2

Theory of Speculation

LOUIS BACHELIER

Introduction

The influences which determine fluctuations on the Exchange are innumerable; past, present, and even discounted future events are reflected in market price, but often show no apparent relation to price changes.

Besides the somewhat natural causes of price changes, artificial causes also intervene: The Exchange reacts on itself, and the current fluctuation is a function, not only of previous fluctuations, but also of the orientation of the current state.

The determination of these fluctuations depends on an infinite number of factors; it is, therefore, impossible to aspire to mathematical prediction of it. Contradictory opinions concerning these changes diverge so much that at the same instant buyers believe in a price increase and sellers in a price decrease.

The calculus of probabilities, doubtless, could never be applied to fluctuations in security quotations, and the dynamics of the Exchange will never be an exact science.

But it is possible to study mathematically the static state of the market at a given instant, i. e., to establish the law of probability of price changes consistent with the market at that instant. If the market, in effect, does not predict its fluctuations, it does assess them as being more or less likely, and this likelihood can be evaluated mathematically.

Research for a formula which expresses the likelihood of a market fluctuation does not appear to have been published to date. Such is the object of this study.

I have thought it necessary first to review some theoretical notions concerning transactions in the Exchange, adding thereto certain new insights indispensable to our ultimate investigations.

---

*A thesis presented to the Faculty of Sciences of the Academy of Paris on March 29, 1900, for the Degree, Docteur ès Sciences Mathématiques. Originally published in Ann. Sci. École Norm. Sup. (3), No. 1018 (Paris, Gauthier-Villars, 1900). Dedicated to Monsieur H. Poincaré, Member of the Institute, Professor in the Faculty of Sciences. [Translated by A. James Boness, Assistant Professor of Finance and Economics, University of Pittsburgh.]

Operations in the Exchange

Operations in the Exchange. There are two kinds of futures operations: fixed futures contracts and option contracts.† These operations can be combined in an indefinitely large number of ways, according to how often one negotiates various kinds of options.

Fixed Futures Contracts.[1] Operations in fixed futures contracts are entirely analogous to spot transactions, except that price differences are settled at a predetermined time called the "liquidation date." This occurs the last day of each month.

The price established on the liquidation date, to which all the transactions of the month are related, is called the "compensation price."

The buyer of a futures contract limits neither his potential gain nor his potential loss. He receives the difference between purchase price and selling price if the sale is made above the purchase price; he pays the difference if the sale is made below.

The seller of a futures contract who repurchases at a higher price than he originally sold loses; he profits in the contrary case.

Contangoes. The spot buyer collects his interest and may retain his certificates indefinitely. Because a futures contract expires on the liquidation date, the buyer of a futures contract, to maintain his position until the following liquidation date, must pay the seller a premium called a "contango."*

The contango varies with each liquidation date. For government bonds, on the average it is 18 centimes per 3% bond of 100 francs par value; but it could be higher, or nothing at all.‡ It could even be negative, in which event it is called a "backwardation." In this case, the seller pays a premium to the buyer of the contract.

On the day when interest becomes due, the buyer of a futures contract received the amount of such interest from the seller. At the same time, the price of the security decreases by an equal amount. Thus, immediately after the payment of interest both

---

†"Fixed futures contracts" are subsequently rendered simply as futures contracts or futures; this is current American usage. [Note: References designated by a dagger or double dagger are those of the translator; superscript numbers refer to the editor's notes given on pp. 75-78; asterisks designate author's notes.]

*For the complete definition of contango I refer to specialized works on the subject [Charles Castelli, The Theory of Options in Stocks and Shares, London, Matheison, 1877].

‡"Government bonds," rentes. Whether the reference is to bills or bonds of public or private issuers are moot points. I follow S. Alexander's interpretation ("Price Movements in Speculative Markets. . .," Ind. Mgmt. Rev., May, 1961, p. 8).

buyer and seller of a futures contract find themselves in the same relative positions as before.

It is apparent that if the buyer benefits by collecting interest, on the other hand he generally must pay contangoes. The seller, on the contrary, receives contangoes, but he must pay interest.

Government Bonds with Contangoes. On government bonds, the coupon amount of 75 centimes per quarter represents 25 centimes per month, while the contango is almost always less than 20 centimes per month. This suggests the idea of buying fixed future contracts on government bonds with the intention of renewing them indefinitely by means of contangoes.

This is called an operation of "government bonds with contangoes." Later we will study its probability of success.

Equivalent Prices. To thoroughly understand the mechanism of interest payments and contangoes, we will abstract from all other causes of price changes.

Since every three months a coupon amount of 75 centimes becomes due on outstanding government bonds, representing interest on the investment of the buyer, the bond logically must increase in value by 25 centimes each month. For each currently quoted price there corresponds a price which will be greater by 25 centimes a month later, greater by 12.5 centimes two weeks later, etc.

All of these prices may be thought of as "equivalents."

Consideration of equivalent prices is much more complicated in the case of futures contracts. To begin with, it is obvious that if the contango is zero, the position in a futures contract must behave like a spot position, and its price must increase by 25 centimes per month.

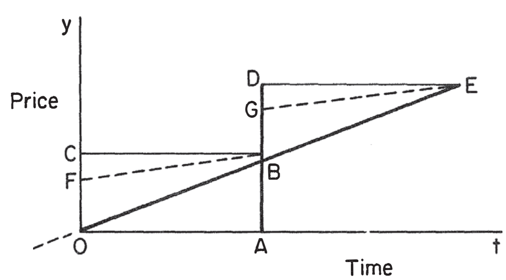

Figure 1

Now consider the case where the contango will be 25 centimes. Let the x-axis represent time in Figure 1; the distance OA represents a month included between the two liquidation dates at O and A.

If AB is equivalent to 25 centimes, the logical course of spot prices of a bond would be represented by the straight line OBE.*

Now consider the case where the contango will be 25 centimes. A little before the liquidation date, the spot and the futures contract operations will be at the same price, O. Then, the buyer of the futures contract having to pay a contango of 25 centimes, the price of the futures contract will jump abruptly from O to C

*I assume that there is no interest due date during the interval, which, however, would not alter the demonstration.

and will follow the horizontal line CB during the following month. At B, it again will be indistinguishable from the price of a spot position, only to increase suddenly by 25 centimes to D, etc.

In the case where the contango is a given amount corresponding to the distance OF, the price must follow the line FG, then GE, and so forth. The price of a futures contract on a bond, then, logically must increase from one liquidation date to another by an amount represented by FC, which could be called the "complement of the contango."

All prices between F and B on the line FB are "equivalents" for the different moments to which they correspond.

Actually, the distance between the prices of the futures contract and the spot position does not decrease in an absolutely regular way, and FB is not a straight line. The construction which was made for the beginning of a one-month period may be repeated for any moment during that month, such as the point N.

Let NA be the time which will elapse between the given moment N and the liquidation date represented by A.

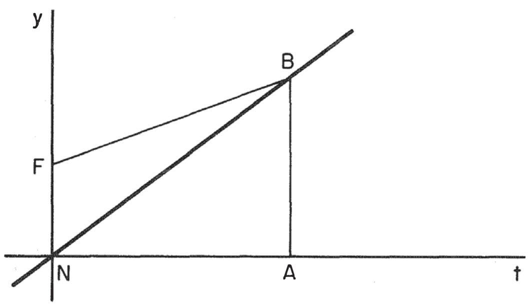

Figure 2

During the time NA, the spot price logically must increase by AB in proportion to NA. Let NF be the distance between spot and futures prices. All the prices corresponding to the line FB are "equivalents."

True Price. We will call the equivalent price corresponding to a moment the "true price" corresponding to that moment.

Understanding true price is very important. I shall now investigate how it is determined.

Designate by b the amount by which the price of a futures contract logically should increase during one day. The coefficient term b generally varies only slightly; its value each day may be exactly determined.

Suppose that n days intervene before the liquidation date, and let C be the distance between spot and futures prices.

In n days, the spot price must increase by 25n/30 centimes, the futures price being greater by the amount C must increase during these n days only by (25n/30 - C), that is, during a single day, by

$$\frac{1}{n}\left(\frac{25n}{30} - C\right) = \frac{1}{6n}\left(5n - 6C\right).$$

Then,

$$b = \frac{1}{6n}(5n - 6C).$$

The average of the five last years gives b = 0.264 centime.

The true price corresponding to m days will be equal to the currently quoted price plus the amount mb.

Geometric Representation of Operations in Futures. An operation may be represented geometrically in a very simple way, the x-axis representing different prices and the y-axis representing the corresponding profits.

Suppose I had bought a futures contract at the price represented by O, taken as origin. At the price x = A, the operation gives a profit of x. Since the corresponding ordinate must be equal to the profit, AB = OA. The purchase of a futures contract is therefore represented by the line OB inclined at 45 degrees from the price axis.

A sale of a futures contract is represented inversely.

Options.† In the purchase or sale of a futures contract, buyers and sellers expose themselves to a theoretically unlimited potential loss. In an option operation, the buyer pays a higher price for his certificate than in the case of a futures contract, but his potential loss on the downside is limited in advance to a certain sum which is the amount of the premium.[2]

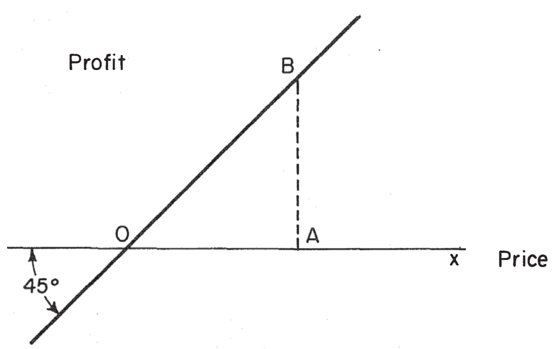

Figure 3.

The seller of an option has the advantage of selling at a higher price, but he can profit only by the amount of the premium.

There also are options at a discount‡ which limit the potential loss of the seller; in this case, the transaction occurs at a price less than that of a futures contract.

Options at a discount are negotiated only in speculations in commodities. In securities speculations, one may obtain an option at a discount by selling a futures contract and simultaneously buying an option.[3] To simplify the exposition, I will be concerned only with options selling at a premium.

Suppose, for example, that 3% bonds cost 104 francs at the

---

†"Options," primes. These are European-type calls, which differ from American calls in that they are generally of shorter duration, are exercisable at a spread from current market price, and are exercisable only on the expiration date. A more detailed technical discussion is given by Charles Castelli, The Theory of Options in Stocks and Shares, London, Matheison, 1877. This work also explains straddles, call-o'-more's, contangoes, etc.

‡Options at a discount are European-type puts.

beginning of the month. If we buy a futures contract on 3000 francs par value, we expose ourselves to a potential loss which may become considerable if a fall in the market occurs.

To avoid this risk, we could buy an option at 50 centimes,† paying for it no longer 104 francs, but 104.15 francs, for example. It is true that our purchase price is higher, but our potential loss remains limited to 50 centimes per 3% bond, or 15 francs, whatever the market decline might be.

The operation is the same as if we had bought a futures contract for 104.15 francs, this future not being able to decrease by more than 50 centimes, that is, descend below 103.65 francs.

The price of 103.65 francs in the present case is the "base of the option."

Clearly the price at the base of the option is equal to the price at which the option is negotiated, less the amount of the premium.

Call Date of Options. The "call date of options" occurs the day before the liquidation date, that is, the second-last day of the month. Let us return to the preceding example and suppose that at the instant of the call the price of the bond is less than 103.65 francs; we then will abandon our option, the premium of which will be the profit to our seller.

If, on the contrary, the price on the call date is greater than 103.65 francs, our operation will be transformed into a futures operation. In this case, the option is said to be "exercised."

In summary, an option is exercised or abandoned according as the price of the bond on the call date is greater than or less than the base of the option.

Clearly, option operations do not run up to the liquidation date. If an option is exercised on the call date, it becomes a futures contract and is liquidated on the following day.

In all that will follow, we will assume that the compensation price is indistinguishable from the price on the call date of options. This assumption is justifiable since there is nothing to prevent the liquidation of transactions on the call date.

The Spread of Options.[4] The spread between the price of a futures contract and that of an option depends on a great number of factors and varies continuously.

---

†"An option at 50 centimes," une prime dont 50^{c}. Bachelier footnotes that this is a peculiar construction even in French. The premium payable on the expiration date of this option is 50 centimes per 100 francs principal amount of optioned bonds. The premium, as in Castelli, is payable only if the option is allowed to expire. If the option is exercised, its holder acquires the principal amount of bonds at the exercisable price, here 104.15 francs, and is able to sell them at the current market price on the expiration date.

At the same instant, the spread is proportionally greater for less restrictive premiums; for example, an option at 50 centimes obviously is higher priced than an option at 25 centimes.

The spread of an option decreases more or less regularly from the beginning of the month to the day before the call date, when the spread becomes very small.

But, depending on circumstances, it could diminish very irregularly and be greater a few days before the call date than at the beginning of the month.

Options Expiring Next Month. Options are negotiated not only for the current month, but also for the next month. The spread of the latter is necessarily greater than that of one-month options, but it is less than one would think in comparing the price of an option with that of a futures contract. In effect, it is necessary to subtract the influence of the contango of the one-month option from the apparent spread.

For example, the average spread of an option at 25 centimes 45 days before its call date is on the order of 72 centimes, but, since the average contango is about 17 centimes, the spread is really only about 55 centimes.

The payment of interest makes the price of an option fall by a sum equal to the amount of the interest. If, for example, on September 2, I buy an option at 25 centimes to expire the current month for 104.50 francs, the price of my option will have become 103.75 francs on September 16 after the payment of interest.

The price at the base of the option will be 103.50 francs.

One-Day Options. One-day options at 5 centimes and sometimes at 10 centimes are negotiated, especially off the Exchange.

The call for these small options occurs every day at 2 o'clock.

Options in General. There are two factors to consider in a transaction for an option of given duration: the amount of the option and the spread from the price of a futures contract.

It is indeed obvious that the smaller the spread, the greater the premium.

To simplify the negotiation of options, they have been classified into three kinds by making three very simple assumptions concerning the amount of the premium and the spread.

1. The amount of the premium is constant and the spread is variable. This is the kind of option which is negotiated on securities; for example, on the 3% bonds, options at 50 centimes, at 25 centimes, and at 10 centimes are traded.
2. The spread is constant and the amount of the premium is variable; this is the case for options at a discount on securities (i.e., the sale of a futures contract against the purchase of an option).
3. The spread is variable as well as the amount of the premium, but the two quantities are always equal. This is

the way that options on commodities are traded. It is obvious that using this last system one can trade at a given moment only at a single premium for the same option duration.

Remark on Options. We shall examine the law which governs the spreads of options; however, even now we are able to make a rather interesting comment:

An option must be more valuable, the smaller its spread. This obvious fact is not sufficient to show that the market in options is rational.

Several years ago, I noticed that it was possible, while admitting the above fact, to imagine operations in which one of the traders would profit regardless of eventual prices.

Without reproducing the calculations, which are elementary but rather tedious, I offer an example.

The following operation:

buy one unit at 1 franc,
sell four units at 50 centimes,
buy three units at 25 centimes,

will yield a profit at all eventual prices provided that the spread between the option at 25 centimes and the option at 50 centimes is at most one-third of the spread between the option at 50 centimes and the option at 1 franc.

We will see that such spreads are never found in practice.

Geometrical Representation of Operations in Options. We propose to represent geometrically the purchase of an option.

[Ed. Note: The following paragraph replaces an unclear paragraph in the original.]

The horizontal axis represents the terminal spot price and the vertical axis, profit. The price of a futures contract at the time of the option purchase is denoted by P. The base, or striking price, is B and what Bachelier refers to as the price of an option is S, equal to B plus the premium, h. When the ending price is less than B, the option is not exercised, and the loss is equal to premium h. Above B loss is reduced by a centime

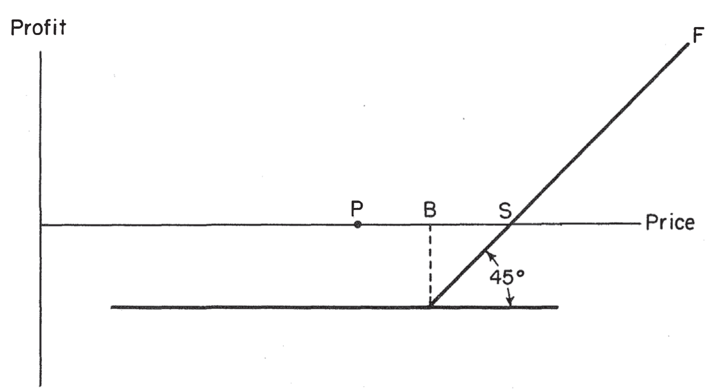

Figure 4

for every centime by which the terminal price rises above B. The loss disappears at a price S, and becomes a profit if the price rises still further.

The sale of an option is represented inversely.

True Spreads. Until now, we have spoken only of quoted spreads, those with which one ordinarily deals. These, however, are not the ones which appear in our theory, but rather the "true spreads," that is, the spreads between the prices of options and the true prices corresponding to the call date of options. These prices are greater than quoted prices (unless the contango is greater than 25 centimes, which is rare), from which it follows that the true spread of an option is less than its quoted spread.

The true spread of an option traded n days before the call date will be equal to its (quoted) spread less the quantity nb.

The true spread of an option to expire at the end of next month will be equal to its (quoted) spread less the quantity $[25 + (n - 30)b]$.

Call-o'-More's. In certain markets there are operations of a kind intermediate between operations in futures contracts and operations in options; these are "call-o'-more's."

Suppose that 30 francs is the price of a commodity. Instead of buying a single futures contract at a price of 30 francs for a given duration, we could buy a call-o'-more for the same duration at 32 francs, for example. For any eventual price below 32 francs, we lose only on a single unit, while for any eventual price above 32 francs, we will profit on two units.

We could have been able to buy a call-twice-o'-more at 33 francs, for example, such that, for any eventual price below 33 francs, we would lose on a single unit, while for any eventual price above 33 francs, we would profit on three units. One can imagine call-o'-more's of multiple orders. The geometric representation of such transactions presents no difficulty.

There also are trades in put-o'-more's, necessarily at the same spread as call-o'-more's of the same multiple order.

Probabilities in Operations on the Exchange

Probabilities in Operations on the Exchange. One can consider two kinds of probabilities:†

1. The probability which might be called "mathematical," which can be determined *a priori* and which is studied in games of chance.
2. The probability dependent on future events and, consequently, impossible to predict in a mathematical manner.

This last is the probability that the speculator tries to predict.

† This classification of probabilities (subsequent to Bayes) foreshadows the currently popular dichotomy of objective and subjective probability.

He analyzes causes which could influence a rise or a fall of market values or the amplitude of market fluctuations. His inductions are absolutely personal, since his counterpart in a transaction necessarily has the opposite opinion.

It seems that the market, the aggregate of speculators, at a given instant can believe in neither a market rise nor a market fall since, for each quoted price, there are as many buyers as sellers.

As a matter of fact, the market believes in a rise resulting from the difference between interest payments and contangoes. The sellers make a slight sacrifice for which they consider themselves compensated.

One could ignore this difference on condition that he consider true prices corresponding to the liquidation date. But transactions are governed by the quoted prices; the seller pays the difference.

From the consideration of true prices; one could say:

At a given instant, the market believes in neither a rise nor a fall of true prices.

But if the market believes in neither a rise nor a fall of true prices, it may suppose more or less likely some fluctuations of a given amplitude.

The determination of the law of probability consistent with the market at a given instant will be the purpose of this study.

Mathematical Expectation. The "mathematical expectation" of an uncertain gain is the product of that gain and the corresponding probability of its occurring.

The "total mathematical expectation" of a player will be the sum of the products of the uncertain gains and the corresponding probabilities of their occurring.

Obviously a player will have neither advantage nor disadvantage if his total mathematical expectation is zero.

Then the game is called a "fair game."

Bets at race tracks and all those games of chance engaged in at casinos are not "fair." The house or the bookie, in the case of track bets, gambles with a positive expectation, and the gamblers wager with a negative expectation.

In these kinds of games the gamblers have no choice between their side of the transaction and its opposite. As it is not the same on the Exchange, it might seem strange that these transactions are not fair, the seller accepting *a priori* a disadvantage if contangoes are less than interest payments.

The existence of a second kind of probability explains this seeming paradox.

Mathematical Advantage. The mathematical expectation shows whether a bet is advantageous or not; moreover, it tells what the bet logically must win or lose, but it does not give a coefficient in some manner representing the intrinsic value of the bet.

This leads us to introduce a new concept, that of mathematical advantage.

We shall call the "mathematical advantage" of a gambler that ratio of his positive expectation to the arithmetic sum of his positive and negative expectations.

Mathematical advantage, like a probability, varies between zero and one. It is equal to 0.5 when the game is a fair game.

Principle of Mathematical Expectation. The spot buyer may be compared with a gambler. In effect, if the price of a security might increase after its purchase, a decrease is equally possible. The causes of this increase or decrease derive from the second type of probabilities.

According to the first type, the price of a security* must increase to a value equal to the amount of its interest payments. It follows that from the point of view of this first type of probabilities:

The mathematical expectation of the spot buyer is positive.

Obviously, it will be the same as the mathematical expectation of the buyer of a futures contract if the contango is zero, because his operation will be reducible to that of a spot buyer.

If the contango on the bond were 25 centimes, the buyer would have no more advantage than the seller.

Thus one may say:

The mathematical expectations of the buyer and the seller [of futures] are zero when the contango is 25 centimes.

When the contango is less than 25 centimes, which is the usual case, the mathematical expectation of the buyer is positive; that of the seller is negative.

It always is necessary to remember that only the first type of probability is in question.

According to the preceding results, one may always consider the contango as being 25 centimes on condition that the quoted price is replaced by the true price corresponding to the liquidation date. Then, if one considers these true prices, he may say that:

The mathematical expectations of the buyer and the seller are zero.

As far as contangoes are concerned, one may consider the call date of options as indistinguishable from the liquidation date; thus:

The mathematical expectations of the buyer and the seller of options are zero.

In sum, the consideration of true prices permits the statement of this fundamental principle:

*I consider the most simple case of a fixed-income security; otherwise the increase in revenue would be a probability of the second class.

The mathematical expectation of the speculator is zero.

It is necessary to evaluate the generality of this principle carefully: It means that the market, at a given instant, considers not only currently negotiable transactions, but even those which will be based on a subsequent fluctuation in prices as having a zero expectation.

For example, I buy a bond with the intention of selling it when it will have appreciated by 50 centimes. The expectation of this complex transaction is zero exactly as if I intended to sell my bond on the liquidation date, or at any time whatever.

The expectation of an operation can be positive or negative only if a price fluctuation occurs—a priori it is zero.

General Shape of the Probability Curve. The probability that the price y will be quoted at a given moment is a function of y.

This probability may be represented by the ordinate of a curve the abscissas of which correspond to various prices.

Clearly the price considered most likely by the market is the current true price:† if the market judged otherwise, it would quote not this price, but another price higher or lower.

In the remainder of this study, we will take as origin the coordinates of the true price corresponding to the given moment. Price could vary between $-x_0$ and $+\infty$, x_0 being the current arithmetic price.

We will assume that it might vary between $-\infty$ and $+\infty$, the probability of a spread greater than x_0 being considered completely negligible, a priori.

Under these conditions, one may allow that the probability of a spread greater than the true price is independent of the arithmetic amount of the price and that the curve of probabilities is symmetrical about the true price.‡

In the following, only relative prices will be in question; the origin of the coordinates will always correspond to the current true price.

The Law of Probability. The law of probability can be derived from the principle of joint probabilities.

Let $p_{x,t}\,dx$ designate the probability that, at time t, price will

---

†"Current true price:" There is an apparent confusion here between true prices and quoted prices. But in the sense that current spot prices are discounted future true prices, current spot prices always are true, i.e., the market is in equilibrium. Quoted prices diverge from true prices only in the markets for futures and options.

‡Bachelier here notes a problem which partly motivates a log-normal (rather than normal, as in this thesis) distribution of price relatives. Cf. M. F. M. Osborne, "Brownian Motion in the Stock Market," Opns. Research, Mar.-Apr., 1959, p. 146.

be included in the elementary interval x, x + dx.

We seek the conditional probability that price z will be quoted at the moment $t_1 + t_2$, price x having been quoted at the moment t_1.

By the principle of joint probabilities, the desired probability will be equal to the product of the probability that price x will be quoted at the moment t_1, i.e., $p_{x,t}\,dx$, and the probability that, given x quoted at the moment t_1, price z will be quoted at the moment $t_1 + t_2$, i.e., multiplied by $p_{z-x,t_2}\,dz$.

The desired probability then is $p_{x,t_1}p_{z-x,t_2}\,dx\,dz$.

It being possible for price at the moment t_1 to be anywhere in the intervals dx between $-\infty$ and $+\infty$, the probability that price z will be quoted at the moment $t_1 + t_2$ will be[5]

$$\int_{-\infty}^{+\infty} p_{x,t_1} p_{z-x,t_2}\,dx\,dz.$$

The probability of this price z at the moment $t_1 + t_2$ may also be expressed as p_{z,t_1+t_2}. Thus

$$p_{z,t_1+t_2}\,dz = \int_{-\infty}^{+\infty} p_{x,t_1} p_{z-x,t_2}\,dx\,dz$$

or

$$p_{z,t_1+t_2} = \int_{-\infty}^{+\infty} p_{x,t_1} p_{z-x,t_2}\,dx.$$

This is the equation which the function p must satisfy. This equation is confirmed, as we shall see, by the function

$$p = Ae^{-B^2x^2}.$$

Observe henceforth that

$$\int_{-\infty}^{+\infty} p\,dx = A\int_{-\infty}^{+\infty} e^{-B^2x^2}\,dx = 1.$$

The classic integral in the first term has a value of $\sqrt{\pi}/B$, thus $B = A\sqrt{\pi}$ and, accordingly

$$p = Ae^{-\pi A^2x^2}.$$

Setting x = 0, one has $A = p_0$, i.e., A equals the probability of the currently quoted price.

It is necessary to establish that the function

$$p = p_0 \exp(-\pi p_0{}^2 x^2)$$

where p_0 depends on time, in fact satisfies the previous equation.

Let p_1 and p_2 be the quantities corresponding to p_0 and relative to times t_1 and t_2. It is necessary to prove that the expression

$$\int_{-\infty}^{+\infty} p_1 \exp(-\pi p_1{}^2 x^2)\, p_2 \exp(-\pi p_2{}^2 x^2)\, dx$$

may be put in the form $Ae^{-B^2x^2}$, A and B depending only on time.

This integral becomes, noting that z is a constant,

$$p_1 p_2 \exp(-\pi p_2{}^2 z^2) \int_{-\infty}^{\infty} \exp[-\pi(p_1{}^2 + p_2{}^2)x^2 + 2\pi p_2{}^2 zx]\, dx$$

or

$$p_1 p_2 \exp\left(-\pi p_2{}^2 z_2{}^2 + \frac{\pi p_2{}^4 z^2}{p_1{}^2 + p_2{}^2}\right) \int_{-\infty}^{\infty} \exp\left[-\pi\left(x\sqrt{p_1{}^2 + p_2{}^2} - \frac{p_2{}^2 z}{\sqrt{p_1{}^2 + p_2{}^2}}\right)^2\right]$$

Setting

$$x\sqrt{p_1{}^2 + p_2{}^2} - \frac{p_2{}^2 z}{\sqrt{p_1{}^2 + p_2{}^2}} = u,$$

we will then have

$$\frac{p_1 p_2}{\sqrt{p_1{}^2 + p_2{}^2}} \exp\left(-\pi p_2{}^2 z^2 + \frac{\pi p_2{}^4 z^2}{p_1{}^2 + p_2{}^2}\right) \int_{-\infty}^{\infty} \exp(-\pi u^2)\, du.$$

The integral having a value of 1, we finally obtain

$$\frac{p_1 p_2}{\sqrt{p_1{}^2 + p_2{}^2}} \exp\left(-\frac{\pi p_1{}^2 p_2{}^2}{p_1{}^2 + p_2{}^2} z^2\right).$$

This expression having the desired form, one must conclude that the probability is expressed by the formula

$$p = p_0 \exp(-\pi p_0{}^2 x^2)$$

in which p_0 depends on time.

Evidently the probability is governed by the Gaussian law, already famous in the calculus of probabilities.

Probability as a Function of Time. The next to last equation above shows that the parameters $p_0 = f(t)$ satisfy the functional relation

$$f^2(t_1 + t_2) = \frac{f^2(t_1)f^2(t_2)}{f^2(t_1) + f^2(t_2)} .$$

We differentiate first with respect to t_1, then with respect to t_2. The first term having the same form in both cases, we obtain[6]

$$\frac{f'(t_1)}{f^3(t_1)} = \frac{f'(t_2)}{f^3(t_2)} .$$

This relation being true for any t_1 and t_2, the common value of the two relations is constant, and

$$f'(t) = Cf^3(t),$$

from which

$$f(t) = p_0 = \frac{H}{\sqrt{t}},$$

H designating a constant.

Thus, for the expression of the probability, we have

$$p = \frac{H}{\sqrt{t}} \exp\left(-\frac{\pi H^2 x^2}{t}\right).$$

Mathematical Expectation. The expectation corresponding to price x has the value

$$\frac{Hx}{\sqrt{t}} \exp\left(-\frac{\pi H^2 x^2}{t}\right).$$

The total positive expectation then is

$$\int_0^{+\infty} \frac{Hx}{\sqrt{t}} \exp\left(-\frac{\pi H^2 x^2}{t}\right) dx = \frac{\sqrt{t}}{2\pi H} .$$

For present purposes, we take as constant the mathematical expectation k corresponding to t = 1; then we will have

$$k = \frac{1}{2\pi H} \quad \text{or} \quad H = \frac{1}{2\pi k} \; .$$

The definitive expression of the probability is then

$$p = \frac{1}{2\pi k\sqrt{t}} \exp\left(-\frac{x^2}{4\pi k^2 t}\right)$$

The mathematical expectation,

$$\int_0^{+\infty} px \, dx = k\sqrt{t},$$

is proportional to the square root of time.

Alternate Determination of the Law of Probability. The expression for the function p may be obtained by following a different path from that which we used.

I suppose that two mutually exclusive events, A and B, have the respective probabilities of p and q = 1 - p. The probability that, of m events, there will be α equal to A and m - α equal to B is expressed by

$$\frac{m!}{\alpha!(m-\alpha)!} p^{\alpha} q^{m-\alpha} \; .$$

This is one of the terms of the expansion of $(p+q)^m$.

The greatest of these probabilities occurs when

$$\alpha = mp \quad \text{and} \quad (m-\alpha) = mq.$$

Consider the term in which the exponent of p is mp + h; the corresponding probability is

$$\frac{m!}{(mp+h)!\,(mq-h)!} p^{mp+h} q^{mq-h} \; .$$

The quantity h is the spread.

We seek the mathematical expectation of a gambler who would wager a sum equal to the spread when the spread is positive.

We have just seen that the probability of a spread is the term of the expansion of $(p+q)^m$ in which the exponent of p is mp+h and that of q is mq - h. To obtain the mathematical expectation corresponding to this term, it will be necessary to multiply this probability by h. Now

$$h = q(mp + h) - p(mq - h);$$

mp + h and mq - h are the exponents of p and q in the term of the expansion of $(p + q)^m$. To multiply a term

$$q^{\mu} p^{\nu}$$

by

$$\nu q - \mu p = pq\left(\frac{\nu}{p} - \frac{\mu}{q}\right),$$

is to take the derivative with respect to p, subtract from it the derivative with respect to q, and multiply the difference by pq.

To obtain the total mathematical expectation, we must then take the terms of the expansion of $(p + q)^m$ for which h is positive, that is,

$$p^m + mp^{m-1}q + \frac{m(m - 1)}{1 \cdot 2} p^{m-2}q^2 + \cdots + \frac{m!}{mp!\, mq!} p^{mp}q^{mq},$$

and subtract the derivative with respect to q from the derivative with respect to p, then multiply the result by pq.

The derivative of the second term with respect to q is equal to the derivative of the first with respect to p, the derivative of the third term with respect to q is the derivative of the second with respect to p, and so forth. The number of terms thus diminishes by pairs, and there remains only the derivative of the last with respect to p,

$$\frac{m!}{mp!\, mq!} p^{mp}q^{mq} mpq.$$

The average value of the distance h will be equal to twice this quantity.

When the number m is sufficiently large, the preceding expressions may be simplified by use of the asymptotic Stirling's formula,

$$n! = e^{-n}n^n\sqrt{2\pi n}\ .$$

In this way one obtains for the mathematical expectation the value

$$\frac{\sqrt{mpq}}{\sqrt{2\pi}}\ .$$

The probability that the spread h will be included in the interval h, h + dh will be expressed by

$$\frac{dh}{\sqrt{2\pi mpq}} \exp\left(-\frac{h^2}{2mpq}\right).$$

We are able to apply the preceding theory to our study. Suppose that time is divided into very small intervals Δt, such that $t = m\Delta t$. During the period Δt price probably will fluctuate very little.

We take the sum of the products of spreads which could occur at the period Δt and the corresponding probabilities, that is, $\int_0^\infty px\,dx$, p being the probability of the spread x.

This integral must be finite since, according to the assumed smallness of Δt, large spreads have a vanishing probability. Moreover, this integral gives a mathematical expectation which must be finite if it corresponds to a very small interval of time.

Designate by Δx twice the value of the above integral; Δx will be the average of the spreads or the mean spread during the time Δt.

If the number of trials, m, is very large and if the probability is constant for all trials, we will be able to assume that price varies during each trial Δt by the mean spread Δx. The increase, Δx, will have a probability of $\frac{1}{2}$, as will also the decrease of $-\Delta x$.

Letting $p = q = \frac{1}{2}$, the preceding formula will give the probability that, at the moment, price will be included between x and x + dx; this will be

$$\frac{2dx\sqrt{\Delta t}}{\sqrt{2\pi}\sqrt{t}} \exp\left(-\frac{2x^2\Delta t}{t}\right)$$

or, letting $H = (2\sqrt{\Delta t}/\sqrt{2\pi})$

$$\frac{Hdx}{\sqrt{t}} \exp\left(-\frac{\pi H^2 x^2}{t}\right).$$

The mathematical expectation will be expressed by

$$\frac{\sqrt{t}}{2\sqrt{2\pi}\sqrt{\Delta t}} = \frac{\sqrt{t}}{2\pi H}.$$

If we take the mathematical expectation, k, corresponding to $t = 1$ as constant, we will find as before,

$$p = \frac{1}{2\pi k\sqrt{t}} \exp\left(-\frac{x^2}{4\pi k^2 t}\right).$$

The preceding formulas give $\Delta t = (1/8\pi k^2)$. The average variation during this interval of time is

$$\Delta x = \frac{\sqrt{2}}{2\sqrt{\pi}}.$$

If we set $x = n\Delta x$, the probability will be expressed by

$$p = \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{m}} \exp\left(-\frac{n^2}{\pi m}\right).$$

Probability Curve. The function

$$p = p_0 \exp(-\pi p_0^2 x^2)$$

can be represented by a curve the ordinate of which is maximized at the origin and which has two points of inflection at

$$x = \pm\frac{1}{p_0\sqrt{2\pi}} = \pm\sqrt{2\pi}\ \ k\sqrt{t}.$$

These same values of x also are the abscissas of the maxima and minima of curves of mathematical expectation, the equation of which is

$$y = \pm px.$$

The probability of price x is a function of t; it increases up to a certain moment and then decreases. The derivative $dp/dt = 0$ when $t = (x^2/2\pi k^2)$. Thus the probability of the price x is at a maximum when this price corresponds to the point of inflection of the probability curve.

Probability in a Given Interval. The integral

$$\frac{1}{2\pi k\sqrt{t}} \int_0^x \exp\left(-\frac{x^2}{4\pi k^2 t}\right) dx = \frac{c}{\sqrt{\pi}} \int_0^x \exp(-c^2x^2)\, dx$$

cannot be expressed in finite terms, but one can give its serial expansion

$$\frac{1}{\sqrt{\pi}}\left[cx - \frac{\frac{1}{3}(cx)^3}{1} + \frac{\frac{1}{5}(cx)^5}{1\cdot 2} - \frac{\frac{1}{7}(cx)^7}{1\cdot 2\cdot 3} + \cdots\right].$$

The series converges rather slowly for large values of cx. For this case, Laplace has given the definite integral in the form of an easily evaluated continuing fraction,

$$\frac{1}{2} - \frac{e^{-c^2x^2}}{2cx\sqrt{\pi}}\;\cfrac{1}{1+\cfrac{\alpha}{1+\cfrac{2\alpha}{1+\cfrac{3\alpha}{1+\ddots}}}},$$

where $\alpha = \dfrac{1}{2c^2x^2}$.

The successive reductions are

$$\frac{1}{1+\alpha},\quad \frac{1+2\alpha}{1+3\alpha},\quad \frac{1+5\alpha}{1+6\alpha+3\alpha^2},\quad \frac{1+9\alpha+8\alpha^2}{1+10\alpha+15\alpha^2}.$$

There is another procedure for calculating the value of the above integral when x is a large number.

$$\int_x^\infty e^{-x^2}\,dx = \int_x^\infty \frac{1}{2x}\,e^{-x^2}\,2x\,dx.$$

Integrating by parts, one obtains

$$\int_x^\infty e^{-x^2}\,dx = \frac{e^{-x^2}}{2x} - \int_x^\infty e^{-x^2}\frac{dx}{2x^2}$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \int_x^\infty e^{-x^2}\frac{1\cdot 3}{4x^4}\,dx$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \frac{e^{-x^2}\,1\cdot 3}{8x^5} - \int_x^\infty e^{-x^2}\frac{1\cdot 3\cdot 5}{8x^6}\,dx.$$

The general term of the series is expressed by

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^{2m-1}\, x^{2m+1}}\, e^{-x^2}.$$

The ratio of one term to the preceding term exceeds unity when $2n + 1 > 4x^2$. The series, thus, diverges after a certain term. One may obtain an upper limit of the integral which takes the place of the remainder.

$$\frac{1\cdot 3\cdot 5 \cdots (2n+1)}{2^{2n-1}} \int_x^{\infty} \frac{e^{-x^2}}{x^{2n+2}}\, dx < \frac{1\cdot 3\cdot 5 \cdots (2n+1)}{2^{2n-1}}\, e^{-x^2} \int_x^{\infty} \frac{dx}{x^{2n+2}}$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^{2n-1}\, x^{2n+1}}\, e^{-x^2}.$$

Now this last quantity is the term which precedes the integral. The complementary term, then, is always smaller than that which precedes it.

There are published tables giving values of the integral

$$\theta(y) = \frac{2}{\sqrt{\pi}} \int_0^{y} e^{-y^2}\, dy.$$

Obviously, $\displaystyle\int_0^{x} p\, dx = \frac{1}{2}\,\theta\!\left(\frac{x}{2k\sqrt{\pi}\sqrt{t}}\right).$

The probability

$$\mathscr{P} = \int_x^{\infty} p\, dx = \frac{1}{2} - \frac{1}{2}\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}\, k\sqrt{t}}} e^{-\lambda^2}\, d\lambda,$$

that price x will be attained or exceeded at the moment t increases steadily with time. If t were infinite, the probability would equal $\frac{1}{2}$, an obvious result.

The probability

$$\int_{x_1}^{x_2} p\, dx = \frac{1}{\sqrt{\pi}} \int_{\frac{x_1}{2\sqrt{\pi}\, k\sqrt{t}}}^{\frac{x_2}{2\sqrt{\pi}\, k\sqrt{t}}} e^{-\lambda^2}\, d\lambda,$$

that price will be included in the finite interval x_2, x_1 at the moment t is zero for $t = 0$ and for $t = \infty$. It is maximized when

$$t = \frac{1}{4\pi k^2} \frac{x_2 - x_1}{\log (x_2/x_1)} .$$

If we assume the interval x_2, x_1 to be very small, we again find the probability maximized at the moment

$$t = \frac{x^2}{2\pi k^2} .$$

Probable Spread. We shall thus name the interval $\pm \alpha$ such that, at the end of time t, price would have as much chance of being included in this interval as of exceeding the interval.

The quantity α is determined by the equation

$$\int_0^{\alpha} p \, dx = \frac{1}{4} ,$$

or

$$\theta\left(\frac{\alpha}{2k\sqrt{\pi}\sqrt{t}}\right) = \frac{1}{2} ;$$

that is,

$$\alpha = 2(0.4769\, k\sqrt{\pi}\sqrt{t}) = 1.688\, k\sqrt{t} .$$

This interval is proportional to the square root of time.

More generally, we consider the interval $\pm \beta$ such that the probability that price will be included in this interval at the moment t is equal to u. We have

$$\int_0^{\beta} p \, dx = \frac{u}{2} ,$$

or

$$\theta\left(\frac{\beta}{2k\sqrt{\pi}\sqrt{t}}\right) = u.$$

We notice that this interval is proportional to the square root of time.

Radiation of Probability. I will directly seek an expression for the probability $\mathscr{P}$ that price x will be attained or exceeded at the moment t. Previously we saw that, by dividing time into very small intervals Δt, one was able during an interval Δt to consider price as changing by the fixed and very small quantity Δx.

I presume that, at the moment t, prices x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}, ... differ among themselves by the quantity Δx and have respective probabilities p_{n-2}, p_{n-1}, p_n, p_{n+1}, p_{n+2}, ... From the knowledge of the distribution of probabilities at the moment t, one easily deduces the distribution of probabilities at the moment $t + \Delta t$. We assume, for instance, that price x_n is quoted at moment t. At moment $t + \Delta t$, price x_{n+1} or x_{n-1} will be quoted. The probability p_n that price x_n will be quoted at the moment t decomposes into two probabilities at the moment $t + \Delta t$. Price x_{n-1} by this event will have a probability of $p_n/2$, and price x_{n+1} by the same event will have a probability of $p_n/2$.

If price x_{n-1} is quoted at the moment $t + \Delta t$, it is because at the moment t price x_{n-2} or x_n had been quoted. The probability of price x_{n-1} at the moment $t + \Delta t$ is therefore $(p_{n-2} + p_n)/2$, the probability of price x_n at the same moment is $(p_{n-1} + p_{n+1})/2$, the probability of price x_{n+1} is $(p_n + p_{n+2})/2$, etc.

During the interval Δt, price x_n has, in a fashion, emitted toward price x_{n+1} the probability $p_n/2$; price x_{n+1} has emitted toward price x_n the probability $p_{n+1}/2$. If p_n is greater than p_{n+1}, the exchange of probabilities is $(p_n - p_{n+1})/2$ from x_n toward x_{n+1}.

Thus one can state:

Each price x during an element of time radiates toward its neighboring price an amount of probability proportional to the difference of their probabilities.

I say proportional because it is necessary to account for the relation of Δx to Δt.

The above law can, by analogy with certain physical theories, be called the law of radiation or diffusion of probability.

I consider the probability $\mathscr{P}$ that price x is in the interval x, ∞ at the moment t and evaluate the increase in this probability during the period Δt.

Let p be the probability of price x at moment t, $p = - d\mathscr{P}/dx$. We evaluate the probability which is in a manner of speaking exchanged through price x during the period Δt. This is, according to what has just been said,

$$\frac{1}{c^2}\left(p - \frac{dp}{dx} - p\right)\Delta t = -\frac{1}{c^2}\frac{dp}{dx}\Delta t = \frac{1}{c^2}\frac{d^2\mathscr{P}}{dx^2}\Delta t,$$

c designating a constant.

This increase in probability also may be expressed as $d\mathcal{P}/dt\,\Delta t$. Then[7]

$$c^2 \frac{\partial \mathcal{P}}{\partial t} - \frac{\partial^2 \mathcal{P}}{\partial x^2} = 0.$$

This is a Fourier equation.

The preceding theory assumes discontinuous changes in prices. One may arrive at the Fourier equation without making this assumption by allowing that in a very small interval of time Δt price changes in a continuous manner but by a very small quantity less than ε, for example.

We designate by ω the probability corresponding to p and relative to Δt.

According to our assumption, price will be able to change only within the limits $\pm\,\varepsilon$ during the period Δt. It follows that

$$\int_{-\varepsilon}^{+\varepsilon} \omega \, dx = 1.$$

Price could be $x - m$ at moment t, m being positive and less than ε. The probability of this event is p_{x-m}.

The probability that price x will be exceeded at the moment $t + \Delta t$, given that it was equal to $x - m$ at moment t, by the principle of joint probabilities has a value of

$$p_{x-m} \int_{\varepsilon - m}^{\varepsilon} \omega \, dx.$$

Price could be $x + m$ at moment t. The probability of this event is p_{x+m}.

The probability that price will be less than x at moment $t + \Delta t$, given that it was equal to $x + m$ at moment t, similarly, will have a value of

$$p_{x+m} \int_{\varepsilon - m}^{\varepsilon} \omega \, dx.$$

The increase in probability $\mathcal{P}$ in the interval of time Δt will be equal to the sum of expressions such as

$$(p_{x-m} - p_{x+m}) \int_{\varepsilon - m}^{\varepsilon} \omega \, dx$$

for all values of m from zero to ε.

Developing the expressions for p_{x-m} and p_{x+m} and ignoring terms which contain m^2, we have

$$p_{x-m} = p_x - m\frac{dp_x}{dx},$$

$$p_{x-m} = p_x + m\frac{dp_x}{dx}.$$

Then the above expression becomes

$$-\frac{dp}{dx}\int_{\varepsilon - m}^{\varepsilon} 2m\omega\, dx.$$

The desired increase in probability, then, has a value of

$$-\frac{dp}{dx}\int_{0}^{\varepsilon}\int_{\varepsilon - m}^{\varepsilon} 2m\omega\, dx\, dm.$$

The integral does not depend on x, t, or p; it is a constant. The increase in probability $\mathscr{P}$, then, is expressed by

$$\frac{1}{c^2}\frac{dp}{dx}.$$

The integral of the Fourier equation is

$$\mathscr{P} = \int_0^{\infty} f\left(t - \frac{c^2x^2}{2\alpha^2}\right)e^{-\alpha^2/2}\, d\alpha.$$

The arbitrary function f depends on the following considerations:

Necessarily, $\mathscr{P} = \frac{1}{2}$ if x = 0, t having any positive value; $\mathscr{P} = 0$ for negative t.

Setting x = 0 in the above integral,

$$\mathscr{P} = f(t)\int_0^{\infty} e^{-\alpha^2/2}\, d\alpha = \frac{\sqrt{\pi}}{\sqrt{2}}\, f(t),$$

that is,

$$f(t) = \frac{1}{\sqrt{2\pi}} \quad \text{for } t > 0,$$

$$f(t) = 0 \quad \text{for } t < 0.$$

This last equation shows that the integral $\mathcal{P}$ will have zero elements when $t - (c^2x^2/2\alpha^2)$ is less than zero, i.e., when $\alpha < (cx/\sqrt{2}\sqrt{t})$; thus the lower limit of the integral $\mathcal{P}$ must be taken as $cx/\sqrt{2}\sqrt{t}$:

$$\mathcal{P} = \frac{1}{\sqrt{2}\sqrt{\pi}} \int_{\frac{cx}{\sqrt{2}\sqrt{t}}}^{\infty} e^{-\alpha^2/2}\, d\alpha = \frac{1}{\sqrt{\pi}} \int_{\frac{cx}{2\sqrt{t}}}^{\infty} e^{-\lambda^2}\, d\lambda,$$

or, replacing $\int_{\frac{cx}{2\sqrt{t}}}^{\infty}$ by $\int_0^{\infty} - \int_0^{\frac{cx}{2\sqrt{t}}}$,

$$\mathcal{P} = \frac{1}{2} - \frac{1}{2}\frac{2}{\sqrt{\pi}} \int_0^{\frac{cx}{2\sqrt{t}}} e^{-\lambda^2}\, d\lambda,$$

a previously derived formula.

Law of the Spreads of Options. To know the law which governs the relation of the amounts of premiums to the spreads of options, we will apply the principle of mathematical expectation to the buyer of an option:

The mathematical expectation of the buyer of an option is zero.

We take the true price of a futures contract as origin (Figure 5).

Let p be the probability of price $\pm$ x, that is, in the present case, the probability that the call date of options will occur at price $\pm$ x.

Let m + h be the true spread of the option at h.

[Ed. Note: h is the premium, x is the terminal price less the futures price at the time of the transaction and m is the difference between the futures price and the base, or striking price. In U. S. markets, m is called the spread. In France, m + h is called the spread.]

We require that the total mathematical expectation will be zero.

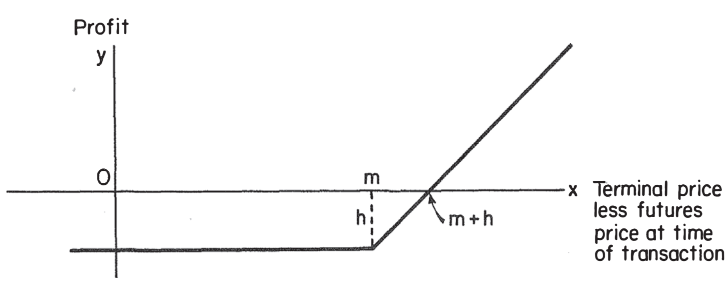

Figure 5

We will evaluate this expectation

1. For prices included between $-\infty$ and m,
2. For prices included between m and m + h,
3. For prices included between m + h and $+\infty$.

1. For all prices between $-\infty$ and m, the option is abandoned, i.e., the buyer suffers a loss of h. His mathematical expectation for a price included in the given interval therefore is -ph, and for the entire interval

$$-h\int_{-\infty}^{m} p\,dx.$$

2. For a price included between m and m +h, the loss of the buyer will be m + h - x. The corresponding mathematical expectation will be - p(m + h - x) and for the entire interval

$$-\int_{m}^{m+h} p(m+h-x)\,dx.$$

3. For a price x included between m + h and $+\infty$, the profit of the buyer will be (x - m - h). The corresponding mathematical expectation will be p(x - m - h) and for the entire interval

$$\int_{m+h}^{+\infty} p(x-m-h)\,dx.$$

The principle of total expectation then will give

$$\int_{m+h}^{\infty} p(x-m-h)\,dx - \int_{m}^{m+h} p(m+h-x)\,dx - h\int_{-\infty}^{m} p\,dx = 0,$$

or, simplifying,

$$h + m\int_{m}^{\infty} p\,dx = \int_{m}^{\infty} px\,dx.$$

Such is the equation in definite integrals which establishes a relation among probabilities, the spread of an option, and the amount of its premium.

In the case where the base of the option would fall in the neighborhood of negative x's, as shown in Figure 6, m would be negative and one would arrive at the relation

$$\frac{2h + m}{2} + m \int_0^m p \, dx = \int_{-m}^{\infty} px \, dx.$$

According to the symmetry of probabilities, the function p necessarily being even, it follows that the two equations above resolve into one.

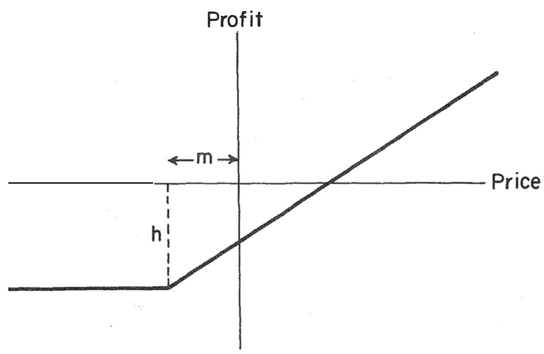

Figure 6

Differentiating, one obtains the differential equation of the spreads of options

$$\frac{d^2h}{dm^2} = p_m,$$

p_m being the expression for probability in which m is substituted for x.

Simple Options. The simplest case in the above equations is that in which m = 0, i. e., in which the amount of the premium is equal to the spread. This kind of option is called a "simple option"; it is the only kind negotiated in speculations in commodities.[8]

Setting m = 0 and designating by a the value of a simple option, the above equations become

$$a = \int_0^{\infty} px \, dx = \int_0^{\infty} \frac{x}{2\pi k \sqrt{t}} \exp\left(-\frac{x^2}{4\pi k^2 t}\right) dx.$$

The equation $a = \int_0^{\infty} px \, dx$ shows that the simple option is equivalent to the positive expectation of the buyer of a futures contract. This fact is obvious, since the buyer of a simple option furnishes the sum a as payment to the seller in order to enjoy the advantages of the buyer of a futures contract, that is, to have his (the buyer of the future's) positive expectation without undertaking his risks.

From the formula

$$a = \int_0^\infty px\,dx = k\sqrt{t},$$

we deduce the following principle, one of the most important of our study:

The value of a simple option must be proportional to the square root of time.

Previously, we saw that the probable spread was given by the formula

$$\alpha = 1.688k\sqrt{t} = 1.688a.$$

The probable spread is obtained, then, by multiplying the average premium by the constant, 1.688. Therefore it is very easy to calculate for speculations on commodities since, in this case, the quantity is known.

The following formula gives the expression for probability as a function of a:

$$p = \frac{1}{2\pi a} \exp\left(-\frac{x^2}{4\pi a^2}\right).$$

The probability in a given interval will be expressed by the integral

$$\frac{1}{2\pi a} \int_0^u \exp\left(-\frac{x^2}{4\pi a^2}\right) dx$$

or

$$\frac{1}{2\pi a}\left(u - \frac{u^3}{12\pi a^2} + \frac{u^5}{160\pi^2 a^4} - \frac{u^7}{2678\pi^3 a^6} + \cdots\right).$$

This probability is independent of a and, consequently, of time if u, instead of being a given number, is a parameter of the form u = ba. For example, if u = a,

$$\int_0^a p\,dx = \frac{1}{2\pi} - \frac{1}{24\pi^2} + \frac{1}{320\pi^3} - \cdots = 0.155.$$

The integral $\int_0^a p\,dx$ represents the probability of success of the buyer of a simple option. Now

$$\int_a^\infty p\,dx = \frac{1}{2} - \int_0^a p\,dx = 0.345.$$

Thus:

The probability of success of the buyer of a simple option is independent of the expiration date; it has a value of

0.345.

The positive expectation of a simple option is expressed by

$$\int_a^\infty p(x - a)\,dx = 0.58a.$$

Straddles. A straddle or double option consists of the simultaneous purchase of an option at a premium and an option at a discount (simple options). It is easily seen that the seller of a straddle profits in the interval - 2a, + 2a. His probability of success is then

$$2\int_0^{2a} p\,dx = \frac{2}{\pi} - \frac{2}{3\pi^2} + \frac{2}{10\pi^3} - \cdots = 0.56.$$

The probability of success of the buyer of a straddle is 0.44. The positive expectation of a straddle is

$$2\int_{2a}^\infty p(x - 2a)\,dx = 0.55a.$$

Coefficient of Instability. The coefficient k, introduced previously, is the "coefficient of instability" or of nervousness of a security. It is a measure of its (the security's) static state. Its extension indicates a disturbed state; its contraction, on the other hand, indicates a tranquil state.

This coefficient is given directly for speculations in commodities by the formula

$$a = k\sqrt{t},$$

but in speculation in securities it can be calculated only approximately, as we shall see.

Series of the Spreads of Options. The equation in definite integrals of the spreads of options cannot be expressed in finite terms when the quantity m, the difference between the spread of the option and the amount of the premium h, is not zero.

This equation leads to the series

$$h - a + \frac{m}{2} - \frac{m^2}{4\pi a} + \frac{m^4}{96\pi^2 a^3} - \frac{m^6}{1920\pi^3 a^5} + \cdots = 0.$$

This relation, in which the quantity a designates the amount of the premium of a simple option, can be solved for a when m is known, or inversely.

Approximate Law of the Spreads of Options. The preceding series may be written

$$h = a - f(m).$$

We consider the product of the premium h and the spread (m + h):

$$h(m + h) = [a - f(m)][m + a - f(m)].$$

Taking the derivative with respect to m, we have

$$\frac{d}{dm}[h(m + h)] = f'(m)[m + a - f(m)] + [a - f(m)][1 - f'(m)].$$

If we set m = 0, whereby f(m) = 0, $f'(m) = \frac{1}{2}$, this derivative cancels out. We must conclude from this that:

The product of the premium and the spread of an option is maximized when the two factors of this product are equal; this condition holds for simple options.

In the neighborhood of its maximum, the product in question should vary only slightly. This often allows the approximate evaluation of a by the formula

$$h(m + h) = a^2.$$

This formula underestimates the value of a.

Considering only the three first terms of the series,

$$h(h + m) = a^2 - \frac{m^2}{4},$$

which overestimates the value of a.

Usually, one will obtain a very satisfactory approximation by taking the first four terms of the series; thus

$$a = \frac{\pi(2h + m) \pm \sqrt{\pi^2(2h + m)^2 - 4\pi m^2}}{4\pi}.$$

With this approximation, one will get the value of m as a function of a,

$$m = \pi a \pm \sqrt{\pi^2 a^2 - 4\pi a(a - h)}.$$

We temporarily allow the simplified formula

$$h(m + h) = a^2 = k^2 t.$$

In speculations on securities, options at a premium have a constant premium h; thus the spread m + h is proportional to time.

The spread of options at a premium in speculations on securities is proportional to the duration of their expiration date and to the square of their coefficient of instability.

Options at a discount, on securities (i. e., the sale of a futures contract against the purchase of an option) have a constant spread, h, and a variable premium m + h. Thus:

The premium of options at a discount, in speculations on securities, is proportional to the duration of the expiration date and to the square of the coefficient of instability.

The two preceding laws are only approximate.

Call-o'-more's. We apply the principle of mathematical expectation to the purchase of the call-o'-more of order n traded at a spread of r.

The call-o'-more of order n can be considered as consisting of two transactions:

1. Purchase of a futures contract on one unit at a price r;
2. Purchase of a futures contract on (n - 1) units at a price r, this purchase being considered only in the interval r, ∞.

The first transaction has a mathematical expectation of -r; the second has the expectation

$$(n - 1) \int_r^{\infty} p(x - r)\, dx.$$

Thus

$$r = (n - 1) \int_r^{\infty} p(x - r)\, dx,$$

or, substituting for the value of p,

$$p = \frac{1}{2\pi a} \exp\left(- \frac{x}{4\pi a^2}\right),$$

and expanding in series,

$$2\pi a^2 - \pi a \frac{n+1}{n-1} r + \frac{r^2}{2} - \frac{r^4}{48\pi a^2} + \cdots = 0.$$

Keeping only the first three terms, one obtains

$$r = a\left[\frac{n+1}{n-1}\pi - \sqrt{\left(\frac{n+1}{n-1}\pi\right)^2 - 4\pi}\right].$$

If $n = 2$, $r = 0.68a$.

The spread of a call-o'-more of order two must be about two-thirds of the value of the corresponding simple option.

If $n = 3$, $r = 1.096a$.

The spread of a call-o'-more of order three must be greater by about one-tenth than the value of the corresponding simple option.

We have just seen that the spreads of call-o'-more's are approximately proportional to the quantity a.

It follows that the probability of success in these transactions is independent of the duration of the expiration date.

The probability of success of a call-o'-more of order two is 0.394; the transaction will pay off four times in ten.

The probability of success of a call-o'-more of order three is 0.33; the transaction will pay off one time in three.

The positive expectation of a call-o'-more of order n is

$$n \int_r^{\infty} p(x - r)\, dx$$

and, as

$$\frac{r}{n-1} = \int_r^{\infty} p(x - r)\, dx,$$

the desired expectation has a value of $[n/(n-1)]r$, that is, 1.36a for the call-o'-more of order two and 1.64a for the call-o'-more of order three.

By selling a futures contract and simultaneously buying a call-o'-more of order two, one obtains an option at a premium of $r = 0.68a$ and a spread of twice r.

The probability of success of this operation is 0.30.

By analogy with transactions in options, we will call the transaction resulting in two call-o'-mores of order n, one at a premium and the other at a discount, "call-o'-more straddles of order n."

The call-o'-more straddle of order two is a very strange operation. Between the prices $\pm$ r the potential loss is constant and equal to 2r. The potential loss then progressively decreases until the price ± 3r, where it disappears.

There is a profit outside of the interval ± 3r.

The probability of success is 0.42.

Operations in Fixed Futures Contracts

Now that we have finished the general study of probabilities, we apply it to the investigation of the probabilities of the principal operations on the Exchange, beginning with the simplest, operations in futures and operations in options. We will finish with the study of combinations of these operations.

The theory of speculation in commodities, much simpler than that in securities, has already been treated. We have, in effect, already calculated the probability of success and the expectation for simple options, for straddles, and for call-o'-mores.

The theory of transactions on the Exchange depends on two coefficients, b and k. From their values at a given instant, the spread between a futures contract and a spot position and between a futures contract and any kind of option can easily be deduced.

In the following study, we will be concerned with the 3% government bond, which is one of the securities on which options are regularly traded.

We will take as the values of b and k their average values over the past five years (1894-1898), i.e.,

$$b = 0.264$$

$$k = 5$$

(time is expressed in days and the unit of price change is the centime).

We will understand "calculated values" as those which are deduced from the formulas of the theory with the values of constants b and k given above.

"Observed values" are those deduced directly from quotation lists during the same period 1894-1898.*

In the following sections, we will need to know the average values of the quantity a at different moments; the formula $a = 5\sqrt{t}$ gives:

For 45 days: $a = 33.54$
For 30 days: $a = 27.38$
For 20 days: $a = 22.36$
For 10 days: $a = 16.13$

For one day, apparently a must equal 5; but in all the calculations of probabilities where averages are involved, t is never set equal to 1 day.

---

* All observations are taken from the Cote de la Bourse et de la Banque.

Actually, there are 365 days in a year, but only 307 trading days. The "average day" of the Exchange is therefore $t = (365/307)$, which gives $a = 5.45$.

The same comment may be made concerning the coefficient b. In all calculations relative to a trading day, b must be replaced by $b_1 = (365/307)b = 0.313$.

Probable Spread. We seek the price interval $(-\alpha, +\alpha)$ such that, at the end of a month, the bond will be as likely to be within this interval as to be outside of it.

It is required that

$$\int_0^{\alpha} p\, dx = \frac{1}{4},$$

whence,

$$\alpha = \pm 46.$$

During the past 60 months, the fluctuation has been contained within this interval 33 times, and 27 times the limits were exceeded.

Similarly, the interval relative to one day might be wanted; it is

$$\alpha = \pm 9.$$

Out of 1452 observations, the fluctuation has been less than 9 centimes 815 times.

In the preceding discussion, we have assumed that quoted price is indistinguishable from true price. Under these conditions, the mathematical expectations and probabilities of success of the buyer and the seller are the same. In fact, quoted price is less than true price by the quantity nb, if n is the number of days until the expiration date.

The probable spread of 46 centimes on both sides of the true price corresponds to the interval between 54 centimes above the quoted price and 38 centimes below this price.

Formula for the Probability of Success in the General Case. To find the probability of a price increase for a period of n days, it is necessary to know the spread, nb, between the true price and the quoted price. The probability then is equal to

$$\int_{-nb}^{\infty} p\, dx.$$

The probability of a price decrease is equal to one less the probability of a price increase.

The Probability of Success of a Spot Purchase.[9] We seek the probability of success of a spot purchase destined to be resold after 30 days.

In the preceding formula 25 must be substituted for nb.

Then the probability is equal to 0.64; the transaction has two chances in three of succeeding.

To find the probability corresponding to one year, substitute 300 for nb. The expression $a = k\sqrt{t}$ gives

$$a = 95.5.$$

The probability is 0.89.

Nine times out of ten, the spot purchase of a bond yields a profit at the end of one year.

The Probability of Success of a Futures Purchase. We seek the probability of success of a futures purchase consummated at the beginning of the month.

$$nb = 7.91, \qquad a = 27.38.$$

One deduces that:

The probability of a price increase is 0.55

The probability of a price decrease is 0.45.

The probability of success of a futures purchase increases with time; for a year,

$$n = 365, \qquad nb = 96.36, \qquad a = 95.5.$$

The probability of success then has a value of 0.65.

When a futures contract is purchased to be resold at the end of one year, there are two chances out of three of success.

Obviously, if the monthly contango were 25 centimes, the probability of success of the purchase would be 0.50.

Mathematical Advantage of Futures Operations. It seems indispensable to me, as I have already remarked, to study the mathematical advantage of a game as soon as it is not a fair game, and that is the case for futures contracts.

If we suppose $b = 0$, the mathematical expectation of the futures purchase is $a - a = 0$. The advantage of the operation is $(a/2a) = \frac{1}{2}$, as elsewhere in all fair games.

We seek the mathematical advantage of a futures purchase of n days while supposing that $b > 0$. The buyer, during this period, will have won the sum nb resulting from the difference between interest payments and contangoes. His expectation will be $a - a + nb$. Thus his mathematical advantage will be

$$\frac{a + nb}{2a + nb}.$$

The advantage of the seller will be

$$\frac{a}{2a + nb}.$$

We consider the case of the buyer in particular:

When $b > 0$, his mathematical advantage increases directly with n; it is persistently greater than the probability of success.

For a month, the advantage of the buyer is 0.563 and his probability of success 0.55. For a year, his advantage is 0.667 and his probability of success 0.65.

Thus one may state that:

The advantage of a futures operation is almost equal to its probability of success.

Option Transactions

The Spread of Options. Knowing the value of a for a given moment, one easily calculates the true spread by the formula

$$m = \pi a \pm \sqrt{\pi^2 a^2 - 4\pi a(a - h)}.$$

Knowing the true spread, one obtains the quoted spread by adding the quantity nb. n is the number of days before the call date.

In the case of an option expiring this month, one adds the quantity $[25 + (n - 30)b]$.

Thus one arrives at the following results:

Options at 50 centimes

	Quoted spread	
	Calculated	Observed
Expiring in 45 days	50.01	52.62
Expiring in 30 days	20.69	21.22
Expiring in 20 days	13.23	14.71

Options at 25 centimes

	Quoted spread	
	Calculated	Observed
Expiring in 45 days	72.70	72.80
Expiring in 30 days	37.78	37.84
Expiring in 20 days	25.17	27.39
Expiring in 10 days	12.24	17.40

Options at 10 centimes

	Quoted spread	
	Calculated	Observed
Expiring in 30 days	66.19	60.93
Expiring in 20 days	48.62	46.43
Expiring in 10 days	26.91	31.89

In the case of the option at 5 centimes for the following day we have

$$h = 5, \qquad a = 5.45$$

from which

$$m = 0.81.$$

The true spread is therefore 5.81; adding $b_1 = (365/307)\,b = 0.31$, one obtains the calculated spread of 6.12.

The average for the past five years is 7.36.

The observed and calculated figures agree on the whole, but they show certain discrepancies which must be explained.

Thus, the observed spread of the option at 10 centimes expiring in 30 days is too small. The explanation is readily understood: In very agitated periods when an option at 10 centimes would have a very large spread, this option was not quoted. The observed average therefore is understated.

Besides, it cannot be denied that, for several years, the market has had a tendency to quote options of short duration with too large spreads.[10] Therefore, it accounts proportionately less for small spreads on options of short duration.

However, it is necessary to add that the market seems to have recognized its error, because in 1898 it appears to have exaggerated in the opposite direction.

Probability of the Exercise of Options. For an option to be exercised, price on the call date must be greater than the price at the base of the option. The probability of exercise therefore is expressed by the integral

$$\int_{\varepsilon}^{\infty} p\,dx,$$

ε being the true price at the base of the option.

This integral is easy to evaluate, as shown previously. It leads to the following results.

Probability of exercise of options at 50

	Calculated	Observed
Expiring in 45 days	0.63	0.59
Expiring in 30 days	0.71	0.75
Expiring in 20 days	0.77	0.76

Probability of exercise of options at 25

	Calculated	Observed
Expiring in 45 days	0.41	0.40
Expiring in 30 days	0.47	0.46
Expiring in 20 days	0.53	0.53
Expiring in 10 days	0.65	0.65

Probability of exercise of options at 10

	Calculated	Observed
Expiring in 30 days	0.24	0.21
Expiring in 20 days	0.28	0.26
Expiring in 10 days	0.36	0.38

One may say that options at 50 centimes are exercised three times out of four, options at 25 centimes two times out of four, and options at 10 one time out of four.

The probability of exercise of an option at 5 centimes for the following day is, according to the calculation, 0.48; the outcome of 1456 observations gives 671 options certainly exercised and 76 of which the exercise is doubtful. Including these last 76 options the probability would be 0.51; excluding them it would be 0.46; the average is 0.48, as indicated by the theory.

Probability of Profit from Options. For an option to yield a profit to its buyer, a price greater than the option price must occur on the call date. The probability of profit is therefore expressed by the integral

$$\int_{\varepsilon_1}^{\infty} p\, dx,$$

ε_1 being the price of the option.

This integral leads to the following results.

Probability of profit from options at 50

	Calculated	Observed
Expiring in 45 days	0.40	0.39
Expiring in 30 days	0.43	0.41
Expiring in 20 days	0.44	0.40

Probability of profit from options at 25

	Calculated	Observed
Expiring in 45 days	0.30	0.27
Expiring in 30 days	0.33	0.31
Expiring in 20 days	0.36	0.30
Expiring in 10 days	0.41	0.40

Probability of profit from options at 10

	Calculated	Observed
Expiring in 30 days	0.20	0.16
Expiring in 20 days	0.22	0.18
Expiring in 10 days	0.27	0.25

Evidently, within the ordinary limits of practicality, the probability of success for the purchase of an option varies little. The purchase at 50 centimes is successful four times out of ten; the purchase at 25 centimes, three times out of ten; and the purchase at 10 centimes, twice out of ten.

According to the calculation, the buyer of an option at 5 centimes for the following day has a probability of success of 0.34; observation of 1456 quotations shows that 410 options would certainly have yielded profits and that 80 others would have been doubtful: The observed probability therefore is 0.31.

Complex Operations

Classification of Complex Operations. Since one trades in a futures contract and often as many as three options of the same expiration date, he could at a single time undertake triple and even quadruple operations.

Triple operations occur in a number of ways which may be considered classical; the study of them is very interesting, but too long to be explained here. We will limit ourselves, then, to double operations.

These may be divided into two groups according as they do or do not include a futures contract.

Operations including a futures contract consist of a futures purchase and an option sale, or the inverse.

Operations in options against options consist of the purchase of a large option followed by the sale of a small one, or the inverse.

The proportion of purchases to sales, moreover, can vary indefinitely. To simplify the problem, we will study only two very simple proportions:

1. The second operation involves the same figure as the first.
2. The second operation involves a figure twice as large as the first.

To clarify the issues, we will assume that transactions occur at the beginning of the month, and we will take as true spreads the average spreads for the past five years: 12.78 for an option at 50 centimes, 29.87 for an option at 25, and 58.28 for an option at 10 centimes.

We also will observe that for operations of a month's duration the true price is greater than the quoted price by the quantity 7.91 = 30b.

<u>Purchase of a Futures Contract Against the Sale of an Option.</u> In the real world one buys futures at the price - 30b = -7.91 and one sells options at 25 at the price 29.87.

It is easy to represent the operation geometrically (Figure 7): the futures purchase is represented by the line AMB.

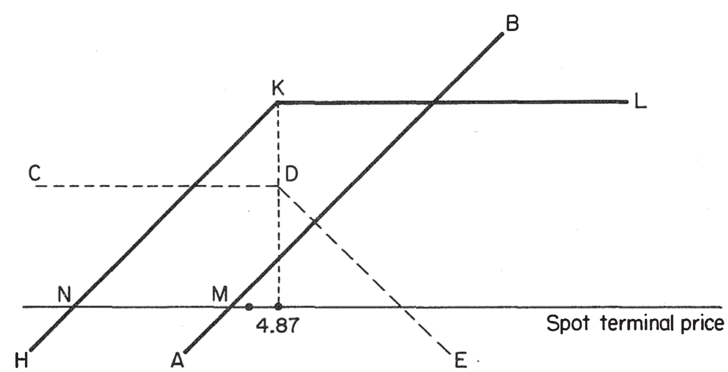

Figure 7

MO = 30b. The option sale is represented by the broken line CDE. The resulting operation will be represented by the broken line HNKL. The abscissa of the point N will be

$$-(25 + 30b).$$

One sees that the operation yields a limited potential profit equal to the spread attached to the option. On the down-side the risk of loss is unlimited.

The probability of success of the operation is expressed by the integral

$$\int_{-25-30b}^{\infty} p\,dx = 0.68.$$

If one had sold an option at 50, the probability of success would have been 0.80.

It is interesting to know the probability of success in the case of a contango of 25 centimes (b = 0).

This probability is 0.64 when an option at 25 is sold and 0.76 when an option at 50 is sold.

If one sells an option against a spot purchase, the probability is 0.76 for an option at 25 and 0.86 for an option at 50.

Sale of a Futures Contract Against the Purchase of an Option. This operation is the inverse of the preceding one. On the up-side it gives a limited potential loss and on the down-side an unlimited potential gain. Consequently, it is an option at a discount, an option the spread of which is constant and the premium variable, which is the opposite of options at a premium.

Purchase of a Futures Contract Against the Sale of Two Options. Here one buys a futures at the true price - 30b and sells two options at 25 at the price 29.87.

Figure 8 represents the operation geometrically; it shows that the potential loss is limited on both sides.

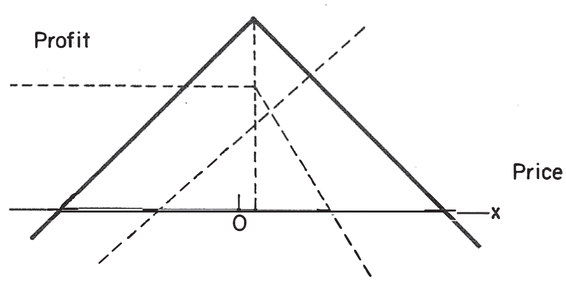

Figure 8

One will profit between the prices - (50 + 30b) and 59.74 + 30b. The probability of success is

$$\int p\,dx = 0.64.$$

Selling two options at 50, the probability of success would be 0.62, and selling two options at 10 would have a probability of success of 0.66.

If one had bought futures on 2 units and sold options at 50 on three units, the probability would have been 0.66.

Sale of a Futures Contract Against Purchase of Two Options. This operation is the opposite of the preceding one; it yields profits in the event of a large price increase and in the event of a large price decrease.

Its probability of success is 0.27.

Purchase of an Option with Large Premium Against the Sale of an Option with a Small Premium. I assume that the following two transactions occurred simultaneously: An option at 50 was bought for 12.78 and an option at 25 was sold for 29.87.

Below the base of the option with a large premium (- 37.22), both options are abandoned, and the loss is 25 centimes.

From the price - 37.22 onward, the option with a large premium is exercised, and at a price of - 12.22 the outcome of the operation is zero. There is a profit until the base of the option at 25, that is, the price 4.87, is attained.

Then the options are liquidated and one profits by the spread. Thus on the down-side the loss is 25 centimes; that is the maximum risk. On the up-side the profit is equal to the spread.

The potential loss is limited; the potential gain is also limited.

Figure 9 represents the operation geometrically.

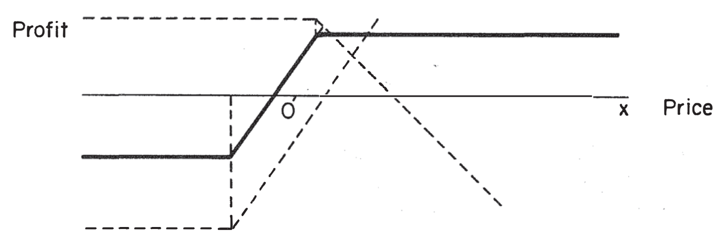

Figure 9

The probability of success is given by the integral

$$\int_{-12.22}^{\infty} p\, dx = 0.59.$$

For buying an option at 25 and selling an option at 10, the probability of success would be 0.38.

Sale of an Option with a Large Premium Against the Purchase of an Option with a Small Premium. This operation, which is the counterpart of the preceding one, may be treated simply: on the down-side the profit is the difference of the amounts of the premium, on the up-side the loss is the spread between the options.

Purchase of an Option with a Large Premium Against the Sale of Two Options with Small Premiums. I assume that the following transactions occurred: purchase of an option at 50 for 12.78, sale of two options at 25 for 29.87 (each).

For large price declines the options are abandoned: The premiums match, and it is a neutral operation. At the base of the option with a large premium, i.e., at the price - 37.22, the option with a large premium is exercised, and profits increase progressively up to the base of the options with small premiums, (4.87).

At this point profit is maximized (42.09 centimes), and the options with small premiums become exercised. Profits decrease progressively, and at the price of 45.96 the profit is zero.

Beyond this price on the up-side losses increase progressively.

In summary, the operation offers a limited potential gain, zero risk on the down-side, and unlimited potential losses on the up-side.

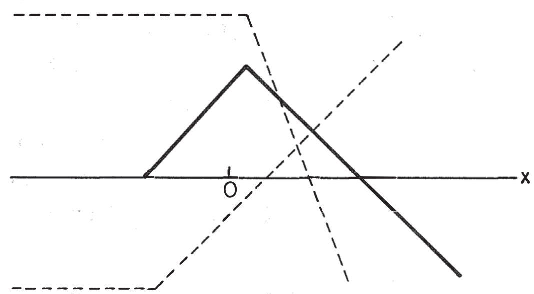

Figure 10

Probability of a neutral outcome	0.30
Probability of a gain	0.45
Probability of a loss	0.25

Sale of an Option with a Large Premium Against the Purchase of Two Options with Small Premiums. The discussion and geometrical representation of this operation, the opposite of the preceding one, offer no difficulty. It is pointless to tarry with it.

Practical Classification of Exchange Operations. From the practical point of view, one may divide exchange operations into four classes:

Operations on the up-side.

Operations on the down-side.

Operations in anticipation of a large fluctuation in any direction.

Operations in anticipation of small fluctuations.

Table 1 summarizes the principal operations on the up-side.

To obtain the scale of operations on the down-side, it is sufficient to invert this table.

Table 1

	Average Probability		
	$b = \frac{25}{30}$ (No Contango)	b = 0.26 (Average Contango)	b = 0 (Contango Equal to Spread)
Purchase option at 10	0.20	0.20	0.20
Purchase option at 25	0.33	0.33	0.33
Purchase option at 25 against sale of option at 10	0.38	0.38	0.38
Purchase option at 50	0.43	0.43	0.43
Purchase futures contract	0.64	0.55	0.50
Purchase option at 50 against sale of option at 25	0.59	0.59	0.59
Purchase futures contract against sale of option at 25	0.76	0.68	0.64
Purchase futures contract against sale of option at 50	0.86	0.80	0.76

Probability That a Price Be Attained in a Given Interval of Time

We seek the probability P that a given price c will be attained or exceeded in an interval of time t.

Assume at first, as a simplification, that time is divided into two units: that t equals two days, for instance.

Let x be the price quoted the first day and let y be the price change on the second day.

For price c to be attained or exceeded, it is necessary that on the first day price be included between c and ∞ or that on the second day it be included between $c - x$ and ∞.

In the present investigation, four cases must be distinguished.

First Day x Included Between	Second Day y Included Between
$-\infty$ and c	$-\infty$ and $c - x$
$-\infty$ and c	$c - x$ and ∞
c and ∞	$-\infty$ and $c - x$
c and ∞	$c - x$ and ∞

Of the four cases, the last three are favorable.

The probability that price will be included in the interval dx the first day and in the interval dy the second day will be $p_x p_y \, dx \, dy$.

The probability P, by definition the ratio of the number of favorable cases to the number of possible cases, will be expressed by

$$P = \frac{\int_{-\infty}^{c}\int_{c-x}^{\infty} + \int_{c}^{\infty}\int_{-\infty}^{c-x} + \int_{c}^{\infty}\int_{c-x}^{\infty}}{\int_{-\infty}^{c}\int_{-\infty}^{c-x} + \int_{-\infty}^{c}\int_{c-x}^{\infty} + \int_{c}^{\infty}\int_{-\infty}^{c-x} + \int_{c}^{\infty}\int_{c-x}^{\infty}}$$

(the element is $p_x p_y \, dx \, dy$).

The four integrals in the denominator represent the four possible cases; the three integrals in the numerator represent the three favorable cases. The denominator being equal to one, one may simplify and write

$$P = \int_{-\infty}^{c}\int_{c-x}^{\infty} p_x p_y \, dx \, dy + \int_{c}^{\infty}\int_{-\infty}^{\infty} p_x p_y \, dx \, dy.$$

One would be able to apply the same reasoning when assuming that he had to consider three consecutive days, then four, etc.

This method leads to more and more complicated expressions, since the number of favorable cases would increase indefinitely. It is much simpler to study the probability 1 - P that price c will never be attained.

Then there is only a single "favorable" case whatever the number of days, which is that where price is not attained on any of the days considered.

The probability 1 - P is expressed by

$$1 - P = \int_{-\infty}^{c}\int_{-\infty}^{c-x}\int_{-\infty}^{c-x_3-x_2} \cdots \int_{-\infty}^{c-x_1 \cdots - x_{n-1}} p_{x_1} \cdots p_{x_n} \, dx_1 \cdots dx$$

where x_1 is price on the first day,
x_2 is price change on the second day,
x_3 is the price change on the third day, etc.

The evaluation of this integral appearing difficult, we will resolve the problem by using a method of approximation.

One may consider time t as divided into small intervals such that $t = m\Delta t$. During the unit of time Δt, price will vary only by the quantity $\pm \Delta x$, the average spread relative to this unit of time.

Of the four cases, the last three are favorable.

The probability that price will be included in the interval dx the first day and in the interval dy the second day will be $p_x p_y \, dx \, dy$.

The probability P, by definition the ratio of the number of favorable cases to the number of possible cases, will be expressed by

$$P = \frac{\int_{-\infty}^{c}\int_{c-x}^{\infty} + \int_{c}^{\infty}\int_{-\infty}^{c-x} + \int_{c}^{\infty}\int_{c-x}^{\infty}}{\int_{-\infty}^{c}\int_{-\infty}^{c-x} + \int_{-\infty}^{c}\int_{c-x}^{\infty} + \int_{c}^{\infty}\int_{-\infty}^{c-x} + \int_{c}^{\infty}\int_{c-x}^{\infty}}$$

(the element is $p_x p_y \, dx \, dy$).

The four integrals in the denominator represent the four possible cases; the three integrals in the numerator represent the three favorable cases. The denominator being equal to one, one may simplify and write

$$P = \int_{-\infty}^{c}\int_{c-x}^{\infty} p_x p_y \, dx \, dy + \int_{c}^{\infty}\int_{-\infty}^{\infty} p_x p_y \, dx \, dy.$$

One would be able to apply the same reasoning when assuming that he had to consider three consecutive days, then four, etc.

This method leads to more and more complicated expressions, since the number of favorable cases would increase indefinitely. It is much simpler to study the probability 1 - P that price c will never be attained.

Then there is only a single "favorable" case whatever the number of days, which is that where price is not attained on any of the days considered.

The probability 1 - P is expressed by

$$1 - P = \int_{-\infty}^{c}\int_{-\infty}^{c-x}\int_{-\infty}^{c-x_3-x_2} \cdots \int_{-\infty}^{c-x_1-\cdots-x_{n-1}} p_{x_1} \cdots p_{x_n} \, dx_1 \cdots$$

where x_1 is price on the first day,
x_2 is price change on the second day,
x_3 is the price change on the third day, etc.

The evaluation of this integral appearing difficult, we will resolve the problem by using a method of approximation.

One may consider time t as divided into small intervals such that $t = m\Delta t$. During the unit of time Δt, price will vary only by the quantity $\pm \Delta x$, the average spread relative to this unit of time.

Each of the spreads $\pm \Delta x$ will have a probability of $\frac{1}{2}$.

We assume that $c = n\Delta x$ and seek the probability that price c will be exactly attained at the moment t, i. e., that this price will be attained at this moment t for the first time. If, during the m units of time, price has varied by the quantity $n\Delta x$, it is because there had been $(m + n)/2$ positive changes and $(m - n)/2$ negative changes.

The probability that, in m changes, there had been $(m+n)/2$ positive ones is

$$\frac{m!}{[(m - n)/2!][(m + n)/2!]}\left(\frac{1}{2}\right)^m .$$

We do not seek this probability, but the product of this probability and the ratio of the number of cases in which price $n\Delta x$ is attained at the moment $m\Delta t$ for the first time to the total number of cases in which price $n\Delta t$ is attained at the moment $m\Delta t$.

We shall calculate this ratio.

During the m units of time which we consider, there had been $(m + n)/2$ positive changes and $(m - n)/2$ negative changes.

We may represent one of the combinations giving an increase of $n\Delta x$ in m units of time by the symbol

$$D_1 U_1 U_2 \cdots D_{(m-n)/2} \cdots U_{(m+n)/2}$$

D_1 indicates that during the first unit of time there was a negative change; U_1, which follows successively, indicates that there was a positive change the second unit of time, etc.

For a combination to be favorable, it is necessary that, reading from right to left, the number of D's be constantly greater than the number of U's. We have returned, apparently, to the following problem:

Of m letters there are $(m + n)/2$ letters U and $(m - n)/2$ letters D; what is the probability that, writing these letters at random and reading them in a predetermined manner, the number of U's at all times during the reading is greater than the number of D's?

The solution of this problem, stated in a slightly different form, has been given by M. André. The desired probability is equal to n/m.

Thus the probability that price $n\Delta x$ will be attained for the first time at the end of m units of time is

$$\frac{n}{m}\frac{m!}{[(m - n)/2!][(m + n)/2!]}\left(\frac{1}{2}\right)^m .$$

This is an approximate expression. We will obtain a more exact formula by substituting the exact value of the probability at the moment t for the quantity by which n/m is multiplied, i.e., by

$$\frac{\sqrt{2}}{\sqrt{m}\sqrt{\pi}} e^{-n^2/\pi m}.$$

The probability we seek is therefore

$$\frac{n\sqrt{2}}{m\sqrt{m}\sqrt{\pi}} e^{-n^2/\pi m},$$

or, substituting $2c\sqrt{\pi}/\sqrt{2}$ for n and $8\pi k^2 t$ for m,

$$\frac{dt\, c\sqrt{2}}{2kt\sqrt{t}\sqrt{\pi}} e^{-c^2/4\pi k^2 t}.$$

This is the expression for the probability that price c will be attained for the first time at the moment dt.

The probability that price c will not be attained before the moment t will have a value of

$$1 - P = A \int_t^{\infty} \frac{c\sqrt{2}}{2kt\sqrt{t}\sqrt{\pi}} e^{-c^2/4\pi k^2 t}\, dt.$$

I have multiplied the integral by an arbitrary constant A because the price can be attained only if the quantity m is even.

Letting

$$\lambda^2 = \frac{c^2}{4\pi k^2 t},$$

one has

$$1 - P = 2\sqrt{2}\, A \int_0^{\frac{c}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda.$$

To determine A, set $c = \infty$, then $P = 0$ and

$$1 = 2\sqrt{2} A \int_0^{\infty} e^{-\lambda^2}\, d\lambda = \sqrt{2}\sqrt{\pi} A,$$

thus

$$A = \frac{1}{\sqrt{2}\sqrt{\pi}},$$

then

$$1 - P = \frac{2}{\sqrt{\pi}} \int_0^{\frac{c}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda.$$

The probability that price x will be attained or exceeded during the interval of time t is expressed by

$$P = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda.$$

The probability that price x will be attained or exceeded at the moment t, as we have seen, is given by

$$\mathcal{P} = \frac{1}{2} - \frac{1}{2}\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda.$$

Evidently $\mathcal{P}$ is one-half of P.

The probability that a price will be attained or exceeded at the moment t is one-half of the probability that this price will be attained or exceeded in the interval of time t.

The interpretation of this result is very simple: The price cannot be exceeded at the moment t without having been attained previously. The probability $\mathcal{P}$ is therefore equal to the probability P multiplied by the probability that, given that the price was quoted previously, it will be exceeded at the moment t, i.e., multiplied by $\frac{1}{2}$. Thus

$$\mathcal{P} = \frac{P}{2}.$$

One may notice that the multiple integral which expresses the probability 1 - P and which seems unamenable to ordinary methods of calculus is determined by a very simple reasoning due to the calculus of probabilities.

Applications. Tables of the function θ allow one very easily to calculate the probability

$$P = 1 - \theta\left(\frac{x}{2\sqrt{\pi}k\sqrt{t}}\right).$$

The formula

$$P = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda$$

shows that the probability is constant when the spread x is proportional to the square root of time, i.e., when there is an expression of the form x = ma. We shall study the probabilities of certain interesting spreads.

Assume at first that $x = a = k\sqrt{t}$. Then the probability P is equal to 0.69. When the spread a is attained, one may hedge a simple option by selling a futures contract.[11]

Thus, there are two chances in three that one might be able to hedge a simple option by selling a futures contract.

We specifically consider the point in appplying it to 3% bills. Over a period of 60 months, 38 times one had been able to sell a futures contract at a spread of a, which corresponds to a probability of 0.63.

Now we study the case where x = 2a.

The preceding formula gives a probability of 0.43.

When the spread of 2a is attained, one can hedge an option with a premium of 2a by selling a futures contract. Thus:

There are four chances in ten that one might be able to hedge an option with a premium 2a by selling a futures contract.

The spread of 0.7a is the spread of call-o'-mores of order two; the corresponding probability is 0.78.

One has three chances in four of being able to hedge a call-o'-more of order two by selling a futures contract.

The call-o'-more of order three is traded at a spread of 1.1a, to which the probability of 0.66 corresponds.

There are two chances in three of being able to hedge a call-o'-more of order three by selling a futures contract.

Finally, we cite as spreads worthy of notice the spread of 1.7a which corresponds to a probability of 0.5 and the spread of 2.9a which corresponds to a probability of 0.25.

Apparent Mathematical Expectation. The mathematical expectation

$$M_1 = P_x = x - \frac{2x}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda$$

is a function of x and of t. We differentiate it with respect to x and have

$$\frac{\partial M_1}{\partial x} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda - \frac{xe^{-x^2/4\pi k^2 t}}{\pi k\sqrt{t}}.$$

If one considers a fixed moment t, this expectation will be maximized when

$$\frac{\partial M_1}{\partial x} = 0,$$

that is, when $x = 2a$, approximately.

Apparent Total Expectation. The total expectation corresponding to time t will be the integral

$$\int_0^{\infty} Px\, dx.$$

Let

$$f(a) = \int_0^{\infty} \left(x - \frac{2x}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda \right) dx.$$

We differentiate with respect to a and have

$$f'(a) = \frac{1}{\pi a^2} \int_0^{\infty} x^2 e^{-x^2/4\pi a^2}\, dx,$$

or $f'(a) = 2\pi a$. Thus

$$f(a) = \pi a^2 = \pi k^2 t.$$

The total expectation is proportional to time.

The Most Likely Moment. The probability

$$P = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda$$

is a function of x and of t.

The study of its variation, considering x as variable, presents no distinctive features: The function steadily decreases as x increases.

Now we assume that x is constant and study the variation of the function considering t as variable. We differentiate and have

$$\frac{\partial P}{\partial t} = \frac{xe^{-x^2/4\pi k^2 t}}{2\pi t\sqrt{t}}.$$

We will evaluate the most likely moment by setting equal to zero the derivative

$$\frac{\partial^2 P}{\partial t^2} = \frac{xe^{-x^2/4\pi k^2 t}}{2\pi t\sqrt{t}} \left(\frac{x^2}{4\pi k^2 t} - \frac{3}{2}\right);$$

then

$$t = \frac{x^2}{6\pi k^2}.$$

Assume, for instance, that $x = k\sqrt{t_1}$; then $t = (t_1/6\pi)$.

The most likely moment at which one may hedge a simple option by selling a futures contract is located at one-eighteenth of the duration until the expiration date.

If we now assume that $x = 2k\sqrt{t_1}$, we obtain $t = (2t_1/3\pi)$.

The most likely moment at which one may hedge an option of premium 2a by selling a futures contract is located at one-fifth of the duration until the expiration date.

The probability P corresponding to the moment $t = (x^2/6\pi k^2)$ has a value of $1 - \theta(\sqrt{6}/2) = 0.08$.

Average Moment. When an event can occur at different moments, we call the average moment of the occurrence of the event the sum of the products of the probabilities corresponding to the given moments and their respective durations.

The average moment is equal to the sum of the expectations of duration.

The average moment at which price x will be exceeded is thus expressed by the integral

$$\int_0^\infty t\frac{dP}{dt}dt = \int_0^\infty \frac{x}{2\pi k\sqrt{t}} e^{-x^2/4\pi k^2 t} dt.$$

Setting $(x^2/4\pi k^2 t) = y^2$,

$$\frac{x^2}{2\pi\sqrt{\pi}k^2}\int_0^\infty \frac{e^{-y^2}}{y^2} dy.$$

This integral is infinite.

Therefore the average moment is infinite.

Absolute Probable Moment. This will be the moment for which $P = \frac{1}{2}$, or

$$\theta\left(\frac{x}{2\sqrt{\pi}k\sqrt{t}}\right) = \frac{1}{2},$$

which reduces to

$$t = \frac{x^2}{2.89k^2}.$$

The absolute probable moment varies, just as the most likely moment, proportionally to the square of the quantity x, and it is about six times as large as the most likely moment.

Relative Probable Moment. It is interesting to know not only the probability that a price x will be quoted in an interval of time t, but also the probable moment T at which this price must be attained. This moment obviously is different from that with which we have just been concerned.

The interval of time 0,T will be such that there will be as many chances that the price will be attained before the moment T as there will be that it will be quoted thereafter, that is, in the interval of time T,t.

T will be given by the formula

$$\int_0^T \frac{\partial P}{\partial t} = \frac{1}{2} \int_0^t \frac{\partial P}{\partial t}\, dt$$

or

$$1 - 2\,\theta\left(\frac{x}{2\sqrt{\pi}k\sqrt{t}}\right) = -\,\theta\left(\frac{x}{2\sqrt{\pi}k\sqrt{t}}\right).$$

As an application, assume that $x = k\sqrt{t}$; the formula gives $T = 0.18t$. Thus:

One has as many chances to be able to hedge a simple option by selling a futures contract during the first fifth of the duration of the option contract as during the remaining four-fifths.

To treat a specific example, we assume that the security concerned is a bond and that t = 30 days; then T will be equal to five days. Thus there are as many chances, the formula states, that one can hedge an option with a premium of a (on the average, about 28 centimes) during the first five days as there are chances that one would be able to hedge it in the subsequent twenty-five days. Among the 60 liquidations on which our observations bear, the spread had been attained thirty-eight times: eighteen times during the first four days, twice on the fifth day, and eighteen times after the fifth day.

The observations are therefore in agreement with the theory.

We now assume that $x = 2k\sqrt{t}$; we find that $T = 0.42t$. Now, the quantity $2k\sqrt{t}$ is the spread of the option with twice the spread of a simple option; thus one may say:

There are as many chances that one would be able to hedge an option with premium 2a by selling a futures contract during the first four-tenths of the duration of the option contract as during the other six-tenths.

Still concerning bonds, our previous observations have shown that, in twenty-three cases out of 60 liquidations, the spread 2a (about 56 centimes on average) had been attained. Of these twenty-three cases, the spread had been attained eleven times before the fourteenth of the month and twelve times after this date.

The probable moment would be $0.11\sqrt{t}$ for the call-o'-more of order two and $0.21\sqrt{t}$ for the call-o'-more of order three.

Finally, the probable moment would be one-half of the total moment (the moment when the transaction expires) if x were equal to $2.5k\sqrt{t}$.

Distribution of Probability. Thus far we have solved two problems:

Discovery of the probability [that price x will be achieved for the first time] at the moment T.

Discovery of the probability that a price will be attained in a given interval of time t.

We shall solve this last problem exhaustively. It is not sufficient to know the probability that a price will be attained before the moment t; it is also necessary to know the law of probability at the moment t given that the price, c, is not attained.[12]

I assume, for example, that we buy a bond to sell it at a profit of c. If at the moment t we had not been able to consummate the sale, what, at this moment, will be the law of probability that applies to our operation?

If price c has not been attained, it is because the maximum positive variation had been less than c, although the negative variation may have been indefinitely great. Thus there is an obvious asymmetry to the probability curve at the moment t.

We seek the form of this curve.

Let ABCEG be the probability curve at the moment t, assuming that the operation had to survive until this moment (Figure 11).

The probability that price c will be exceeded at the moment t is represented by the area DCEG which, obviously, will not be part of the probability curve for the case of a possible sale.

We can even affirm a priori that the area under the probability curve, in this case, must be decreased by twice the quantity equal to DCEG, since the probability P is twice the probability represented by DCEG.

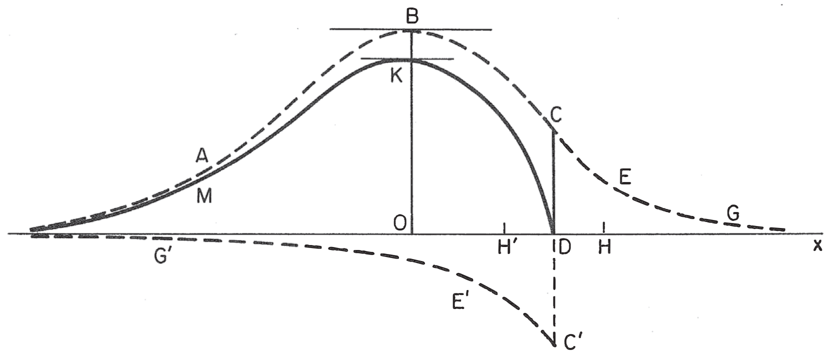

Figure 11

If price c is attained at the moment t, price H will have the same probability as the symmetrical price H' at this instant.

The possibility of sale at price c thus simultaneously decreases the probability of H' by an amount equal to the suppressed probability of H, and to have the probability at moment t we must reduce the ordinates of the curve ABC by those of the curve G'E'C' symmetrical to GEC. The desired probability curve then will be the curve DKM.

The equation of this curve is

$$P = \frac{1}{2\pi k\sqrt{t}} \left\{ \exp\left(- \frac{x^2}{4\pi k^2 t}\right) - \exp\left[-\frac{(2c - x)^2}{4\pi k^2 t}\right] \right\}.$$

Curve of Maximum Probability. To obtain the curve of the greatest probability in the case where price c has not been attained it is sufficient to set $dp/dx = 0$; thus one obtains

$$\frac{x}{2c - x} + \exp\left[- \frac{c(c - x)}{\pi k^2 - t}\right] = 0.$$

If one assumes that $c = a = k\sqrt{t}$, he obtains

$$x_m = - 1.5a.$$

If one assumes that $c = 2a$, he obtains

$$x_m = - 0.4a.$$

Finally, one would obtain

$$x_m = -c$$

if c were equal to 1.33a.

Probable Price. We seek the expression for the probability in the interval 0, u. This will be

$$\frac{1}{2\pi k\sqrt{t}}\int_0^u \exp\left(-\frac{x^2}{4\pi k^2 t}\right)dx - \frac{1}{2\pi k\sqrt{t}}\int_0^u \exp\left[-\frac{(2c-x)^2}{4\pi k^2 t}\right]dx.$$

The first term has a value of

$$\frac{1}{2}\theta\left(\frac{u}{2\sqrt{\pi}k\sqrt{t}}\right).$$

In the second term, set

$$2\sqrt{\pi}k\sqrt{t}\lambda = 2c - x;$$

this term will become

$$-\frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\frac{2c}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\,d\lambda + \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\frac{2c-u}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\,d\lambda.$$

The desired expression for the probability is therefore

$$\frac{1}{2}\theta\left(\frac{u}{2\sqrt{\pi}k\sqrt{t}}\right) - \frac{1}{2}\theta\left(\frac{2c}{2\sqrt{\pi}k\sqrt{t}}\right) + \frac{1}{2}\theta\left(\frac{2c-u}{2\sqrt{\pi}k\sqrt{t}}\right).$$

It is interesting to study the case where u = c to learn the probability of gain for a purchase of a futures contract when the (predetermined re-sale) price has not been attained.

The above formula, under the hypothesis that u = c, becomes

$$\theta\left(\frac{c}{2\sqrt{\pi}k\sqrt{t}}\right) - \frac{1}{2}\theta\left(\frac{2c}{2\sqrt{\pi}k\sqrt{t}}\right).$$

Assume that c = a; then the probability is 0.03.

If the spread a has never been attained in the interval t, there are only three chances in one hundred that, at the moment t, price will be included between zero and a.

One could buy a simple option with the intention of selling a futures against this option when its spread will be attained.

The probability of that sale is, as we have seen, 0.69. The probability that the sale will not occur and that there will be a profit is 0.03, and the probability that there will be a loss is 0.28.

Assume that $c = 2a$; then the probability is 0.13.

If the spread $2a$ has never been attained in the interval t, there are thirteen chances in one hundred that, at the moment t, price will be included in the interval zero and $2a$.

The probable price is that one the ordinate of which divides the area of the probability curve into two equal parts. It is not possible to express its value in finite terms.

Real Expectation. The mathematical expectation $k\sqrt{t} = a$ expresses the expectation of an operation which must continue up to the moment t.

If one proposes to cash in on an operation in the case where a certain spread will be attained before the moment t, the mathematical expectation has an entirely different value, obviously varying between zero and $k\sqrt{t}$ when the chosen spread varies between zero and infinity.

Let c be the price at which a purchase will be made, for instance. To obtain the real positive expectation of the operation, one must add to the expectation of sale, cP, the positive expectation corresponding to the case where the sale does not occur, i.e., the quantity

$$\int_0^c \frac{x}{2\pi k\sqrt{t}} \left\{ \exp\left(-\frac{x^2}{4\pi k^2 t}\right) - \exp\left[-\frac{(2c-x)^2}{4\pi k^2 t}\right] \right\} dx.$$

If one carries out the integration of the first term and adds the entire integral of the expectation of resale

$$cP = c - c\frac{2}{\sqrt{\pi}} \int_0^{\frac{c}{2\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda,$$

one obtains for the expression of the real expectation

$$M = c + k\sqrt{t}\left[1 - \exp\left(-\frac{c^2}{\pi k^2 t}\right)\right] - c\frac{2}{\sqrt{\pi}} \int_0^{\frac{c}{\sqrt{\pi}k\sqrt{t}}} e^{-\lambda^2}\, d\lambda,$$

or

$$M = c + k\sqrt{t}\left[1 - \exp\left(-\frac{c^2}{\pi k^2 t}\right)\right] - c\theta\left(\frac{c}{\sqrt{\pi}k\sqrt{t}}\right).$$

If one assumes $c = \infty$, he again finds $M = k\sqrt{t}$. One could easily expand M as a series, but the preceding formula is more convenient. It is evaluated with tables of logarithms and of the function θ.

For $c = a$, one obtains

$$M = 0.71a.$$

Similarly, for $c = 2a$

$$M = 0.95a.$$

For the same spreads, the expectations of sale would be 0.69a and 0.86a.

The value of the average spread on the down-side, when price c is not attained, is

$$\frac{\int_{-\infty}^{0} px\,dx}{\int_{-\infty}^{0} p\,dx} = \frac{M}{1 - P - P_1},$$

P_1 designating the quantity

$$\int_{0}^{c} p\,dx.$$

Thus the average spread has the value 2.54a when $c = a$ and 2.16a when $c = 2a$.

If one assumes $c = \infty$, he sees that the average spread equals 2a, a previously obtained result.

We again take up, as an example, the general problem relating to the spread a. I buy a futures contract with the intention of reselling it with the spread $a = k\sqrt{t}$. If the sale has not been effected by the moment t, I will sell regardless of the price.

What are the principal findings furnished by the calculus of probability for this operation?

The real positive expectation of the operation is 0.71a.

The probability of resale (at the specified spread) is 0.69.

The most likely moment of the resale is t/18.

The probable moment of resale is t/5.

If the resale does not occur, the conditional probability of success is 0.03, the probability of failure is 0.28, the positive expectation is 0.02a, the negative expectation is 0.71a, and the expected loss is 2.54a.

The total probability of success is 0.72.

I do not believe it necessary to present other examples. It is evident that the present theory resolves the majority of problems in the study of speculation by the calculus of probability.

Perhaps a final remark will not be pointless. If, with respect to several questions treated in this study, I have compared the results of observation with those of theory, it was not to verify formulas established by mathematical methods, but only to show that the market, unwittingly, obeys a law which governs it, the law of probability.

Seen and approved:
Paris, January 6, 1900.
THE DEAN OF THE FACULTY OF SCIENCES,

G. DARBOUX.

Seen and allowed for publication:
Paris, January 6, 1900.
THE VICE-RECTOR OF THE ACADEMY OF PARIS,

GRÉARD.

Editor's Notes

1. Since the institutional arrangements on Continental security markets differ in several critical regards from American security markets, the reader may find this section of Bachelier's monograph quite confusing. To avoid this, a digression at this point might be helpful.

   In U. S. markets, payment for securities purchased is made shortly (usually within four days) after the transaction, and no allowance is made for the opportunity cost of funds during this short delay. In Europe, however, transactions are frequently made for settlement at some fixed date in the future. In Bachelier's futures transactions, price is specified at the time of transaction but payment is made at the end of the month. Since the buyer of a "future" acquires all the rights of ownership immediately, he receives any interest paid on the bond between the time of the transaction and the settlement, or "liquidation," date. Since the seller receives neither his interest nor the selling price, he must be recompensed for his opportunity loss by the payment of contango.

   Another difference between French and American bond markets may confuse readers. In the U. S., bonds are normally quoted "net" of accumulated interest. That is, the quoted price is the value of the bond excluding all accumulated interest. If an American buys a bond between interest payment dates, he must pay, in addition to the price quoted in his newspaper, an amount equal to the accumulated interest on the bond. In Bachelier's France, the quoted price included the accumulated interest. As a result, the quoted price dropped by the amount of the interest payment on the date it was payable, similar to the procedure used with dividends on American stocks.

2. Here again, Bachelier's terminology is confusing to an American reader. Specifically, an option is an option to buy a security at a fixed price at a fixed time in the future. If the option is not exercised, the buyer of an option must pay a forfeit, called a premium, to the seller. It is like an American call but differs in several ways:

   a. The American call is exercisable at any time between the transaction date and the expiration of the call. The French option is exercisable only on the day before the termination date.
   b. The American call is paid for at purchase; the French option is paid for at termination. If unexercised, the payment is made in cash. If exercised, it is included in the purchase price. It is this difference which creates the biggest terminological difficulties to an American reader, a point to which we will return in a moment.
   c. In American calls, the exercise price is typically the current market price, and the premium paid varies with the stock being considered. In French options, the premium is typically fixed and the exercise price varies with the stock being considered.

We compared the terminology of Bachelier with American usage: In his example, the phrase "an option at 50 centimes" refers to an option with premium fixed at 50 centimes per 100 francs par value; the 50 centimes is equivalent to what Americans loosely refer to as "the price" of the call. Since so many different "prices" are involved in this discussion we will call it the premium. Bachelier (loosely) says the option is purchased at 104.15: What is actually purchased, of course, is the right to buy bonds (in a spot transaction) at 104.15. Since, if the option is not exercised, the buyer must pay at 50 centime forfeit, it is to his advantage to exercise if the spot price at expiration is greater than 103.65. American terminology stresses the lower price, 103.65, (called the striking price) rather than the price including premium, largely because the premium is paid at purchase. Thus, an American would refer to a call on these terms as having a striking price of 103.65 and a premium of .50.

Although Bachelier does not make it clear at this point, French options, like U. S. calls, grant the buyer the right to receive all payments made on the security, if it is exercised.

3. This procedure, known as conversion in the U. S. market, is described at greater length in Kruizenga, below.

4. The spread is equal to the difference between what Bachelier calls the "price" of an option and the price of a futures contract. Remember, the price of an option is the striking price plus the premium.

5. This is one of, if not the, first expression of what is now known as the Chapman-Kolmogorov-Smoluchowski equation.

6. What Bachelier had done here is to set

$$\frac{\partial f^2(t_1+t_2)}{\partial t_2} = \frac{\partial f^2(t_1+t_2)}{\partial t_1} .$$

7. This is the first recognition that the Wiener process satisfies the Fourier, or diffusion equation. This point was rediscovered by Albert Einstein in his renowned paper on Brownian motion in 1905.

8. This is the usual kind of U. S. call, with striking price equal to market price.

9. Bachelier's definition of success is not really very satisfactory for an economist. He makes no allowance for opportunity costs of funds. Thus a spot purchase of bonds at (say) 100 francs which yields, one month later, 100.01 including interest, is by Bachelier's definition "success" even though the implied return on funds is .12% per annum. This is true throughout the paper.

10. The reader might be interested to know, at this point, that this pattern also tends to occur in the U. S. market, and is reflected, in the Boness and Kruizenga papers, by the showing of poorer results from purchase of shorter term calls.

11. It may be hard for students of the American put and call market to understand the motivation for what follows. Since the French option is only exercisable at expiration, a buyer who has an option on a bond whose price has risen by a given amount (in this case, a) may wish to exercise the option, but will be unable to. He can achieve almost the same purpose by selling a future with the same expiration date. Then if the

futures price continues to rise, the loss on the sale offsets the gain on the option and his profit cannot increase. On the other hand, if the futures price falls by a or less, the profit on the sale offsets the loss on the option transaction. If, however, the futures price falls more than a, the loss on the option is limited to the premium, but the sale of futures continues to show a profit.

The sale of a future to offset an option purchase converts the option at a premium into an option at a discount. The reader can either verify this himself or refer to Kruizenga, below.

12. Although Bachelier's statement of the problem is to find the probability of x at time t, given that the price c is not attained in the interval (0, t), this probability is also the solution to the now classical case of diffusion in the presence of a single absorbing barrier at c. (By an absorbing barrier at c, I mean that, if the price reaches c, it stops, i. e., it is "absorbed.") The proof used by Bachelier uses the now classic "method of images" or "method of reflection."