be verified a little further. First we prove condition 4. We have

$$h' = B' K \eta' = \frac{\frac{1}{2}B_0 K \Delta^2 + \eta_0 + \Delta}{(1 - B_0 K \Delta)^2} B_0 K \Delta = \frac{h_0 + B_0 K \Delta + \frac{1}{2}B_0^2 K^2 \Delta^2}{(1 - B_0 K \Delta)^2}$$

$$= \frac{1}{2} + \frac{h_0 - \frac{1}{2} + 2B_0K\Delta}{(1 - B_0K\Delta)^2} = \frac{1}{2} + \frac{h_0 - \frac{1}{2} + (\frac{1}{2} - h_0)}{(1 - B_0K\Delta)^2} = \frac{1}{2}.$$

Thus, condition 4 also holds. This allows us to conclude that the sequence of approximations x'_n beginning with $x'_0 = x'$ converges to some solution $x^{*\prime}$ contained in the ball

$$\|x - x'\| \leq N(h')\eta' \leq \frac{\eta'}{h'} = \frac{1}{B'K} = \frac{1 - B_0K\Delta}{B_0K}. \tag{39}$$

But this ball is located inside the ball (27). Indeed,

$$\|x - x_0\| \leq \|x' - x_0\| + \|x' - x\|$$

$$\leq \Delta + \frac{1 - B_0K\Delta}{B_0K} = \frac{1}{B_0K} = \frac{\eta_0}{h_0} < L(h_0)\eta_0.$$

Therefore, the solution $x^{*\prime}$ is also contained in the ball (27) and by the uniqueness theorem, which we have already proved this solution must coincide with x^*. Thus $\lim_{n\to\infty} x'_n = x^*$.

Note in passing that the fact that the ball (39) is inside (27), where the inequality $\|P''(x)\| \leq K$ holds, means condition 3 is verified for the point $x = x'$, so that our proof is complete. Let us make several remarks on both proofs of the theorems.

Remark 1. Note that condition 2 could have been replaced by the following simpler condition 2′ consisting in inequality

$$\|P(x_0)\| \leq \eta'_0, \tag{40}$$

which directly characterizes the order of accuracy of the initial value as compared with a solution of equation (1). Condition 2′ implies condition 2 for $\eta_0 = B_0\eta'_0$. Indeed,

$$\|\Gamma_0 P(x_0)\| \leq \|\Gamma_0\| \cdot \|P(x_0)\| \leq B_0\eta'_0 = \eta_0.$$

In this connection, if instead of condition 2 we have condition 2′, then the number h_0 in inequality (8) will appear as

$$h_0 = B_0K\eta_0 = B_0^2K\eta'_0 \leq \frac{1}{2}. \tag{41}$$

In passing let us also note that condition 2 itself possesses a different form that is sometimes more convenient in the applications. Namely, using (13), condition 2 can be written as

$$\|\Gamma_0 P(x_0)\| = \|x_1 - x_0\| \leq \eta_0 \tag{42}$$

which is easy to check once we have found the first approximation.

Remark 2. The domain in which the solution x^* must be contained under the assumptions of the theorem is given by inequality (9). However, it is relatively easy to obtain a more exact inequality for x^*. Namely, knowing Γ_0, we can determine that $x_1 = x_0 - \Gamma_0 P(x_0)$ and $P(x_1)$; then applying Theorem 1 to x_1 instead of x_0 (compare (9) and (6)) and using the estimate for $\|\Gamma_1\|$ (16), we get

$$\|x - x^*\| \leq N(h_1)\|\Gamma_1 P(x_1)\| \leq N(h_1)\|\Gamma_1\| \cdot \|P(x_1)\|$$

$$\leq N(h_1)\frac{B_0}{1-h_0}\|P(x_1)\| = N(h_1)\eta_1 \frac{1}{2}\frac{h_0\eta_0}{1-h_0}\frac{2B_0}{h_0\eta_0}\|P(x_1)\|$$

$$= N(h_1)\eta_1 \frac{2B_0}{h_0\eta_0}\|P(x_1)\| = [\eta_0 N(h_0) - \eta_0]\frac{2B_0}{h_0\eta_0}\|P(x_1)\|$$

$$= 2\frac{1 - h_0 + \sqrt{1-2h_0}}{h_0^2}B_0\|P(x_1)\| = \frac{2}{1 - h_0 + \sqrt{1-2h_0}}B_0\|P(x_1)\|.$$

The estimate obtained

$$\|x - x^*\| \leq \frac{2}{1 - h_0 + \sqrt{1-2h_0}}B_0\|P(x_1)\| \tag{43}$$

will be used later on. Note that for small h_0 it is of order $B_0\|P(x_1)\|$.

Remark 3. Note that the estimates (8), (9) and (27) obtained in Theorems 1 and 2 cannot be improved even in the case of a real equation of the second degree, as we see from the following example

$$P(x) = \frac{1}{2}x^2 - x + h = 0, \quad h > 0, \qquad x_0 = 0.$$

Indeed, in the given case

$$P'(x_0) = -1, \qquad \frac{1}{|P'(x_0)|} = 1, \qquad B_0 = 1,$$

$$\|\Gamma_0 P(x_0)\| = 1 \cdot h = h, \quad \eta_0 = h, \quad \|P''(x)\| = 1 = K, \quad h_0 = B_0 K \eta_0 = h.$$

But then roots of the equation will be

$$x^*_{1,2} = 1 \pm \sqrt{1-2h} = N(h_0)\eta_0 \text{ or } L(h_0)\eta_0;$$

and therefore they can be real numbers only under the condition that $h_0 = h \leq 1/2$. But then the first root is located at the distance $N(h_0)\eta_0$ and the second, at the distance $L(h_0)\eta_0$. This shows that the bounds of the domain where the root is contained and the uniqueness domain have been determined exactly.

Remark 4. Note that in the case of real and complex equations theorems similar to Theorem 1 were obtained by A. Ostrowski [1, 2]. These theorems follow from Theorem 1 proved above in the case of a real equation directly and, in the case of complex equations, if we take into consideration the meaning of the operators and the value of the norm in this case in accordance with the examples considered above (Example 3 in subsection 1 and Example 2 in subsection 3), namely,

$$\|P(x_0)\| = |P(x_0)|, \qquad \|\Gamma_0\| = \|[P'(x_0)]^{-1}\| = \frac{1}{|P'(x_0)|},$$

$$K = \max \|P''(x)\| = \max |P''(x)|,$$

$$\eta_0 = |P(x_0)|, \qquad \eta_0 = \frac{|P(x_0)|}{|P'(x_0)|}.$$

In accordance with this, Theorems 1 and 2 for the case of real and complex equations may be stated as follows:

If we have the condition

$$h_0 = \frac{|P(x_0)|K}{|P'(x_0)|^2} \leq \frac{1}{2}, \qquad K = \max |P''(x)|,$$

then the Newton process converges to a solution of the equation $P(x) = 0$ which exists and is contained in the domain

$$|x - x_0| \leq N(h_0)\eta_0 = \frac{(1 - \sqrt{1 - 2h_0})|P'(x_0)|}{K}$$

and unique in the domain

$$|x - x_0| < L(h_0)\eta_0 = \frac{(1 + \sqrt{1 - 2h_0})|P'(x_0)|}{K}.$$

Note that the convergence of the Newton method was also considered under other conditions ([15, 3], and [16], where this theorem was extended to general functional equations); but in these theorems the boundedness of the expression $|P'(x_0)|^{-1}$ was assumed in on entire interval rather than only at the point x_0, which is less convenient for verification.

Remark 5. Note that in particular, if the value $x_0 = x^*$ is a root of the equation, then $h_0 = 0$ and $\eta_0 = 0$, but $L(h_0)\eta_0$ must be replaced by $2(B_0K)^{-1}$. In this connection we must conclude that x_0 serves as the unique solution to the equation in the domain

$$\|x - x^*\| < \frac{2}{B_0 K} \tag{44}$$

and that the method of successive approximations converges to the solution if the initial value is chosen in the domain

$$\|x - x^*\| \le \Delta = \frac{1}{4B_0K}. \tag{45}$$

Remark 6. In the case of a compact operator P, the existence of the solution inside the ball (9) under conditions 1–4 may be obtained also on the basis of the Schauder fixed-point theorem [17]. Namely, the equation $P(x) = 0$ is replaced by the equivalent equation $x = F_0(x)$ and, as may be checked, the operator $F_0(x)$ maps the ball (9) into itself. It is necessary to say, however, that this verification requires most of the arguments used in the theorem and at the same time this method does not yield several other important results — the convergence of the Newton process and the uniqueness theorem. The Cacciopolli-Banach [17] principle would allow the proof of existence of a solution under essentially rougher conditions, approximately $h_0 = 0.1$.

Remark 7. Note finally certain important aspects of the proof of the theorem. Namely, not only the convergence of a certain algorithm is established, but in fact existence and uniqueness theorems are proved and the domain where the solutions are located is found. Here the essential condition for applying these theorems is that we must know an initial approximation x_0 which is a rough approximation to the solution. This initial value x_0 satisfying the conditions of Theorems 1 and 2 always exists if the solution exists and is simple, i.e., if it satisfies $\|[P'(x^*)]^{-1}\| < +\infty$. Indeed, in this case all the conditions will be met by every point x_0 sufficiently close to x^*.

Such an initial value x_0 can in practice be obtained by using rough numerical or approximate solutions of the problem. In particular, in mechanics and other fields of applied mathematics such an approximation may be obtained by considering the problem under simplified assumptions. After an approximate solution has been found and it turns out to satisfy conditions 1–4, we can conclude on the basis of the theorems obtained above that the exact solution exists, is unique and is contained in a certain domain, i.e., we can carry out sufficiently complete theoretic study of the problem.

Thus the theorems found show that the approximate solution of the problem is useful not only for obtaining certain numeric results, but also for the theoretic analysis of the question.

5. Modifications of the Newton Process

Applying the Newton process, we must at each successive approximation find the inverse operator $[P'(x_n)]^{-1}$ or at least solve the equation

$$P'(x_n)(x_n - x_{n+1}) = P(x_n), \tag{1}$$

which may turn out to be difficult. Therefore, for the practical application of the Newton process, it is recommended to modify it, replacing each time $[P'(x_n)]^{-1}$ by $[P'(x_0)]^{-1} = \Gamma_0$ or, which is the same, by successively determining the approximations from the equation

$$P'(x_0)(x'_n - x'_{n+1}) = P(x_n), \tag{2}$$

whence

$$x'_{n+1} = x'_n - [P'(x_0)]^{-1}P'(x_n). \tag{3}$$

Obviously, the first step in the two processes coincides, i.e., $x'_1 = x_1$. In the following theorem we establish the convergence of such a modified process.

Theorem 3. If the assumptions of Theorem 1 hold and $h_0 < 1/2$, then the Newton process for which the successive approximations are determined by (3) converges to the solution

$$\lim_{n\to\infty} x'_n = x^*. \tag{4}$$

Here the convergence has the rate of a geometric progression, to be more exact

$$\|x'_n - x^*\| < q^{n-1}\|x_1 - x^*\|, \qquad q = 1 - \sqrt{1 - 2h_0} < 1. \tag{5}$$

Proof. To prove this theorem, let us first establish a preliminary statement: if for the element x we have the conditions

$$\|x - x^*\| \le q\|x_1 - x^*\|, \tag{6}$$

$$\|x - x_0\| \le N(h_0)\eta_0, \tag{7}$$

then for the element $x' = F_0(x)$ we shall obtain

$$\|x' - x^*\| \le q\|x - x^*\|, \tag{6a}$$

$$\|x' - x_0\| \le N(h_0)\eta_0. \tag{7a}$$

Indeed, we have

$$\|x' - x^*\| = \|F'_0(x) - F'_0(x^*)\| \le \|F'_0(\bar{x})\| \sup \|x - x^*\|, \tag{8}$$

$$\bar{x} = x + \theta(x^* - x).$$

Further, (compare (13), (7), and (9) in Section 4):

$$\|F'_0(\bar{x})\| = \|F'_0(x) - F'_0(x_0)\| \le \sup \|F''_0(\tilde{x})\| \, \|\bar{x} - x^*\|$$

$$\le B_0 K \max\{\|x - x_0\|, \|x_0 - x^*\|\} \le B_0 K N(h_0)\eta_0$$

$$= 1 - \sqrt{1 - 2h_0} = q, \qquad \tilde{x} = x_0 + \theta_1(\bar{x} - x_0).$$

This inequality, together with the previous (8), yields (6a). It remains to establish (7a). Indeed,

$$\|x' - x_0\| = \|F_0(x) - x_0\| = \|x - \Gamma_0 P(x) - x_0\|$$
$$= \|\Gamma_0[P(x) - P'(x_0)(x - x_0)]\|\|$$
$$\leq \|\Gamma_0 P(x_0)\| + \|\Gamma_0[P(x) - P(x_0) - P'(x_0)](x - x_0)\|$$
$$\leq \eta_0 + \frac{1}{2}B_0 K\|x - x_0\|^2 \leq \eta_0 + \frac{1}{2}B_0 K\,[N(2h_0)\eta_0]^2$$
$$=\leq \eta_0 \left[1 + \frac{1}{2}B_0 K \left(\frac{1 - \sqrt{1 - 2h_0}}{h_0}\right)^2 \eta_0\right] = \eta_0 N(h_0). \tag{9}$$

Thus the preliminary statement is proved. Determining the successive approximations beginning with x_0, we find $x_1' = F_0(x_0) = x_1$. For this approximation the conditions (6) and (7) are obviously valid. Then for the next approximation $x_2 = F_0(x_1')$, according to the preliminary statement used for $x = x_1'$, we find

$$\|x_2' - x^*\| \leq q\|x_2' - x^*\|.$$

Continuing in the same manner, we come to (5), which again implies (4).

D. A. Grave [18] proposed and studied a different modification of the Newton process for the case of equations with real variables: Namely, the modification in which the successive approximations for the root of the equation $f(x) = 0$ are determined by the formula

$$x_{n+1} = x_n - Q\frac{f(x_n)}{f'(x_n)}, \tag{10}$$

where Q is a number, $0 < Q < 1$. For abstract equations a similar formula relating successive approximations acquires the form

$$x_{n+1} = x_n - Q[P'(x_n)]^{-1}P(x_n). \tag{11}$$

Convergence of this process may be proved under the same conditions as in Theorem 1, namely, for $h_0 \leq 1/2$.

Theorem 4. Under the assumptions of Theorem 1, the generalized process due to D. A. Grave (see (11)) converges to a solution of the equation.

Proof. Introduce the function $F_Q(x) = x - Q\Gamma_0 P(x)$. Then the approximation of x_1 in terms of x_0 may be expressed as:

$$x_1 = x_0 - Q\Gamma_0 P(x_0) = F_Q(x_0).$$

We first have

$$\|x_1 - x_0\| \leq Q\|\Gamma_0 P(x_0)\| \leq Q\eta_0. \tag{12}$$

Arguing as in the proof of Theorem 1, we find

$$\|F_Q(x_1) - F_Q(x_0)\| \le \|F'_Q(x_0)(x_1 - x_0)\| + \frac{1}{2}\sup\|F''_Q(\bar{x})\|\,\|x_1 - x_0\|^2$$

$$\le \|I - QI\|\,\|x_1 - x_0\| + \frac{1}{2}QB_0K\|x_1 - x_0\|^2$$

$$= (1-Q)Q\eta_0 + \frac{1}{2}QB_0K(Q\eta_0)^2, \quad \bar{x} = x_0 + \theta(x_1 - x_0),$$

$$\|\Gamma_0 P(x_1)\| = \frac{1}{Q}\|F_Q(x_1) - x_1\| = \frac{1}{Q}\|F_Q(x_1) - F_Q(x_0)\|$$

$$\le (1-Q)\eta_0 + \frac{1}{2}Q^2h_0\eta_0.$$

Further,

$$\|\Gamma_0[P'(x_0) - P'(x_1)]\| < K(\eta_0)B_0 = Qh_0 < 1,$$

$$\Gamma_1 = [P'(x_1)]^{-1} = [I - \Gamma_0(P'(x_1))]^{-1} = H\Gamma_0$$

$$\|H\| \le \frac{1}{1-Qh_0}, \qquad \|\Gamma_1\| \le \frac{B_0}{1-Qh_0} = B_1.$$

Finally,

$$\left.\begin{aligned} \|\Gamma_1 P(x_1)\| = \|H\Gamma_0 P(x_1)\| &\le \frac{1}{1-Qh_0}[1 - Q + \tfrac{1}{2}Q^2h_0]\eta_0 = \eta_1, \\ h_1 = B_1K\eta_1 &= \frac{B_0K}{(1-Qh_0)^2}[1 - Q + \tfrac{1}{2}Q^2h_0]\eta_0 \\ &= h_0\frac{1 - Q + \frac{1}{2}Q^2h_0}{(1-Qh_0)^2} = \frac{1}{2} - \frac{\frac{1}{2} - h_0}{(1-Qh_0)^2}. \end{aligned}\right\} \tag{13}$$

This shows that since $h_0 \le 1/2$, we also have $h_1 \le 1/2$, i.e., the process of successive approximations can proceed. To establish its convergence and estimate its rate, we confine ourselves to the least advantageous case when $h_0 = 1/2$. Setting $h_0 = 1/2$ in (13), we find

$$\eta_1 = \frac{1}{1 - Q/2}(1 - Q + \frac{1}{4}Q^2)\eta_0 = (1 - Q/2)\eta_0.$$

Therefore, continuing the process, we get

$$\eta_n \le (1 - Q/2)^n\eta_0$$

and since we have (12),

$$\|x_{n+1} - x_n\| \le Q\eta_n;$$

it is clear that the following limit exists

$$\lim_{n\to\infty} x_n = x^*$$

and

$$\|x_n - x^*\| \le Q(\eta_n + \eta_{n+1} + \cdots) \le 2(1 - Q/2)^n \eta_0.$$

Thus even in this case (the worst one) the rate of convergence has the order of a geometric progression.

6. The Newton Method for Systems of Algebraic Equations

The Newton method for solving a system of m algebraic equations in m unknowns

$$\eta^{(i)} = f_i(\xi^{(1)}, \ldots, \xi^{(m)}) = 0, \quad i = 1, \ldots, m \tag{1}$$

is a natural generalization of the same method for one equation. Successive approximations for the root — the first approximation $(\xi_1^{(1)}, \ldots, \xi_1^{(m)})$ obtained from the initial one $(\xi_0^{(1)}, \ldots, \xi_0^{(m)})$ — are found via the system of equations

$$\sum_{k=1}^{m} \left(\frac{\partial f_i}{\partial \xi^{(k)}}\right)_0 (\xi_1^{(k)} - \xi_0^{(k)}) + f_i(\xi_0^{(1)}, \ldots, \xi_0^{(m)}) = 0, \quad i = 1, \ldots, m.$$

In this form the method is indicated in a work of Runge. A certain attempt to give conditions for convergence of the process was undertaken by Willer [4] for two equations using derivatives of the third order. In the general case of n equations, certain sufficient conditions using only derivatives of the first and the second orders were given by N. P. Stenin ([5], see also [19]). In the case of two equations with two unknowns, the most precise conditions were given by A. Ostrowski [6]. This theorem is presented as Corollary 1 to Theorem 5.

Here we apply general theorems on the convergence of the Newton process to systems of algebraic equations. The given system may be treated as one equation

$$y = P(x) = 0, \tag{2}$$

where P is an operator mapping an m-dimensional space into itself. In particular, applying the metric from Example 1 of Section 1, we obtain the following theorem.

Theorem 5. If we have the conditions

1) $|f_i(\xi_0^{(1)}, \ldots, \xi_0^{(m)})| \le \overline{\eta}, \quad 1 = 1, \ldots, m,$ (3)

2) the matrix $\left\|\left(\frac{\partial f_i}{\partial \xi^{(k)}}\right)_0\right\|$ has a non-zero determinant Δ and if A_{ik} denotes the algebraic complement to its elements we have the condition

$$\frac{1}{|\Delta|}\max_i \sum_{k=1}^{m} |A_{ik}| \leq B, \tag{4}$$

3) $\left|\frac{\partial^2 f_i}{\partial \xi^{(j)} \partial \xi^{(k)}}\right| \leq L$ in the domain under consideration, (5)

4) $h = B^2 \overline{\eta} L m^2 \leq 1/2,$ (6)

then the given algebraic system has a solution which may be obtained by the Newton process.

Proof. Indeed, let us check that the stated conditions guarantee conditions 1–4 of Theorem 1 (concerning condition 1, see Remark 1).

For the given case, the operator $P'(x_0)$ is determined by the matrix $\left\|\left(\frac{\partial f_i}{\partial \xi^{(k)}}\right)_0\right\|$ or the transformation (compare (1)) in Section 3):

$$\Delta\eta^{(i)} = \sum_{k=1}^{m}\left(\frac{\partial f_i}{\partial \xi^{(k)}}\right)_0, \qquad i = 1, \ldots, m.$$

The inverse operator $\Gamma_0 = [P'(x)]^{-1}$ is determined by the inverse matrix $\|A_{ik}/\Delta\|_{i,k=1,\ldots,m}$, its norm, according to (4) of Section 1, may be estimated as follows:

$$\|\Gamma_0\| \leq \max_i \sum_k \frac{|A_{ik}|}{|\Delta|} \leq B,$$

i.e., condition 1 holds.

Further, we have

$$\|P(x_0)\| = \max_i |f_i(\xi_0^{(1)}, \xi_0^{(2)}, \ldots, \xi_0^{(1)})| \leq \overline{\eta},$$

so that condition 2′ also holds (cf. (40) in Section 4).

Further still, according to (12) of Section 3 and (4) of Section 2, we have the following estimate for $\|P''(x)\|$

$$\|P''(x)\| \leq \max_k \sum_{i,j=1}^{m}\left|\frac{\partial^2 f_k}{\partial \xi^{(i)} \partial \xi^{(j)}}\right| \leq m^2 L\,(=K),$$

i.e., condition 3 also holds.

Finally, comparing the estimates obtained, we can check that condition 4 also holds. Therefore on the basis of Theorem 1 we conclude that the Newton process converges. In the same way we can state for the given case the other conclusions of Theorems 1–4.

Corollary 1. Consider the case of a system of two equations in two unknowns. Here the statement of the conditions may be somewhat simplified; namely, let us denote the maximum of absolute values by l

$$\left|\frac{\partial f_i}{\partial \xi^{(k)}}\right| \leq l;$$

then, taking into consideration the fact that in the given case the determinant Δ is of second order and so its minors are its own elements, we have $|A_{ik}| \leq l$. Therefore, we can set [2]

$$\max_i \sum_{k=1}^{2} \frac{|A_{ik}|}{|\Delta|} = \max_i \sum_{k=1}^{2} \frac{\left|\left(\frac{\partial f_i}{\partial \xi^{(k)}}\right)_0\right|}{|\Delta|} \leq \frac{2l}{|\Delta|} = B.$$

In accordance with this, condition (6) will acquire the form

$$h_0 = \frac{4l^2}{|\Delta|^2}\overline{\eta} \cdot 2^2 \leq \frac{1}{2}$$

or

$$32 l^2 L \overline{\eta} \leq |\Delta|^2.$$

In this form the theorem was obtained by A. Ostrowski [6]. It is curious that in applying a similar argumentation in this particular case his proof turns out to be more complicated than that of the more general Theorem 1 and apparently this complexity compels him to confine his research to the case of two equations. Also in his paper he draws no conclusions about the domain of uniqueness.

Remark. Note that we could have applied condition 2 as well. It has the form

$$\|x_1 - x_0\| = \|\Gamma_0 P(x_0)\| \leq \eta.$$

In the given case this will be an estimate of the first increments

$$\|x_1 - x_0\| = \max_i |\xi_1^{(i)} - \xi_0^{(i)}| \leq \eta.$$

When we introduce such an expression, condition 4 must be replaced by

$$h_0 = BLm^2\eta \leq 1/2.$$

[2] As a more exact value of B we can take

$$B = \max\left\{\left|\left(\frac{\partial f_1}{\partial \xi^{(1)}}\right)_0\right| + \left|\left(\frac{\partial f_1}{\partial \xi^{(2)}}\right)_0\right|, \left|\left(\frac{\partial f_2}{\partial \xi^{(1)}}\right)_0\right| + \left|\left(\frac{\partial f_2}{\partial \xi^{(2)}}\right)_0\right|\right\}.$$

Another theorem about systems of algebraic equations will be obtained if we use the metric of the space R^m.

Theorem 5a. Assume the following conditions:

1) the matrix $\|(\partial f_i/\partial \xi^{(k)})_0\|$ has the inverse matrix $\|A_{ik}/\Delta\|$ and

$$\frac{1}{|\Delta|}\left(\sum_{i=1}^m \sum_{j=1}^m A_{ik}^2\right)^{1/2} \leq B$$

(for B we may take any estimate of the norm of the inverse matrix),

2) $\sum_{i=1}^m |\xi_1^{(i)} - \xi_0^{(i)}| \leq \eta^2$,

3) $\sum_{i,j,k=1}^m \left(\frac{\partial^2 f_i}{\partial \xi^{(k)} \partial \xi^{(j)}}\right)^2 \leq K^2$

(we can take $K = m\sqrt{m}L$, where $\left|\frac{\partial^2 f_i}{\partial \xi^{(k)} \partial \xi^{(j)}}\right| \leq L$),

4) $\eta_0 = BK\eta \leq 1/2$.

Then the Newton process converges. The solution of the system $x^* = (\xi^{(1)*}, \dots, \xi^{(m)*})$ lies in the domain

$$\|x^* - x_0\| = \left[\sum_{i=1}^m |\xi^{(i)*} - \xi_0^{(i)}|\right]^{1/2} \leq \frac{1-\sqrt{1-2h_0}}{h_0}\eta.$$

Proof. The proof is carried out as in Theorem 5 by reducing it to Theorem 1. Only in the given case we must use the expressions for the norm of operators mapping R^m into R^m (cf. (7) of Section 1 and (5) of Section 2).

As an example, consider the following system [20, 6]

$$f \equiv 2x^3 - y^2 - 1 = 0, \qquad g \equiv 2xy^3 - y - 4 = 0.$$

For the first approximation take the point T_0 with $x_0 = 1.2$, $y_0 = 1.7$. Then

$$f(T_0) = -0.434, \qquad g(T_0) = 0.1956\,.$$

The system for finding the first increments $\Delta x_1 = x_1 - x_0$ and $\Delta y_1 = y_1 - y_0 = 0.0390.$ is of the form

$$f_x'(T_0)\Delta x_1 - f_y'(T_0)\Delta y_1 + f(T_0) = 0, \qquad 8.64\Delta x_1 - 3.4\Delta y_1 + 0.434 = 0,$$

$$g_x'(T_0)\Delta x_1 - g_y'(T_0)\Delta y_1 + g(T_0) = 0, \qquad 4.913\Delta x_1 - 9.404\Delta y_1 + 0.1956 = 0,$$

so that their values will be

$$\Delta x_1 = x_1 - x_0 = 0.0349, \qquad \Delta y_1 = y_1 - y_0 = 0.0390.$$

Here the determinant of the system is $\Delta = 97.95\,.$

First let us estimate h_0 according to Ostrowski. Let us take

$$l = \max_{i,k} \left| \left(\frac{\partial f_i}{\partial \xi^{(k)}} \right)_0 \right| = 9.404, \qquad \overline{\eta} = \max\{|f(T_0)|, |g(T_0)|\} = 0.434 .$$

In order to find L, let us compute the second derivatives

$$f''_{xx} = 12x, \quad f''_{xy} = 0, \quad f''_{yy} = -2, \quad g''_{xx} = 0, \quad g''_{xy} = 3y^2, \quad g''_{yy} = 6xy,$$

and estimate them in the rectangles $0 \leq x \leq 1.3$, $0 \leq y \leq 1.8$ within which the sequence of approximations must lie; we obtain $L = 12 \cdot 1.3 = 15.6$. Therefore, the value of h_0 will be

$$h_0 = \frac{16 l^2}{\Delta^2} = \frac{16 \cdot 9.404^2 \cdot 0.434 \cdot 15.6}{97.95^2} = 0.998 > 0.5 .$$

Thus on the basis of Ostrowski theorem we cannot conclude that the convergence of the process occurs.

Let us try to apply Theorem 5a.

To find B, consider the matrix

$$\Gamma_0 = \begin{Vmatrix} (f'_x)_0 & (f'_y)_0 \\ (g'_x)_0 & (g'_y)_0 \end{Vmatrix}^{-1} = \begin{Vmatrix} 8.64 & -3.4 \\ 4.913 & 9.404 \end{Vmatrix}^{-1} . \tag{7}$$

The norm of the linear transformation in R^2 effected by matrix (7) is according to Example 2 of Section 1.

$$||\Gamma_0|| = \sqrt{\Lambda_{\max}},$$

where $\Lambda_{\max}$ is the greatest eigenvalue of the matrix $\Gamma_0 \Gamma_0^*$, i.e., the greatest root of the equation

$$\Lambda^2 - \frac{8.64^2 + 3.4^2 + 4.913^2 + 9.404^2}{\Delta^2}\Lambda + \frac{1}{\Delta^2} = 0,$$

i.e., the equation

$$\Lambda^2 - 0.02072\Lambda + 0.00010422 = 0$$

whence

$$\Lambda_{\max} = 0.01036 + \sqrt{0.00010733 - 0.00010422} = 0.0121$$

and

$$B = ||\Gamma_0|| = \sqrt{\Lambda_{\max}} = 0.11 .$$

As η we take the magnitude

$$\eta = ||T_1 - T_0|| = \sqrt{\Delta x_1^2 + \Delta y_1^2} = 0.0524 .$$

Finally estimate K (in the same rectangle) according to Example 2 of Section 2

$$K \leq (15.6^2 + 2^2 + 2 \cdot 9.72^2 + 14.04^2)^{1/2} = \sqrt{631.72} = 25.14 .$$

Now find h_0

$$h_0 = BK\eta = 0.11 \cdot 25.14 \cdot 0.0524 = 0.15 < 0.5 .$$

This shows that the process converges quickly.

7. The Newton Method for Non-linear Integral Equations

We consider non-linear integral equations of the form

$$x(s) = \int_0^1 K(s,t,x(t))\,dt, \tag{1}$$

where K is a continuous function in its arguments. If we introduce the operator

$$y = P(x), \qquad y(s) = x(s) - \int_0^1 K(s,t,x(t))\,dt, \tag{2}$$

the equation will acquire the form considered in Section 4.

The Newton process for this equation is constructed as follows. Choose an initial approximation $x_0(s)$. Then the next approximation, the function $x_1(t)$, must be determined from the linear integral equation

$$x_1(s) - x_0(s) - \int_0^1 K'_x(s,t,x_0(t))(x_1(t) - x_0(t))\,dt = \varepsilon_0(s), \tag{3}$$

where

$$\varepsilon_0(s) = \int_0^1 K(s,t,x_0(t))\,dt - x_0(s).$$

This equation may be obtained directly or from the general formula relating x_1 and x_0 (see (2) in Section 4): $P'(x_0)(x_1 - x_0) = -P(x_0)$ if we also take into consideration the meaning of $P'(x_0)$ in the given case (compare (16) in Section 3).

Thus in order to find successive approximations, it is necessary to solve a linear integral equation at each step.

The convergence of this process for the given case was studied directly at my suggestion by D. M. Zagadsky in [7, 8]. However, the condition of convergence obtained there ($h_0 \le 1/10$) is stricter than the ones that we get by using the general theorems in Section 4. They imply Theorem 6 below if we consider the operator (2) as an operator in the space C.

Theorem 6. If we have the following conditions

1) for the initial value $x_0(s)$, the kernel

$$K'_x(s,t,x_0(t)) = K(s,t). \tag{4}$$

has the resolvent $G(s,t)$ and

$$\int_0^1 |G(s,t)|\,dt \le B, \qquad 0 \le s \le 1, \tag{5}$$

2) $|\varepsilon_0(s)| = |x_0(s) - \int_0^1 K(s,t,x_0(t))\,dt| \le \overline{\eta}$, (6)

3) $|K''_{u^2}(s,t,u)| \leq K$ (7) in the domain determined by the equation (9) (or (10)),

4) $h = (B+1)^2\eta K \leq 1/2,$ (8)

hold, then the Newton process for the integral equation (1) with initial value $x_0(s)$ converges to a solution of the equation which exists and is contained in the domain

$$|x^*(s) - x_0(s)| \leq N(h)(B+1)\overline{\eta} \tag{9}$$

and is unique in the domain

$$|x(s) - x_0(s)| \leq L(h)(B+1)\overline{\eta}. \tag{10}$$

Proof. For the proof it suffices to apply Theorems 1 and 2 (with condition 2′ instead of 2), choosing the space C as the spaces X and Y. Here for the estimate of the operator $[P'(x_0)]^{-1}$ we must take in consideration the fact that in the given case the operator $P'(x)_0$ is the one on the left-hand side of equation (3) and the operator inverse to Γ_0 gives the expression for the solution involving the constant term and can be written by using the resolvents

$$x = \Gamma_0(y), \qquad x(s) = y(s) - \int_0^1 G(s,t)y(t)\,dt.$$

Hence, Γ_0 is easily estimated according to (11) of Section 1 or directly

$$|x(s)| \leq |y(s)| + \int_0^1 |G(s,t)y(t)|\,dt \leq \left(1 + \max_s \int_0^1 |G(s,t)|\,dt\right)\|y\|.$$

Therefore, $\|\Gamma_0\| \leq 1 + B$ and it follows that condition 1 is satisfied with $B_0 = 1 + B$.

Condition 2′ is obviously valid too because ofcondition 2 of this theorem, since $\varepsilon_0(s)$ coincides with $P(x_0)$. Finally, if we have in mind the fact that the second derivative of the identity operator is zero, it follows according to Example 3 of Section 3 that $P''(x)$ is a bilinear operator with kernel $K''_{x^2}(s,t,x_0(t))$ and, therefore, according to (9) of Section 2 and condition 3 of the present theorem we have $\|P''(x)\| \leq K$.

Finally, condition 4 is equivalent to the form of condition 4 indicated in Example 1 of Section 4. Thus all the conditions of Theorems 1 and 2 are satisfied; therefore, if we apply these theorems, we come to the statement of the theorem under consideration.

As we mentioned above, a theorem similar to Theorem 6 was obtained by D. M. Zagadsky. It differs from the one proved here only in that the number 1/2 is replaced by 1/10 in condition 4.

Designing the space L^2 as the spaces X and Y, we obtain the following theorem.

Theorem 7. Suppose we have the following conditions

1) $\int_0^1 [x_0(s) - \int_0^1 K(s,t,x_0(t))\,dt]^2 ds \le \eta^2$.

2) We have the inequality

$$\frac{|\lambda_n|}{|1-\lambda_n|} \le B,$$

where λ_n are the eigenvalues of the kernel

$$K'_u(s,t,x_0(t)) = K(s,t)$$

if the latter is symmetric; if the kernel is not symmetric, then we must require the inequality

$$\frac{|\Lambda_n|}{|1-\Lambda_n|} \le B^2,$$

where Λ_n are the eigenvalues of the symmetric kernel constructed from the given one according to formula (15) from Section 1.

3) $|K''_{u^2}(s,t,u)| \le K$ for all finite values u.

4) $B^2 K\eta \le 1/2$.

Then equation (1) has a solution which may be found by the Newton process.

The proof of Theorem 7 is obtained directly from Theorem 1, since it is possible to verify without difficulty that all the assumptions of the latter are satisfied.

Example. For an example of the application of Theorems 6 and 7 consider the integral equation (compare [7])

$$x(s) = 1 - 0.4854\,s + s^2 + \int_0^1 s\,t \arctan x(t)\,dt,$$

whose exact solution is $x^*(s) = 1 + s^2$.

Take the initial approximation $x_0(t) \equiv 3/2$ and apply the Theorem 6. The kernel is

$$K(s,t) = K'_u(s,t,x_0(t)) = 4/(13)st;$$

hence, its resolvent $G(s,t)$ must be of the form $G(s,t) = Cst$. The constant C can be determined from the integral equation for the resolvent

$$G(s,t) = K(s,t) + \int_0^1 K(s,u)G(u,t)\,du$$

which gives

$$C = \frac{4}{13} + \frac{4}{39}C, \qquad C = \frac{12}{35}.$$

Now we find B

$$B = \max_s \int_0^1 |G(s,t)|\,dt = \frac{6}{35}.$$

Determine $\overline{\eta}$. Since

$$\varepsilon_0(s) = 1 - 0.4854\, s + s^2 + s \int_0^1 t \left(\arctan \frac{3}{2}\right) dt - \frac{3}{2} = s^2 + 0.006\, s - 0.5,$$

we have

$$\overline{\eta} = \max_s |\varepsilon_0(s)| = \varepsilon_0(1) = 0.506.$$

Finally,

$$K = \max_{0 \le s,t \le 1} |K''_{u^2}(s,t,u)| = \max \left| \frac{2stu}{(1+u^2)^2} \right| = \frac{2\sqrt{1/3}}{(1+1/3)^2} = \frac{3\sqrt{3}}{8}.$$

Computing h gives

$$h = (B+1)^2 K \overline{\eta} = \left(\frac{41}{35}\right)^2 \cdot 0.506 \cdot \frac{3\sqrt{3}}{8} = 0.451 < 0.5\,.$$

Thus the convergence of the process is guaranteed.[3]

Let us find the next approximation. According to the general theory, the increment can be determined from the equation

$$\Delta x(s) = \int_0^1 \frac{st}{1+[x_0]^2} \Delta x(t)\, dt + \varepsilon(s),$$

i.e.,

$$\Delta x(s) = \frac{4}{13} s \int_0^1 t \Delta x(t)\, dt + s^2 + 0.006\, s - 0.5\,.$$

Therefore,

$$\Delta x(s) = s^2 + 0.006\, s - 0.5 + \int_0^1 G(s,t)(t^2 + 0.006\, t - 0.5)\, dt$$

$$= s^2 + 0.006\, s - 0.5 + \int_0^1 \frac{12}{35} st(t^2 + 0.006\, t - 0.5)\, dt = s^2 + 0.006\, s - 0.5\,.$$

Thus, the first approximation is

$$x_1(s) = x_0(s) + \Delta x(s) = s^2 + 0.0067\, s + 1,$$

which differs from the exact solution less than by 0.001.

Note that in the cases when the kernel $K'_x(s,t,x_0(t))$ is especially simple, the next approximation can be found more conveniently by using the modified process. Actually for the given example this is not essential since it is purely illustrative in character. The exact solution can easily be found by reducing it to an algebraic equation.

In the solution of many problems related to non-linear differential equations it is also possible to apply the Newton method. Doing this, we can study the process directly by applying Theorem 1 or first reduce the problem to a non-linear integral or integral-differential equation.

[3]The application of Theorem 7 yields a considerably lesser value for h, namely, $h = 0.141$.

8. The Newton Method for Eigenvalue Problems

We shall consider the problem of finding eigenvalues in the case when $X = H$ is a Hilbert space (for example L^2 or R^n) while the operator A is self-adjoint and compact (with isolated eigenvalues). The problem consists in finding pairs (x, λ) satisfying the conditions

$$Ax - \lambda x = 0,$$

where $x \neq 0$. Such a value x is called an eigenvector and the λ an eigenvalue. If it is possible to find a solution $x \neq 0$, then it is also possible to find a solution x such that[4] $\|x\|^2 = (x, x) = 1$. As a rule, under this supplementary condition as a rule the problem ceases to be undetermined and we can state it in the following form:

$$\left.\begin{aligned} Ax - \lambda x &= 0, \\ [(x, x) - 1]/2 &= 0. \end{aligned}\right\} \tag{1}$$

If we introduce the operator

$$P\begin{pmatrix} \lambda \\ x \end{pmatrix} = \begin{pmatrix} [(x, x) - 1]/2 \\ Ax - \lambda x \end{pmatrix}, \tag{2}$$

which maps the space X' of pairs (x, λ), where $x \in H$ and λ is a real number, into the same space, then the system of equations (1) may be replaced by one equation

$$P\begin{pmatrix} \lambda \\ x \end{pmatrix} = 0. \tag{3}$$

Let us consider the application of the Newton method just to this equation. As a preliminary step, let us find the derivatives of the operator $P\binom{\lambda}{x}$. We have

$$P\begin{pmatrix} \lambda + \Delta\lambda \\ x + \Delta x \end{pmatrix} = \begin{pmatrix} (x, x)/2 + (x, \Delta\lambda) + (\Delta x, \Delta x)/2 - 1/2 \\ Ax + A\Delta x - x\Delta\lambda - \lambda\Delta x - \lambda x - \Delta\lambda\Delta x \end{pmatrix},$$

whence without losing magnitudes of the first order, we get

$$dP\begin{pmatrix} \lambda \\ x \end{pmatrix} = P'\begin{pmatrix} \lambda \\ x \end{pmatrix} = \begin{pmatrix} \Delta\lambda \\ \Delta x \end{pmatrix} = \begin{pmatrix} (x, \Delta x) \\ (A - \lambda I)\Delta x - x\Delta\lambda \end{pmatrix}. \tag{4}$$

[4] In general (x, y) denotes the scalar product of the vectors x and y. In the case when $X = L^2$, we have

$$(x, y) = \int_0^1 x(t)y(t)\,dt.$$

Similarly for the second derivative written as a bilinear operator, we obtain

$$P''\begin{pmatrix}\lambda\\x\end{pmatrix}\left[\begin{pmatrix}\Delta\lambda\\\Delta x\end{pmatrix},\begin{pmatrix}\Delta'\lambda\\\Delta' x\end{pmatrix}\right]=\left[P'\begin{pmatrix}\lambda+\Delta'\lambda\\x+\Delta' x\end{pmatrix}-P'\begin{pmatrix}\lambda\\x\end{pmatrix}\right]=\begin{pmatrix}\Delta\lambda\\\Delta x\end{pmatrix}$$

$$=\begin{pmatrix}(\Delta' x,\Delta x)\\-\Delta'\lambda\Delta x-\Delta' x\Delta\lambda\end{pmatrix}. \qquad (5)$$

For the initial values of λ and x let us choose the following. We assume that x is normalized or at least that $(x,x)\approx 1$. Then after its substitution into equation (3), we get

$$P\begin{pmatrix}\lambda\\x\end{pmatrix}=-\begin{pmatrix}t\\y\end{pmatrix},\qquad \begin{array}{l}y=-(Ax-\lambda x),\\ t=-[(x,x)-1]/2.\end{array} \qquad (6)$$

The equation for finding the increment $\Delta x = x_{1-x_0}$ by the Newton method in the general case is of the form: $P'(x_0)(x_1-x_0)+P(x_0)=0$ (see (2) in Section 4) and hence in the case under study it can be written as

$$P'\begin{pmatrix}\lambda\\x\end{pmatrix}\begin{pmatrix}\Delta\lambda\\\Delta x\end{pmatrix}=-P\begin{pmatrix}\lambda\\x\end{pmatrix}=\begin{pmatrix}t\\y\end{pmatrix}$$

or keepng in mind the expression for $P'\binom{\lambda}{x}\binom{\Delta\lambda}{\Delta x}$,

$$\left.\begin{array}{l}(A-\lambda I)\Delta x-x\Delta\lambda=y,\\ (x,\Delta x)=t.\end{array}\right\} \qquad (7)$$

These linear equations can be used to determine the succeeding increment. Here in the case of the Newton method the values of x and λ on the left-hand side are successively replaced by their updated values, but in the case of the modified process (Section 5) the changes are introduced only into the constant terms of system(7).

In order to study the convergence of the Newton process, it is necessary first of all to constitute the inverse operator

$$\Gamma=\left[P'\begin{pmatrix}\lambda\\x\end{pmatrix}\right]^{-1} \qquad (8)$$

that can be found by solving system (7) in general form. Namely, for arbitrary y and t it turns out that

$$\begin{pmatrix}\Delta\lambda\\\Delta x\end{pmatrix}=\Gamma\begin{pmatrix}t\\y\end{pmatrix}. \qquad (9)$$

If such an expression is found, we have an opportunity to estimate it directly

$$\left\| \left[P' \begin{pmatrix} \lambda \\ x \end{pmatrix} \right]^{-1} \right\| = \|\Gamma\|.$$

Then $\|\Gamma\|$ is estimated as the norm of the operator mapping the space of pairs $\binom{\lambda}{x}$ into itself and the norm of the element in this space is taken to be

$$\left\| \begin{pmatrix} \lambda \\ x \end{pmatrix} \right\| = \sqrt{\|x\|^2 + |\lambda|^2} = \sqrt{(x, x) + \lambda^2}. \tag{10}$$

Further, to apply the theorem it is necessary to estimate the norm

$$\left\| P'' \begin{pmatrix} \lambda \\ x \end{pmatrix} \right\|.$$

This may be done in general form on the basis of the expression (5) for P''. We have

$$\left\| P'' \begin{pmatrix} \lambda \\ x \end{pmatrix} \begin{pmatrix} \Delta\lambda \\ \Delta x \end{pmatrix} \begin{pmatrix} \Delta'\lambda \\ \Delta' x \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} (\Delta' x, \Delta x) \\ -\Delta'\lambda\Delta x - \Delta' x \Delta\lambda \end{pmatrix} \right\|^2$$

$$= (\Delta' x, \Delta x)^2 + (\Delta'\lambda\Delta x + \Delta' x\Delta\lambda, \Delta'\lambda\Delta x + \Delta' x\Delta\lambda)$$

$$\leq \|\Delta' x\|^2 \|\Delta x\|^2 + |\Delta'\lambda|^2 \|\Delta x\|^2 + |\Delta\lambda|^2 \|\Delta' x\|^2 + 2\|\Delta x\| \cdot \|\Delta' x\|$$

$$\leq \|\Delta' x\|^2 \|\Delta x\|^2 + |\Delta'\lambda|^2 \|\Delta x\|^2 + |\Delta\lambda|^2 \|\Delta' x\|^2$$

$$+2 \frac{|\Delta'\lambda|^2 + |\Delta\lambda|^2}{2} \frac{\|\Delta' x\|^2 + \|\Delta x\|^2}{2}$$

$$\leq \frac{3}{2} (\|\Delta x\|^2 + |\Delta\lambda|^2)(\|\Delta' x\|^2 + |\Delta'\lambda|^2) = \frac{3}{2} \left\| \begin{pmatrix} \Delta\lambda \\ \Delta x \end{pmatrix} \right\|^2 \left\| \begin{pmatrix} \Delta'\lambda \\ \Delta' x \end{pmatrix} \right\|^2.$$

This implies

$$\left\| P'' \begin{pmatrix} \lambda \\ x \end{pmatrix} \right\| \leq \sqrt{\frac{3}{2}}. \tag{11}$$

Thus in accordance with Theorem 3 the condition of convergence of the Newton process and the modified process may be written in the form

$$h_0 = \sqrt{\frac{3}{2}} \|\Gamma\| \left\| \begin{pmatrix} \Delta\lambda \\ \Delta x \end{pmatrix} \right\| < \frac{1}{2}, \tag{12}$$

where, as we said before, $\|\Gamma\|$ must be estimated directly from its expression (9); Δx and $\Delta\lambda$ are the values of the increments when we pass from the initial values to the first approximation, i.e., solutions of system (7), while the norm is defined according to (10).

It is necessary to note that if condition (12) holds, we may claim the existence of an eigenpair and the convergence of the Newton process to it. However, we cannot guarantee that this is a pair which satisfies certain specific conditions, for example, that this pair corresponds to the smallest eigenvalue.

The given process, in particular, may be applied to finding the eigenvalues of a matrix operator which, as is known, is equivalent to solving of the so-called secular equation.

In this case the application of the process requires solving a system of $n+1$ linear algebraic equation in each step, since this is precisely the form that the system (7) will have in this case. Here, when applying the modified process, the coefficients of this system will be the same in each step of the process. This system may also be somewhat simplified if the second of the equations (1) can be replaced by the simpler one $(x, U) = 1$, where U is a constant vector, for example $U = (1, 0, \ldots, 0)$.

Let us prove that the condition of convergence of the Newton process (12) will be necessarily satisfied if the operator A possesses isolated simple eigenvalues and the initial approximation is close enough to it. Actually, for a certain simplification of the expression we shall carry out our arguments when the spectrum of the operator A is discrete.

Suppose λ_0 and x_0 are the given eigenvalue and element. The other eigenvalues and elements will be denoted by λ_n and x_n $(n = 1, 2, \ldots)$, among the λ_n some may be equal but the elements x_n are all orthogonal to each other. Suppose λ_1 is the value which is nearest to λ_0. Further add elements $x_{-1}, x_{-2}, \ldots$ so that together with the elements $x_0, x_1, x_2, \ldots$ they constitute a complete orthogonal system. Assume that $\lambda_0 = 0$ $(n = -1, -2, \ldots)$. Let us try to estimate the norm of the operator $\Gamma = \left[P'\binom{\lambda}{x}\right]^{-1}$. This operator is an expression of the solution of system (7) (compare (9)) which in the given case is of the form

$$(A - \Delta\lambda I)\Delta x - x_0 \Delta\lambda = y, \qquad (x_0, \Delta x) = t. \tag{7a}$$

Suppose that

$$y = \sum_{-\infty}^{+\infty} \alpha_n x_n.$$

We shall look for the solution in the form $\Delta x = \sum \beta_n x_n$. Then the system (7a), if we keep in mind that $Ax_n = \lambda_n x_n$, is of the form

$$\sum_{-\infty}^{+\infty} \beta_n(\lambda_n - \lambda_0)x_n - x_0 \Delta\lambda = \sum \alpha_n x_n, \qquad \beta_0 = t.$$

Hence, setting the appropriate coefficients equal to each other, we get

$$\beta_0 = t, \qquad \beta_n = \frac{1}{\lambda_n - \lambda_0}\alpha_n, \qquad n = 1, 2, \ldots,$$

$$\beta_n = \frac{-1}{\lambda_0}\alpha_n, \qquad n = -1, -2, \ldots;$$

and, therefore,

$$\left\| \Gamma \begin{pmatrix} t \\ y \end{pmatrix} \right\|^2 = \left\| \begin{matrix} \Delta x \\ \Delta\lambda \end{matrix} \right\|^2 = \|\Delta x\|^2 + \|\Delta\lambda\|^2 = \alpha_0^2 + \sum \beta_n^2$$

$$= \alpha_0^2 + t^2 + \sum_{n=1}^{\infty} \frac{1}{(\lambda - \lambda_0)^2}\alpha_n^2 \frac{1}{\lambda_0^2} \sum_{n=-1}^{-\infty} \alpha_n^2$$

$$\leq t^2 + \max\left\{1, \frac{1}{|\lambda_0|}^2, \frac{1}{|\lambda_1 - \lambda_0|^2}\right\} \sum_{-\infty}^{+\infty} \alpha_k^2$$

$$\leq \max\left\{1, \frac{1}{|\lambda_0|^2}, \frac{1}{|\lambda_1 - \lambda_0|^2}\right\} \left\| \begin{matrix} y \\ t \end{matrix} \right\|^2.$$

This clearly shows that

$$\|\Gamma\| = \left\| \left[P' \begin{pmatrix} \lambda \\ x \end{pmatrix} \right]^{-1} \right\| \leq \max\left\{1, \frac{1}{|\lambda_0|}, \frac{1}{|\lambda_1 - \lambda_0|}\right\}.$$

If the initial values x and λ are close to x_0 and λ_0, then $\|\Gamma\|$ will be bounded while η will be small, because Δx and $\Delta\lambda$ are small so that condition (12) will certainly be satisfied.

The bounds for the domain of convergence may be determined differently by using Remark 5 in Section 4. Here the corresponding inequality ((45) in Section 4) will acquire the form

$$\left\| \begin{pmatrix} \lambda \\ x \end{pmatrix} - \begin{pmatrix} \lambda_0 \\ x_0 \end{pmatrix} \right\| = \sqrt{\|x - x_0\|^2 + (\lambda - \lambda_0)^2} \leq \frac{1}{4\|\Gamma\|\sqrt{3/2}},$$

i.e., the domain satisfies this inequality for the pair of values $\binom{\lambda}{x}$ chosen as the initial value and guaranteeing convergence to the eigenpair $\binom{\lambda_0}{x_0}$.

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