# Independence and Bayesian Networks (Part 1)

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Lecture 7

Readings: RN 12.4, 13.1, & 13.2. PM 8.2 & 8.3.

# Outline

# Learning Goals

- Given a probabilistic model,
  determine if two variables are unconditionally independent, or conditionally independent given a third variable.
- Give examples of deriving a compact representation of a joint distribution by using independence assumptions.
- Describe components of a Bayesian network.
- Compute a joint probability given a Bayesian network.
- Explain the independence relationships in the three key structures.

# (Unconditional) Independence

Definition ((unconditional) independence)

$X$ and $Y$ are (unconditionally) independent iff

$$P(X|Y) = P(X)$$

$$P(Y|X) = P(Y)$$

$$P(X \wedge Y) = P(X)P(Y)$$

Learning $Y$ does NOT influence your belief about $X$.

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Learning $Y$ does NOT influence your belief about $X$.

$\rightarrow$ Convert between the two equations.

To specify joint probability, it is sufficient to specify the individual probabilities.

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Learning $Y$ does NOT influence your belief about $X$.

$\rightarrow$ To justify that

$$P(X \wedge Y) = P(X)P(Y)$$

we need to make four comparisons.

# Conditional Independence

## Definition (conditional independence)

$X$ and $Y$ are conditionally independent given $Z$ if

$$P(X|Y \land Z) = P(X|Z).$$

$$P(Y|X \land Z) = P(Y|Z).$$

$$P(Y \land X|Z) = P(Y|Z)P(X|Z).$$

Learning $Y$ does NOT influence your belief about $X$ if you already know $Z$.

# Conditional Independence

## Definition (conditional independence)

$X$ and $Y$ are conditionally independent given $Z$ if

$$P(X|Y \wedge Z) = P(X|Z).$$

$$P(Y|X \wedge Z) = P(Y|Z).$$

$$P(Y \wedge X|Z) = P(Y|Z)P(X|Z).$$

Learning $Y$ does NOT influence your belief about $X$ if you already know $Z$.

$\rightarrow$ $X$ is conditionally independent of $Y$ given $Z$.

Independence does not imply conditional independence, and vice versa.

# Conditional Independence

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$X$ and $Y$ are conditionally independent given $Z$ if

$$P(X|Y \land Z) = P(X|Z).$$

$$P(Y|X \land Z) = P(Y|Z).$$

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$$P(Y \land X|Z) = P(Y|Z)P(X|Z).$$

Learning $Y$ does NOT influence your belief about $X$ if you already know $Z$.

$\rightarrow$ To justify that

$$P(X \land Y|Z) = P(X|Z)P(Y|Z)$$

we need to make eight comparisons.

# Q #1: Deriving a compact representation

**Q:** Consider a model with three random variables, $A, B, C$. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

# Q #1: Deriving a compact representation

**Q:** Consider a model with three random variables, $A, B, C$. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

$\rightarrow$ (C) $P(A), P(B|A), P(C|A \wedge B)$. $1 + 2 + 4 = 7$ probabilities

Draw a graph to prove it to yourself.

# Q #2: Deriving a compact representation

**Q:** Consider a model with three random variables, $A, B, C$. Assume that $A$, $B$, and $C$ are independent. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

# Q #2: Deriving a compact representation

**Q:** Consider a model with three random variables, $A, B, C$. Assume that $A$, $B$, and $C$ are independent. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

$\rightarrow$ (A) $P(A), P(B), P(C)$. $1 + 1 + 1 = 3$ probabilities

Draw a graph to prove it to yourself.

# Q #3: Deriving a compact representation

**Q:** Consider a model with three boolean random variables, $A, B, C$. Assume that $A$ and $B$ are conditionally independent given $C$. What is the minimum number of probabilities required to specify the joint distribution?

(A) 4

(B) 5

(C) 7

(D) 11

# Q #3: Deriving a compact representation

**Q:** Consider a model with three boolean random variables, $A, B, C$. Assume that $A$ and $B$ are conditionally independent given $C$. What is the minimum number of probabilities required to specify the joint distribution?

(A) 4

(B) 5

(C) 7

(D) 11

$\rightarrow$ (B) $P(C), P(A|C), P(B|C)$. $1 + 2 + 2 = 5$ probabilities

Draw a graph to prove it to yourself.

# Q #3a: Deriving a compact representation

**Q:** Given the joint probability distribution about A, B, and C, how many comparisons do we need to make to justify A and B are independent given C?

(A) 1

(B) 4

(C) 8

(D) 10

# Q #3a: Deriving a compact representation

**Q:** Given the joint probability distribution about A, B, and C, how many comparisons do we need to make to justify A and B are independent given C?

(A) 1

(B) 4

(C) 8

(D) 10

$\rightarrow$ (C)
$p(B = T, A = T|C = T) = p(B = T|A = T) * p(C = T|C = T)$
$p(B = T, A = F|C = T) = p(B = T|C = T) * p(B = F|C = T)$
... A total of 8 eqaulities!

# Q #3b: Deriving a compact representation

**Q:** Read the table to understand whether B and C are independent given A.

| A | B | C | Prob |
|---|---|---|---|
| T | T | T | 0.16 |
| T | T | F | 0.16 |
| T | F | T | 0.24 |
| T | F | F | 0.24 |
| F | T | T | 0.012 |
| F | T | F | 0.008 |
| F | F | T | 0.108 |
| F | F | F | 0.072 |

(A) B and C are independent given A

(B) B and C are not independent given A

# Q #3b: Deriving a compact representation

Read the table to understand whether B and C are independent given A.

| A | B | C | Prob |
|---|---|---|---|
| T | T | T | 0.16 |
| T | T | F | 0.16 |
| T | F | T | 0.24 |
| T | F | F | 0.24 |
| F | T | T | 0.012 |
| F | T | F | 0.008 |
| F | F | T | 0.108 |
| F | F | F | 0.072 |

- Compute $p(B, C|A)$
- Compute $p(B|A)$ and $p(C|A)$
- Verify $p(B, C|A) = p(B|A) * p(C|A)$

# Q #3b: Step-by-Step Derivation $p(B, C|A)$

| A | B | C | Prob |
|---|---|---|---|
| T | T | T | 0.16 |
| T | T | F | 0.16 |
| T | F | T | 0.24 |
| T | F | F | 0.24 |
| F | T | T | 0.012 |
| F | T | F | 0.008 |
| F | F | T | 0.108 |
| F | F | F | 0.072 |

Table: Merging $p(A, B, C)$.

- $p(B, C|A) = p(B, C, A)/p(A)$
- $p(A) = (0.16 + 0.16 + 0.24 + 0.24, 0.012 + 0.008 + 0.108 + 0.072) = (0.8, 0.2)$

# Q #3b: Step-by-Step Derivation $p(B, C|A)$

| B | C | (A) | Prob |
|---|---|---|---|
| T | T | T | 0.16 / 0.8 = 0.2 |
| T | F | T | 0.16 / 0.8 = 0.2 |
| F | T | T | 0.24 / 0.8 = 0.3 |
| F | F | T | 0.24 / 0.8 = 0.3 |
| T | T | F | 0.012 / 0.2 = 0.06 |
| T | F | F | 0.008 / 0.2 = 0.04 |
| F | T | F | 0.108 / 0.2 = 0.54 |
| F | F | F | 0.072 / 0.2 = 0.36 |

Table: Computing $p(B, C|A)$.

- $p(B, C|A) = p(B, C, A)/p(A)$
- $p(A) = (0.8, 0.2)$
- $p(B, C|A)$ is displayed in the table

# Q #3b: Step-by-Step Derivation

| A | B | C | Prob |
|---|---|---|---|
| T | T | T | 0.16 |
| T | T | F | 0.16 |
| T | F | T | 0.24 |
| T | F | F | 0.24 |
| F | T | T | 0.012 |
| F | T | F | 0.008 |
| F | F | T | 0.108 |
| F | F | F | 0.072 |

Table: Merge $p(A, B, C)$

- Marginalizing over variable $C$

# Q #3b: Step-by-Step Derivation $p(B|A)$

| A | B | Prob |
|---|---|---|
| T | T | 0.32 |
| T | F | 0.48 |
| F | T | 0.02 |
| F | F | 0.18 |

Table: Computing $p(A, B)$

- Marginalizing over variable $C$
- Joint $p(A, B)$ is displayed in the table

# Q #3b: Step-by-Step Derivation $p(B|A)$

| B | (A) | Prob |
|---|---|---|
| T | T | $0.32 \ / \ 0.8 = 0.4$ |
| F | T | $0.48 \ / \ 0.8 = 0.6$ |
| T | F | $0.02 \ / \ 0.2 = 0.1$ |
| F | F | $0.18 \ / \ 0.2 = 0.9$ |

Table: Computing $p(B|A)$

- Marginalizing over variable $C$
- Conditional $p(B|A)$ is displayed in the table

# Q #3b: Step-by-Step Derivation $p(C|A)$

| A | B | C | Prob |
|---|---|---|---|
| T | T | T | 0.16 |
| T | F | T | 0.24 |
| T | T | F | 0.16 |
| T | F | F | 0.24 |
| F | T | T | 0.012 |
| F | F | T | 0.108 |
| F | T | F | 0.008 |
| F | F | F | 0.072 |

Table: Merging $p(A, B, C)$

- Marginalizing over variable $B$

# Q #3b: Step-by-Step Derivation $p(C|A)$

| A | C | Prob |
|---|---|---|
| T | T | 0.4 |
| T | F | 0.4 |
| F | T | 0.12 |
| F | F | 0.08 |

Table: Computing $p(A, C)$

- Marginalizing over variable $B$

# Q #3b: Step-by-Step Derivation $p(C|A)$

| C | (A) | Prob |
|---|---|---|
| T | T | 0.4 / 0.8 = 0.5 |
| F | T | 0.4 / 0.8 = 0.5 |
| T | F | 0.12 / 0.2 = 0.6 |
| F | F | 0.08 / 0.2 = 0.4 |

Table: Computing $p(C|A)$

- Marginalizing over variable $B$
- Computing $p(C|A)$

# Q #3b: Step-by-Step Derivation (Verification)

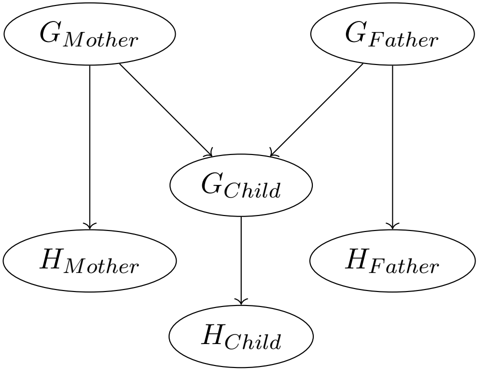
| B | (A) | Prob |
|---|---|---|
| T | T | 0.4 |
| F | T | 0.6 |
| T | F | 0.1 |
| F | F | 0.9 |

| C | (A) | Prob |
|---|---|---|
| T | T | 0.5 |
| F | T | 0.5 |
| T | F | 0.6 |
| F | F | 0.4 |

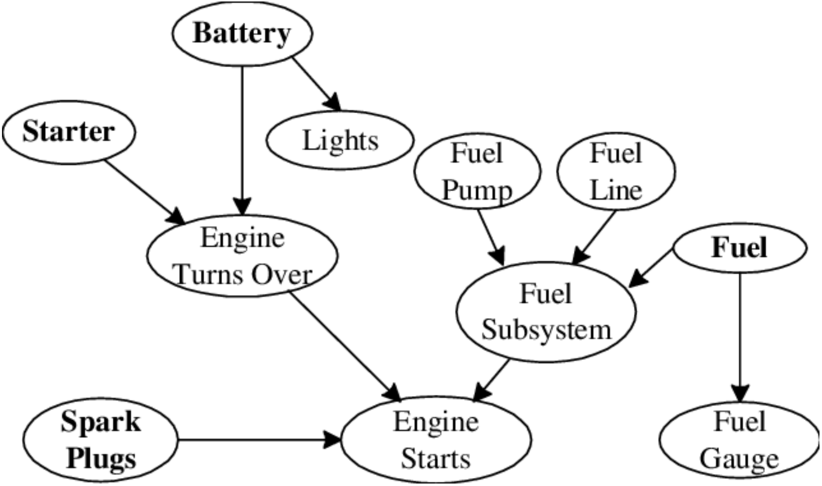
| B | C | (A) | Prob |
|---|---|---|---|
| T | T | T | 0.5 * 0.4 == 0.2 |
| T | F | T | 0.5 * 0.4 == 0.2 |
| F | T | T | 0.5 * 0.6 == 0.3 |
| F | F | T | 0.5 * 0.6 == 0.3 |
| T | T | F | 0.6 * 0.1 == 0.06 |
| T | F | F | 0.4 * 0.1 == 0.04 |
| F | T | F | 0.6 * 0.9 == 0.54 |
| F | F | F | 0.4 * 0.9 == 0.36 |

All of the probabilities are equal, therefore $B$ and $C$ are independent given A.

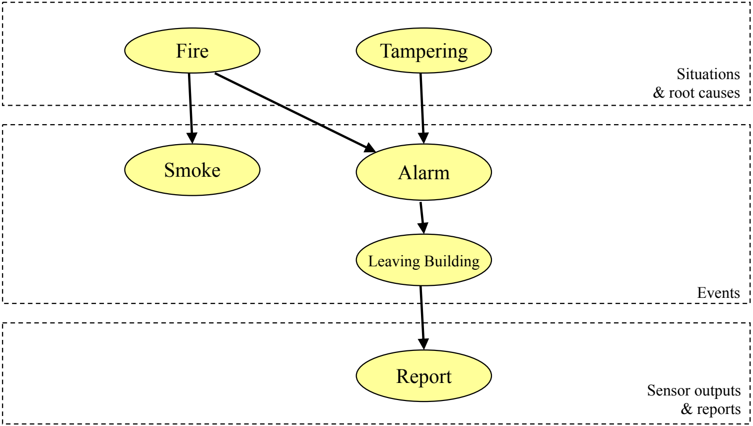
# Inheritance of Handedness



# Car Diagnostic Network

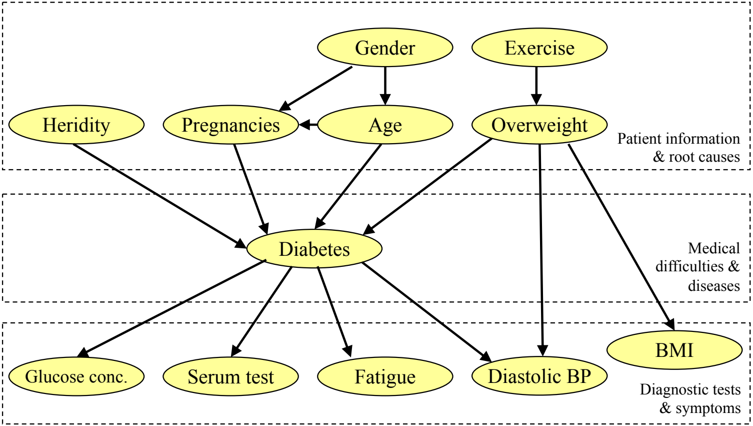


# Example: Fire alarms




Report: “report of people leaving building because a fire alarm went off”

# Example: Medical diagnosis of diabetes



# Why Bayesian Networks?

A probabilistic model of the Holmes scenario:

- The random variables:
  Earthquake, Radio, Burglary, Alarm, Watson, and Gibbon.
- \# of probabilities in the joint distribution: $2^6 = 64$.
- For example,

  $P(E \wedge R \wedge B \wedge A \wedge W \wedge G) =?$

  $P(E \wedge R \wedge B \wedge A \wedge W \wedge \neg G) =?$

  ... etc ...

We can compute any probability using the joint distribution, but

- Quickly become intractable as the number of variables grows.
- Unnatural and tedious to specify all the probabilities.

# Why Bayesian Networks?

A Bayesian Network

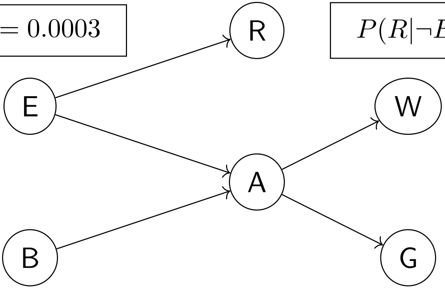
- is a compact version of the joint distribution
- takes advantage of the unconditional and conditional independence among the variables.

# Reminder: Modelling the Holmes Scenario

→ The random variables:

- B: A Burglary is happening.
- A: The alarm is going.
- W: Dr. Watson is calling.
- G: Mrs. Gibbon is calling.
- E: Earthquake is happening.
- R: A report of earthquake is on the radio news.

# A Bayesian Network for the Holmes Scenario




$P(E) = 0.0003$

$P(R|\neg E) = 0.0002,\ P(R|\ E) = 0.9$

$P(W|\neg A) = 0.4$
$P(W|\ A) = 0.8$

$P(G|\neg A) = 0.04$
$P(G|\ A) = 0.4$

$P(B) = 0.0001$

$P(A|\neg B \wedge \neg E) = 0.01,\ P(A|\neg B \wedge\ E) = 0.2$
$P(A|\ B \wedge \neg E) = 0.95,\ P(A|\ B \wedge\ E) = 0.96$

How many probabilities do we need to encode the Network?

# Bayesian Network

A Bayesian Network is a *directed acyclic graph* (DAG).

- Each node corresponds to a random variable.
- $X$ is a parent of $Y$ if there is an arrow from node $X$ to node $Y$.

  $\rightarrow$ Like a family tree, there are parents, children, ancestors, descendants.

- Each node $X_i$ has a conditional probability distribution $P(X_i|Parents(X_i))$ that quantifies the effect of the parents on the node.

# The Semantics of Bayesian Networks

Two ways to understand Bayesian Networks:

- A representation of the joint probability distribution
- An encoding of the conditional independence assumptions

# Representing the joint distribution

The idea is that, given a random variable X, a small set of variables may exist that directly affect the variable's value in the sense that X is conditionally independent of other variables given values for the directly affecting variables.

- The set of locally affecting variables is called **Markov blanket**.
- Start with a set of random variables representing all the features of the model.
- Define the **parents** of random variable $X_i$, written as $parents(X_i)$.
- $X_i$ is independent from others given the $parents(X_i)$.

# Representing the joint distribution

Markov Blanket: a boundary of a random variable.

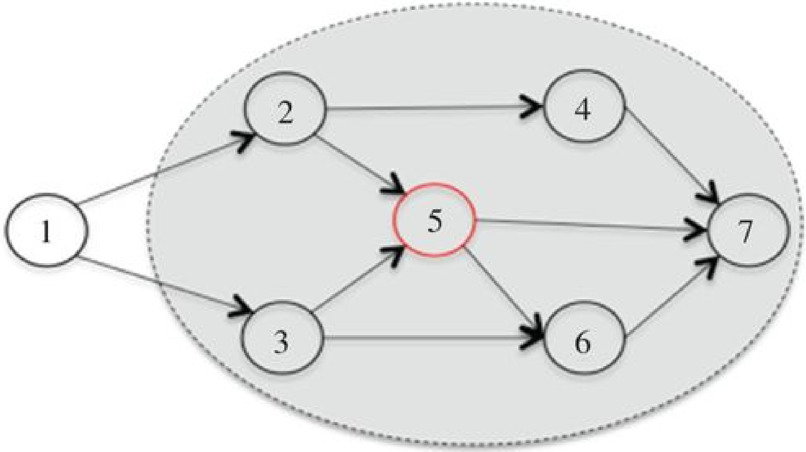



Figure: Markov Blanket for random variable $5$.

# Representing the joint distribution

We can compute the full joint probability using the following formula.

$$P(X_n \wedge \cdots \wedge X_1) = \prod_{i=1}^{n} P(X_i | Parents(X_i))$$

# Representing the joint distribution

**Example:** What is the probability that all of the following occur?

- The alarm has sounded
- Neither a burglary nor an earthquake has occurred
- Both Watson and Gibbon call and say they hear the alarm
- There is no radio report of an earthquake

# Representing the joint distribution

**Example:** What is the probability that all of the following occur?

- The alarm has sounded
- Neither a burglary nor an earthquake has occurred
- Both Watson and Gibbon call and say they hear the alarm
- There is no radio report of an earthquake

$\rightarrow$ Formulate as a joint probability:

$$
\begin{aligned}
&P(\neg B \wedge \neg E \wedge A \wedge \neg R \wedge G \wedge W) \\
&= P(\neg B)P(\neg E)P(A|\neg B \wedge \neg E)P(\neg R|\neg E)P(G|A)P(W|A) \\
&= (1-0.0001)(1-0.0003)(0.01)(1-0.0002)(0.4)(0.8) \\
&= 3.2 \times 10^{-3}
\end{aligned}
$$

# Q #4: Calculating the joint probability

**Q:** What is the probability that all of the following occur?

- NEITHER a burglary NOR an earthquake has occurred,
- The alarm has NOT sounded,
- NEITHER of Watson and Gibbon is calling, and
- There is NO radio report of an earthquake?

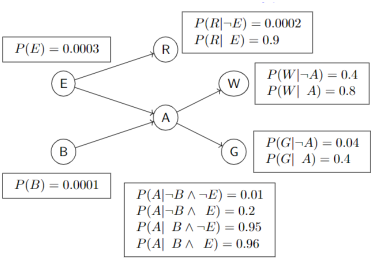
(A) 0.5699

(B) 0.6699

(C) 0.7699

(D) 0.8699

(E) 0.9699



# Q #4: Calculating the joint probability

**Q:** What is the probability that all of the following occur?

- NEITHER a burglary NOR an earthquake has occurred,
- The alarm has NOT sounded,
- NEITHER of Watson and Gibbon is calling, and
- There is NO radio report of an earthquake?

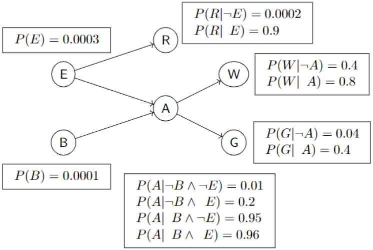
(A) 0.5699

(B) 0.6699

(C) 0.7699

(D) 0.8699

(E) 0.9699




$\rightarrow$ (A)

$$(1-0.0001)(1-0.0003)(1-0.01)(1-0.4)(1-0.04)(1-0.0002) = 0.5699$$

# Burglary, Alarm and Watson

# Q #5: Unconditional Independence

**Q:** Are Burglary and Watson independent?


(A) Yes

(B) No

(C) Can't tell.

# Q #5: Unconditional Independence

**Q:** Are Burglary and Watson independent?


(A) Yes

(B) No

(C) Can't tell.

→ Correct answer is *No*.

If you learned the value of B, would your belief of W change? If B is true, then Alarm is more likely to be true, and W is more likely to be true.

# Q #6: Conditional Independence

**Q:** Are Burglary and Watson conditionally independent given Alarm?


(A) Yes

(B) No

(C) Can't tell

# Q #6: Conditional Independence

**Q:** Are Burglary and Watson conditionally independent given Alarm?


(A) Yes

(B) No

(C) Can't tell

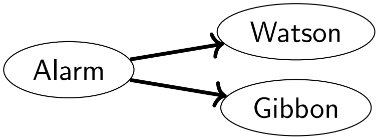
$\rightarrow$ Correct answer is *Yes*.

Assume that W does not observe B directly. W only observes A.

B and W could only influence each other through A.

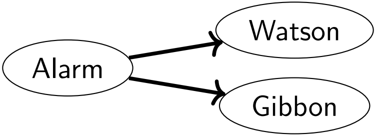
If A is known, then B and W do not affect each other.

# Alarm, Watson and Gibbon



# Q #7: Unconditional Independence

**Q:** Are Watson and Gibbon independent?



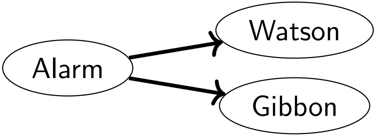

(A) Yes

(B) No

(C) Can't tell

# Q #7: Unconditional Independence

**Q:** Are Watson and Gibbon independent?




(A) Yes

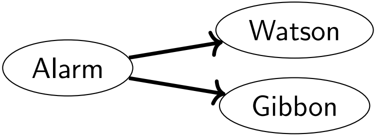
(B) No

(C) Can't tell

→ Correct answer is No. If Watson is more likely to call, then Alarm is more likely to go off, which means that Gibbon is more likely to call.

# Q #8 Conditional Independence

**Q:** Are Watson and Gibbon conditionally independent given Alarm?



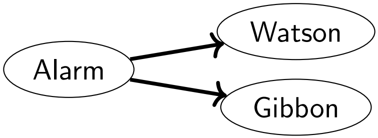

(A) Yes

(B) No

(C) Can't tell

# Q #8 Conditional Independence

**Q:** Are Watson and Gibbon conditionally independent given Alarm?



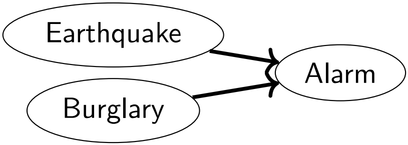

(A) Yes

(B) No

(C) Can't tell

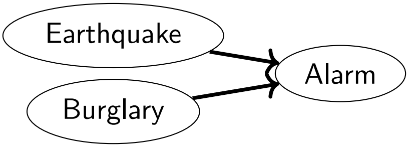
$\rightarrow$ Correct answer is Yes. Watson and Gibbon are both unreliable sensors for Alarm. If Alarm is known, then Watson and Gibbon do not affect each other.

# Earthquake, Burglary, and Alarm



# Q #9 Unconditional Independence

**Q:** Are Earthquake and Burglary independent?



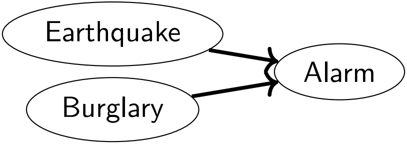

(A) Yes

(B) No

(C) Can't tell

# Q #9 Unconditional Independence

**Q:** Are Earthquake and Burglary independent?




(A) Yes

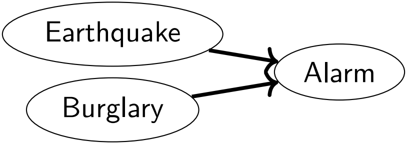
(B) No

(C) Can't tell

$\rightarrow$ Correct answer is Yes.

# Q #10: Conditional Independence

**Q:** Are Earthquake and Burglary conditionally independent given Alarm?



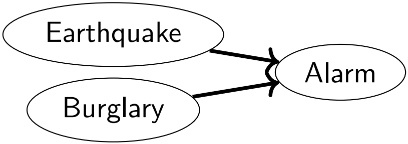

(A) Yes

(B) No

(C) Can't tell

# Q #10: Conditional Independence

**Q:** Are Earthquake and Burglary conditionally independent given Alarm?




(A) Yes

(B) No

(C) Can't tell

→ Correct answer is No. Suppose that the Alarm is going. If there is an Earthquake, then it is less likely that the Alarm is caused by Burglary. If there is a Burglary, it is less likely that the Alarm is caused by Earthquake.

# Revisiting Learning Goals

- Given a probabilistic model,
  determine if two variables are unconditionally independent, or conditionally independent given a third variable.
- Give examples of deriving a compact representation of a joint distribution by using independence assumptions.
- Describe components of a Bayesian network.
- Compute a joint probability given a Bayesian network.
- Explain the independence relationships in the three key structures.