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CESARI CONVERGENCE OF THE GRADIENT METHOD OF APPROXIMATING SADDLE POINTS OF CONVEX-CONCAVE FUNCTIONS

UDC 517

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Let G and G_I be nonempty closed convex subsets of Hilbert spaces E and E_I, and let $F(x, l)$: $G \times G_I \longrightarrow R$ be a continuous function convex in $x \in G$ and concave in $l \in G_I$. Assume that the set of saddle points is nonempty. We pose the problem of an approximate construction of any point of this set. This is a well-known problem. In order to solve it, one most often employs various versions of the Arrow-Hurwicz method [1]. In the simplest of these versions, a sequence $(x_i, l_i) \in G \times G_I$ of successive approximations of a saddle point is constructed according to the rules

$$x_{i+1}=\pi_G(x_i-\rho_i F_x'(x_i, l_i)), \quad l_{i+1}=\pi_{G_I}(l_i+\rho_i F_l'(x_i, l_i)). \tag{1}$$

Here, $\pi_G(x)$ is the point of G closest to x, π_{G_I} is defined in a similar way, $F_x'(u, v)$ ($F_l'(u, v)$) is a functional supporting F as a convex function of x (concave function of l), and $\rho_i > 0$ are properly regulated shifts.

Unfortunately, the version indicated does not, generally speaking, converge for any rule of shift adjustment (a simple example: $F(x, l) = xl$, $-1 \leqslant x \leqslant 1$, $-1 \leqslant l \leqslant 1$). In order to guarantee the convergence, one must impose additional requirements of strict convexity-concavity [2], [3]. If these conditions do not hold, then one must modify the method by either regularizing F (perturbing it so that it becomes strictly convex-concave), or choosing distinct shifts in the x- and l-coordinates (even by an order of the rate of convergence to 0 as $i \longrightarrow \infty$).

The purpose of this work is to show that the methods indicated are, in essence, superfluous. The point is that, under sufficiently general assumptions, the trajectory (1) converges to the set of saddle points in Cesari's sense. Namely, let

$$(x^i, l^i)=\frac{1}{\sum_{j=1}^{i}\rho_j}\sum_{j=1}^{i}\rho_j(x_j, l_j) \tag{2}$$

be the trajectory of convex combinations of points of trajectory (1). It turns out that, under natural assumptions, the trajectory (x^i, l^i) is bounded and converges to the set of saddle points of the function F in the following sense. Let

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$$\bar{F}(x)=\sup_{l\in G_I} F(x,l), \qquad \underline{F}(l)=\inf_{x\in G} F(x,l).$$

Then

$$\bar{F}(x^i)\underset{i\to\infty}{\longrightarrow}\inf_x \bar{F}(x), \qquad \underline{F}(l^i)\underset{i\to\infty}{\longrightarrow}\sup_l \underline{F}(l)$$

(the convergence described is, so to say, convergence with respect to the functional; in the finite-dimensional case, it immediately implies the convergence of (x^i, l^i) to the set of saddle points of the function F). From the practical viewpoint, the convergence of the trajectory (2) fully replaces the convergence of (1).

The precise assertion consists in the following.

THEOREM. *Let G, G_I, and F satisfy all the conditions formulated at the beginning of the section and, moreover, let the values $\|F'_x(x, l)\|$, $\|F'_l(x, l)\|$ of the norms of supporting functionals in* (1) *be bounded on bounded subsets of $G \times G_I$ (in particular, F is Lipschitzian on such subsets). Let the shifts ρ_i in* (1) *satisfy the following conditions:*(1)

(A) *if G and G_I are bounded, then $\rho_i \underset{i\to\infty}{\longrightarrow} 0$, $\Sigma_i \rho_i = \infty$;*

(B) *if $G \times G_I$ is unbounded, then $\rho_i \underset{i\to\infty}{\longrightarrow} 0$, $\Sigma_i \rho_i = \infty$,*

$$\sum_i \rho_i^2 (\|F_x'(x_i, l_i)\|^2 + \|F_l'(x_i, l_i)\|^2) < \infty. \tag{3}$$

Then the trajectory (x^i, l^i) defined by (2) *is bounded, and*

(i) *if $G \times G_I$ is bounded, then the values*

$$\nu^i = \max\{\bar{F}(x) - \inf_x \bar{F}(x), \ \sup_l F(l) - F(l^i)\}$$

tend to 0 *as $i \longrightarrow \infty$ and*

$$\nu^i \leqslant \frac{d^2(G\times G_I)}{2\sum_{j=1}^{i}\rho_i} + \frac{\sum_{j=1}^{i}\rho_i^2}{2\sum_{j=1}^{i}\rho_i} a^2(F), \tag{4}$$

where $d(G \times G_I)$ is the diameter of $G \times G_I$ in the Hilbert space $E \times E_I$ and

$$a^2(F) = \sup_{x,l} (\|F_x'(x, l)\|^2 + \|F_l'(x, l)\|^2);$$

(ii) *if $G \times G_I$ is unbounded, then every limit point of the trajectory (x^i, l^i) in the weak topology of $E \times E_I$ is a saddle point of F.*

PROOF. By obvious symmetry, it is sufficient to verify that, in the case (i), the value of $\bar{F}(x^i) - \inf_x \bar{F}(x) \equiv \bar{F}(x^i) - \bar{F}(x^*)$ converges to 0 and satisfies the estimate (4) (here, x^* is the first component of the saddle point of F) and, in the case (ii), any weak limit point of the sequence $\{x^i\}$ is the first component of a saddle point of F.

In order to prove this, we fix an $l \in G_I$ and look at the changes of $V_l(x_j, l_j) \equiv$

(1) It will be shown that such a choice of shifts is possible in employing the method.

$(x_j - x^*)^2 + (l_j - l)^2$. Denoting for brevity $\xi_j = F_x'(x_j, l_j)$ and $\eta_j = F_l'(x_j, l_j)$, and taking into account the convexity of F in x, we have

$$(x_{j+1}-x^*)^2 \leqslant (x_j-\rho_j\xi_j-x^*)^2 = (x_j-x^*)^2+2\rho_j(\xi_j, x^*-x_j)+\rho_j^2\xi_j^2 \leqslant (x_j-x^*)^2$$
$$+2\rho_j(F(x^*, l_j)-F(x_j, l_j))+\rho_j^2\xi_j^2 \leqslant (x_j-x^*)^2+2\rho_j(\bar{F}(x^*)-F(x_j, l_j))+\rho_j^2\xi_j^2.$$

In the same way,

$$(l_{j+1}-l)^2 \leqslant 2\rho_j(F(x_j, l_j)-F(x_j, l))+\rho_j^2\eta_j^2+(l_j-l)^2.$$

Hence,

$$V_l(x_{j+1}, l_{j+1}) \leqslant V_l(x_j, l_j)+2\rho_j(\bar{F}(x^*)-F(x_j, l))+\rho_j^2(\xi_j^2+\eta_j^2). \tag{5}$$

In particular,

$$V_{\bar{l}}(x_{j+1}, l_{j+1}) \leqslant V_{\bar{l}}(x_j, l_j)+\rho_j^2(\xi_j^2+\eta_j^2) \tag{6}$$

($\bar{l}$ is the l-component of the saddle point of F).

The relations (6) and (3) imply at once that the trajectory (x_i, l_i) is bounded. Clearly, the condition (B) can easily be satisfied, e.g., by the rule

$$\rho_j = \frac{1}{j}\frac{1}{\sqrt{\xi_j^2+\eta_j^2+1}}.$$

With this choice of ρ_i, (3) holds and $\rho_j \underset{j\to\infty}{\longrightarrow} 0$. Therefore, the trajectory (x_j, l_j) is bounded. But then $\xi_j^2 + \eta_j^2 \leqslant c(F) < \infty$ by the condition imposed on F, and $\rho_j = O(1/j)$, i.e., $\Sigma_j\rho_j = \infty$, as required by (B).

We now add the inequalities (5) for $j = 1, 2, \ldots, i$. After obvious transformations, we obtain

$$\frac{1}{\sum_{j=1}^{i}\rho_j}\sum_{j=1}^{i}\rho_j F(x_j, l)-\bar{F}(x^*) \leqslant \frac{V_l(x_1, l_1)-V_l(x_{i+1}, l_{i+1})}{2\sum_{j=1}^{i}\rho_j}$$
$$+\frac{1}{\sum_{j=1}^{i}\rho_j}\sum_{j=1}^{i}\rho_j^2(\xi_j^2+\eta_j^2) \leqslant \frac{V_l(x_1, l_1)}{2\sum_{j=1}^{i}\rho_j}+c(F)\frac{\sum_{j=1}^{i}\rho_j^2}{2\sum_{j=1}^{i}\rho_j}. \tag{7}$$

Here, $c(F) = a^2(F)$ for (i); for (ii), $c(F) = \sup_j(\xi_j^2 + \eta_j^2) < \infty$, since we have established already that the trajectory $\{x_i, l_i\}$ is bounded. By the definition of x^i and by the convexity of F in x, it follows from (7) that

$$F(x^i, l)-\bar{F}(x^*) \leqslant \frac{V_l(x_1, l_1)}{2\sum_{j=1}^{i}\rho_j}+c(F)\frac{\sum_{j=1}^{i}\rho_j^2}{2\sum_{j=1}^{i}\rho_j}. \tag{8}$$

When G and G_I are bounded, the maximization of both sides of (8) with respect to l yields the estimate (4) for $\overline{F}(x^i) - \overline{F}(x^*)$, which proves (4) in view of the symmetry already noted. Clearly, under the conditions of (A) the right-hand side of (4) tends to 0 as $i \to \infty$, which proves (i).

It remains to verify that, under the conditions of (B), any weak limit point of the sequence $\{x^i\}$ is the first component of a saddle point of F. Let $\bar{x}$ be such a limit point, i.e., $x^{i(k)}$ weakly converge to $\bar{x}$, $i(k) \to \infty$ as $k \to \infty$. Assume that $\bar{x}$ is not the first component of a saddle point of F, i.e., that $\overline{F}(\bar{x}) > \overline{F}(x^*)$. We choose an l so that $F(\bar{x}, l) > \overline{F}(x^*)$ (obviously this is possible). Since $F(x, l)$ is weakly lower semicontinuous in x, we have $F(x, l) > \overline{F}(x^*) + \epsilon$, $\epsilon > 0$, in a neighborhood U (in the weak topology) of the point $\bar{x}$. For this l and $i = i(k)$, consider the inequality (8). Passing to the limit in this inequality as $k \to \infty$, we obtain

$$\overline{\lim_{k\to\infty}} F(x^{i(k)}, l) \leqslant \overline{F}(x^*).$$

But, for all k sufficiently large, $x^{i(k)} \in U$, and the last inequality contradicts the definition of U. The contradiction thus obtained completes the proof.

REMARKS. 1. It is clear from the proof of the theorem that, if one of the sets is bounded, say G_I for definiteness, then we have convergence with respect to the functional of the corresponding components of (x^i, l^i) (in our case x^i) to the corresponding component of the saddle point of F. In our case,

$$\overline{F}(x^i) - \overline{F}(x^*) \leqslant \frac{(x_1 - x^*)^2 + d^2(G_I)}{2\sum_{j=1}^{i} \rho_j} + \frac{c(F)\sum_{j=1}^{i} \rho_j^2}{2\sum_{j=1}^{i} \rho_i}$$

(here, $d(G_I)$ is the diameter of G_I; of course, we assume that the condition (B) is satisfied).

2. Assume that G and G_I are bounded and that, instead of precise information concerning F_x' and F_l', information distorted by noise is used, so that the trajectory (1) is described by the stochastic difference equations

$$x_{i+1} = \pi_G(x_i - \rho_i \xi(x_i, l_i; \omega_i)), \quad l_{i+1} = \pi_{GI}(l_i + \rho_i \eta(x_i, l_i; \omega_i)),$$

where ω_i is a point of a Polish space Ω with Borel probability measure F_ω, where the random variables ω_i are independent and each has the distribution F. The functions $\xi(x, l, \omega)$ and $\eta(x, l, \omega)$ are assumed to be Borel with respect to the totality of arguments and such that

$$M_\omega \xi(x, l, \omega) = F_x'(x, l), \quad M_\omega \eta(x, l, \omega) = F_l'(x, l);$$

where M_ω is the result of averaging with respect to F_ω.

Further, assume that the noise is bounded, e.g., by the condition

$$M_\omega \|\xi(x, l, \omega)\|^2 \leqslant L^2, \quad M_\omega \|\eta(x, l, \omega)\|^2 \leqslant L^2, \quad (x, l,) \in G \times G_I.$$

It turns out (and it can easily be derived from the proof of the theorem) that in this situation (x^i, l^i) converges "in the mean" to a saddle point of F. Namely, if we denote by $\omega^i = (\omega_1, \ldots, \omega_i)$ the realization of the noise during the first i steps of the run of the

method, and by $(x_l(\omega^{l-1}), l_l(\omega^{l-1}))$ the lth point of the trajectory (2) ((x_1, l_1) is deterministic), then the following inequality holds:

$$\nu^t = M_{\omega^{t-1}} \max\{F(x^t(\omega^{t-1})) - \inf_x F(x), \sup_l E(l) - E(l^t(\omega^{t-1}))\}$$

$$\leqslant \frac{5d^2(G \times G_1)}{\sum_{j=1}^{t} \rho_j} + \frac{5\sum_{j=1}^{t} \rho_j^2 L^2}{\sum_{j=1}^{t} \rho_j} \tag{9}$$

(here, $M_{\omega^{t-1}}$ is the average with respect to the noise during the first t steps of the run of the method). Thus we have $\bar{\nu}^t \underset{t\to\infty}{\longrightarrow} 0$ if (A) holds.

3. Let us say a few words concerning the rate of convergence of the method. Assume that, under the conditions of the theorem, G and G_l are bounded. We choose ρ_j in the form $O(1/\sqrt{j})$. Then we obtain $\nu^t \leqslant O(\ln t/\sqrt{t})$. If the accuracy ν with which the problem is to be solved is given a priori, then we can take $\rho_t \equiv \rho(\nu) = O(\nu)$ (we assume that an estimate of $\|F_x'\|$ and $\|F_l'\|$ from above on $G \times G_l$ is known a priori) and make $N = O(1/\nu^2)$ steps. With a proper choice of ρ and N of the form indicated, we shall have $\nu^N \leqslant \nu$. Similar facts hold also in the stochastic situation (see the previous remark) with respect to the values $\bar{\nu}^t$.

We note that, in application to the general convex-concave game, the rate of convergence of the traditional versions of the Arrow-Hurwicz method (based on regularization type devices or on making use of distinct shifts in x and l) turns out to be less by an order of magnitude than the above-indicated rate of convergence of the version proposed.

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