Neural Networks - Part 1

Wenhu Chen

Lecture 19

Readings: RN 21.1, PM 7.5.

Outline

Learning Goals

- Describe the simple mathematical model of a neuron.
- Describe desirable properties of an activation function.
- Distinguish feedforward and recurrent neural networks.
- Learn a perceptron that represents a simple logical function.
- Determine the logical function represented by a perceptron.
- Explain why a perceptron cannot represent the XOR function.
- Understanding recurrent neural networks.

Learning complex relationships

- Image interpretation, speech recognition, and translation.
- The relationship between inputs and outputs can be extremely complex.
- How can we build a model to learn such complex relationships?

$\rightarrow$ We need a model that can learn complex relationships, that can be learned efficiently, and does not overfit the data.

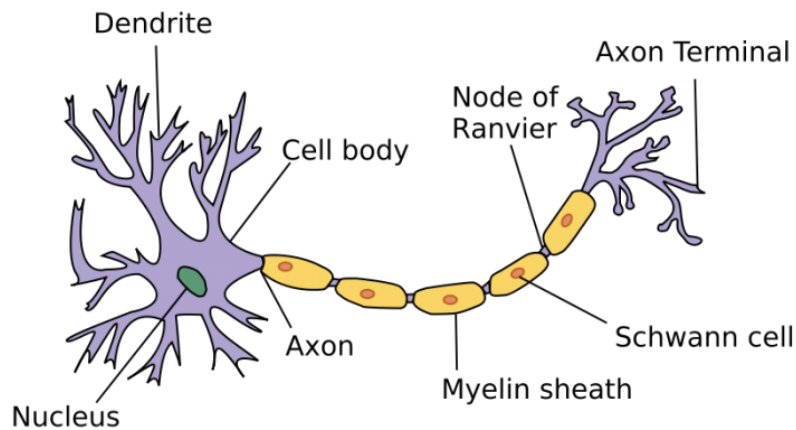
Humans can learn complex relationships well.

Can we build a model that mimics the human brain?

Human brains

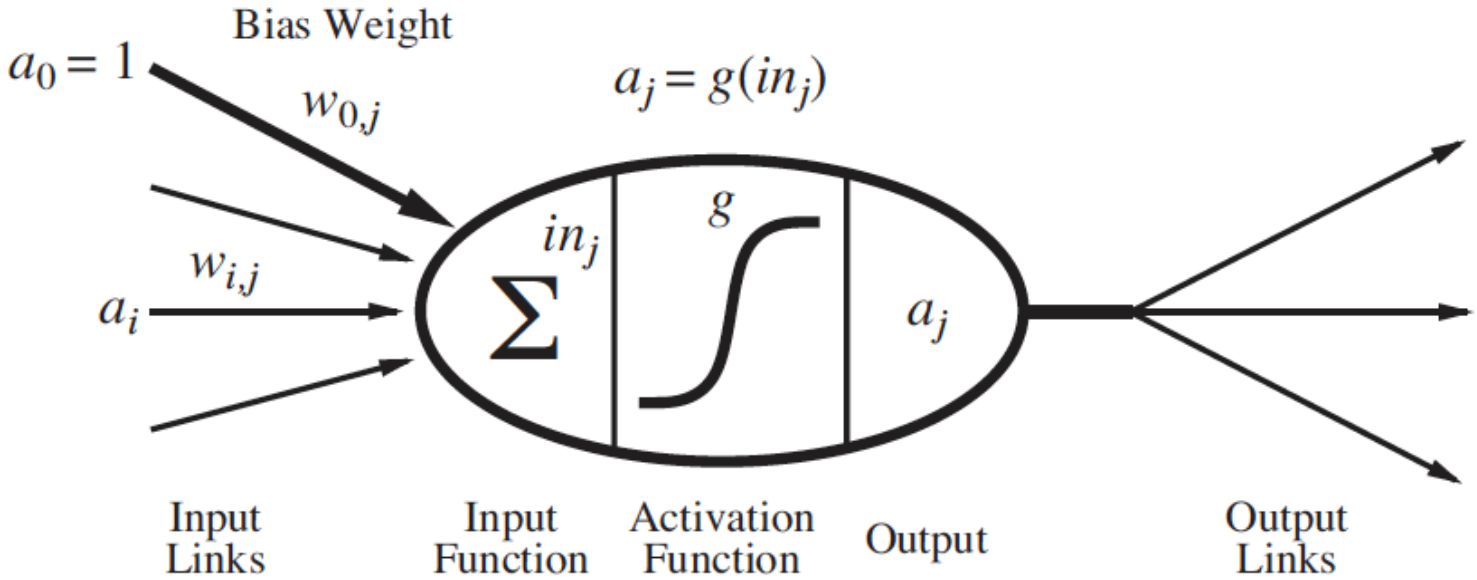
- A brain is a set of densely connected neurons.
- Components of a neuron: dendrites, soma, axon, synapse
  - *Dendrites* receive input signals from other neurons.
  - *Soma* controls activity of the neuron.
  - *Axon* sends output signals to other neurons.
  - *Synapses* are the links between neurons.
- Depending on the input signals, the neuron performs computations and decides to fire or not.

→ Conventional models of neurons have few complex components. Neural networks have many simple components.

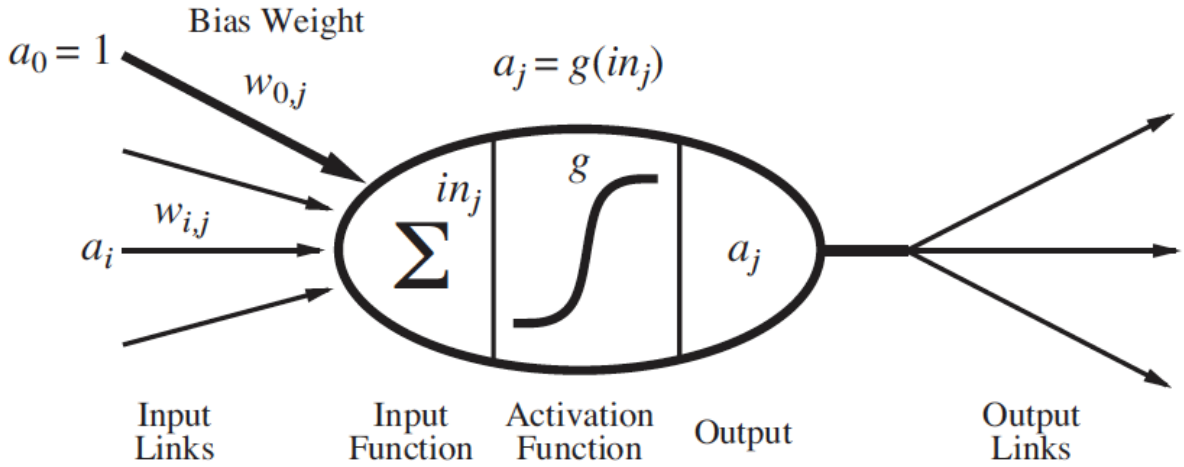


A simple mathematical model of a neuron

- McCulloch and Pitts 1943.
- A linear classfier — it "fires" when a linear combination of its inputs exceeds some threshold.



A simple mathematical model of a neuron



- Neuron j computes a weighted sum of its input signals. $in_j = \sum_{i=0}^{n} w_{ij} a_i$.
- Neuron j applies an activation function g to the weighted sum to derive the output. $a_j = g(in_j) = g\left(\sum_{i=0}^{n} w_{ij} a_i\right)$.

$\rightarrow$ Neuron i sends input signal a_i to neuron j. The link between i and j has weight w_{ij}, which is the strength of the connection. The neuron has a dummy input $a_0 = 1$ with an associated weight w_{0j}.

Desirable Properties of The Activation Function

What are some desirable properties of the activation function?

- It should be nonlinear.

  $\rightarrow$ Complex relationships are often nonlinear. Combining linear functions will not give us a nonlinear function. We can interleave linear and nonlinear functions to represent complex relationships.

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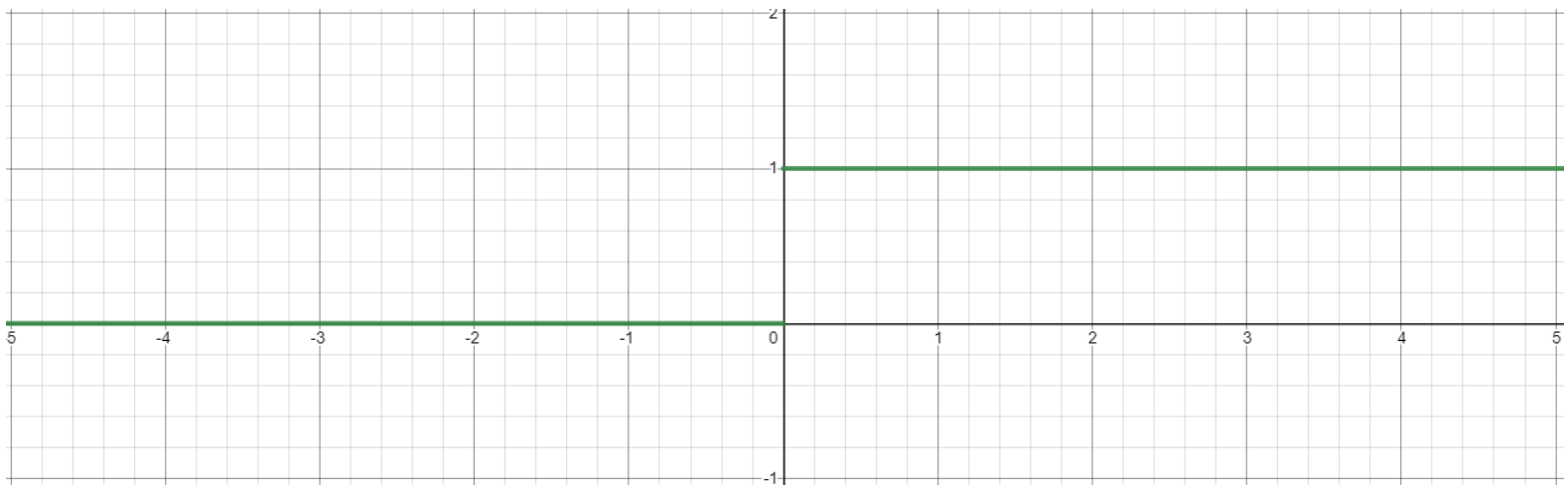
- It should be differentiable almost everywhere.

  $\rightarrow$ We learn a neural network using optimization algorithms such as gradient descent. Many such optimization algorithms require a function to be differentiable.

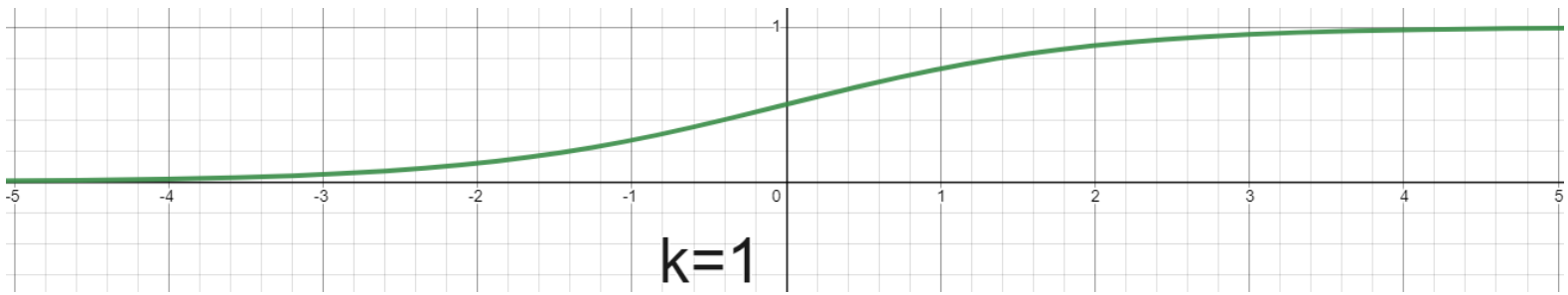
Common activation functions

- **Step function:** $g(x) = 1$ if $x > 0$. $g(x) = 0$ if $x \leq 0$.
  - Simple to use, but not differentiable.
  - Not used in practice, but useful to explain concepts.



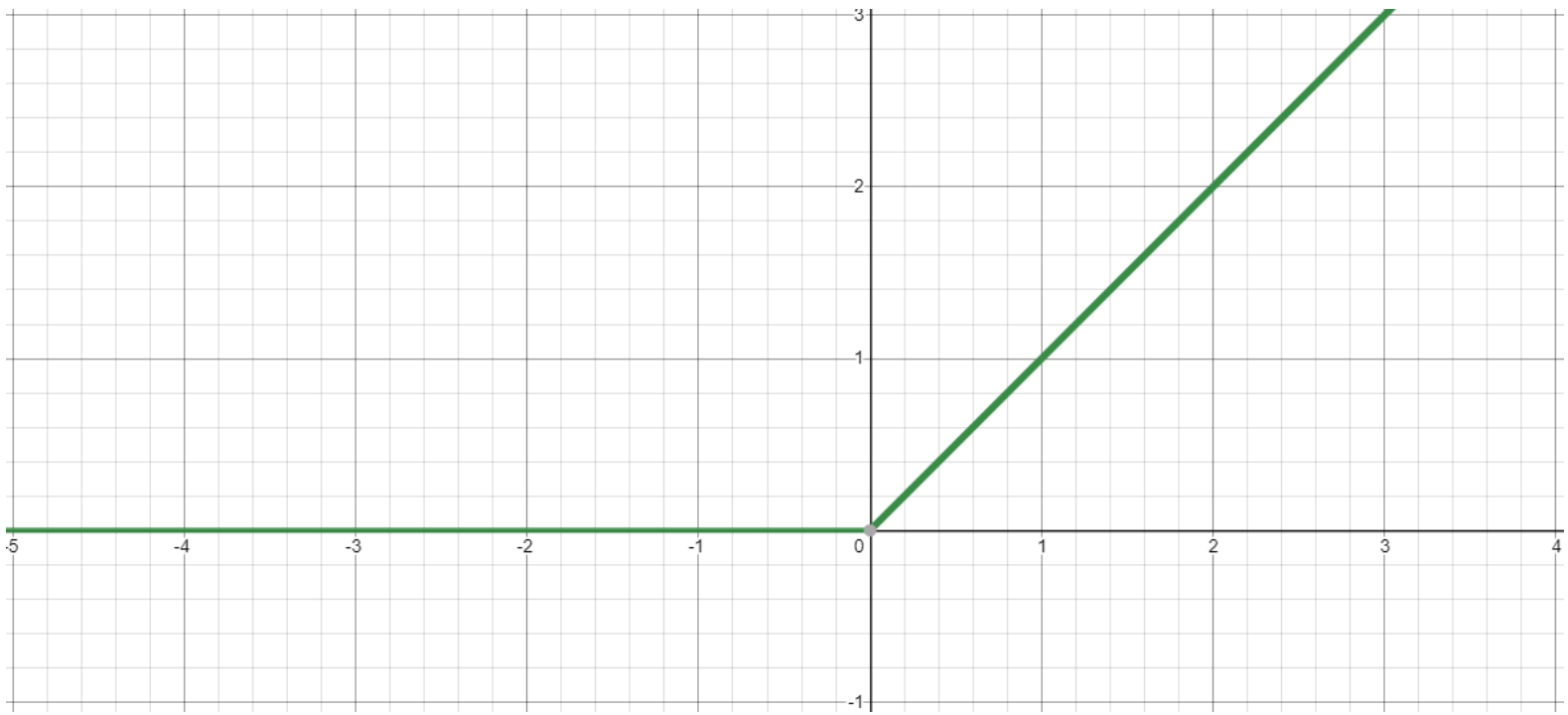
Common activation functions (continued)

- **Sigmoid function:** $g(x) = \dfrac{1}{1 + e^{-kx}}$.
  - For very large or very small x, $g(x)$ is very close to 1 or 0.
  - Can approximate the step function by tuning k. As k increases, the sigmoid function becomes steeper and is closer to the step function. Usually in practice $k = 1$.
  - Differentiable.
  - **Vanishing gradient problem:** when x is very large or very small, $g(x)$ responds little to changes in x. The network does not learn further or learns very slowly.
  - Computationally expensive.



Common activation functions (continued)

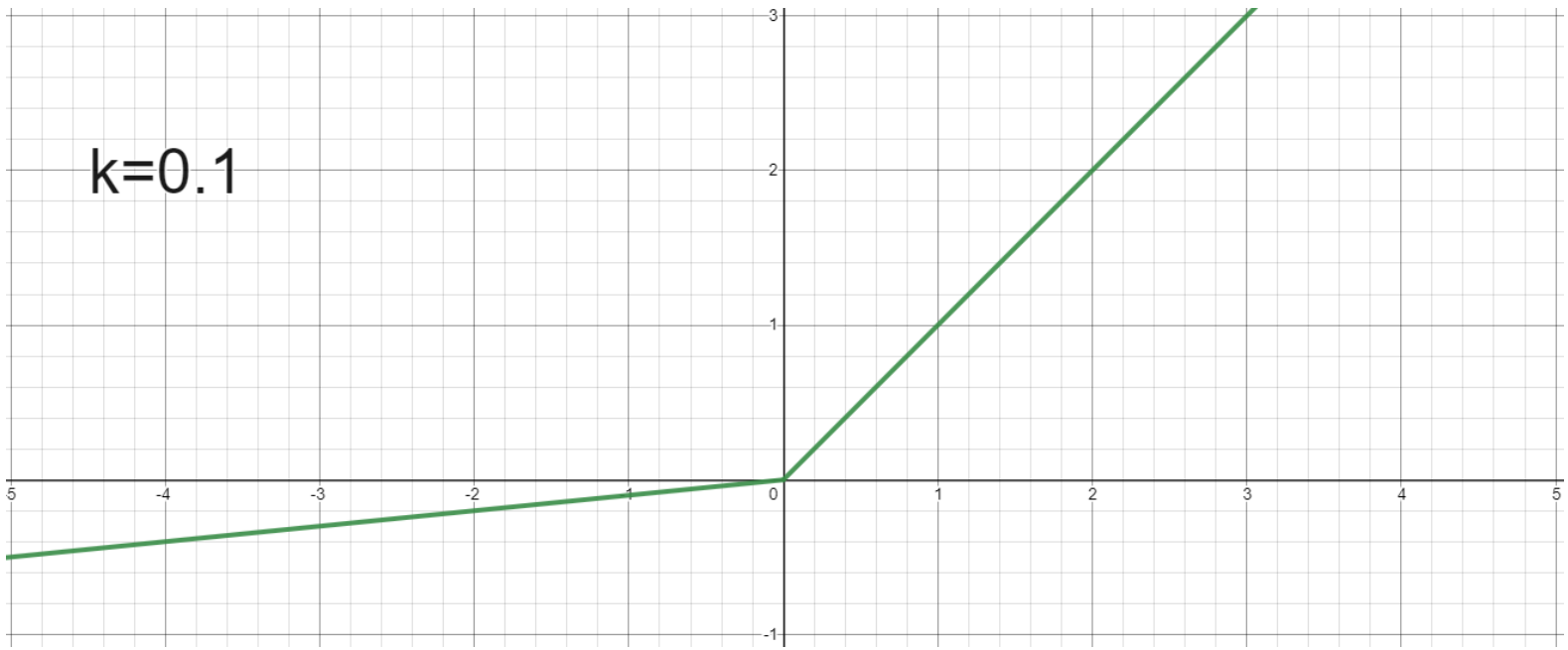
- **Rectified linear unit (ReLU):** $g(x) = max(0, x)$.
  - Computationally efficient; network converges quickly.
  - Differentiable.
  - **The dying ReLU problem:** when inputs approach 0 or are negative, the gradient becomes 0 and the network cannot learn.



Common activation functions (continued)

- **Leaky ReLU:** $g(x) = max(0, x) + k \cdot min(0, x)$

  $\rightarrow$ Small positive slope k in the negative area. Enables learning for negative input values.



Connecting the neurons together into a network

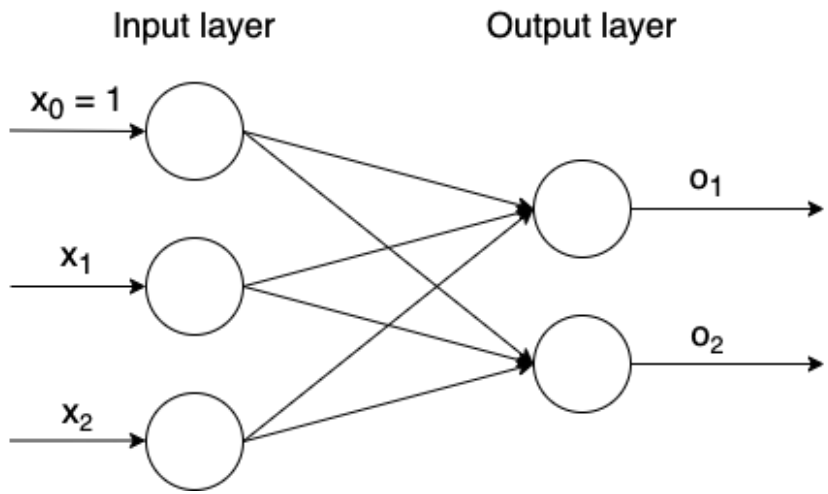
- **Feedforward network**
  - Forms a directed acyclic graph (no loops).
  - Have connections only in one direction.
  - Represents a function of its inputs.

- **Recurrent network**
  - Feeds its outputs back into its inputs.
  - Can support short-term memory. For the given inputs, the behaviour of the network depends on its initial state, which may depend on previous inputs.
  - The model is more interesting, but more difficult to understand and to learn.

Perceptrons

- Single-layer feedforward neural network
- The inputs are connected directly to the outputs.
- Can represent some logical functions, e.g. AND, OR, and NOT.

  $\rightarrow$ A big deal at the time. People believed that AI is solved if computers could perform formal logical reasoning.

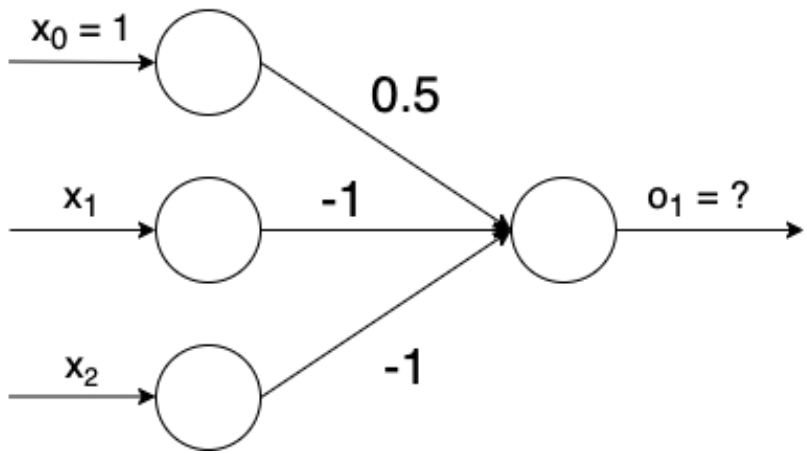


$\rightarrow$ Learn the perceptron for each output separately.

Q: What does the perceptron compute?

Q #1: Consider the following perceptron, where the activation function is the step function. ($g(x) = 1$ if $x > 0$. $g(x) = 0$ if $x \leq 0$.). Which of the following logical functions does the perceptron compute?

(A) $x_1 \wedge x_2$
(B) $\neg(x_1 \wedge x_2)$
(C) $x_1 \vee x_2$
(D) $\neg(x_1 \vee x_2)$



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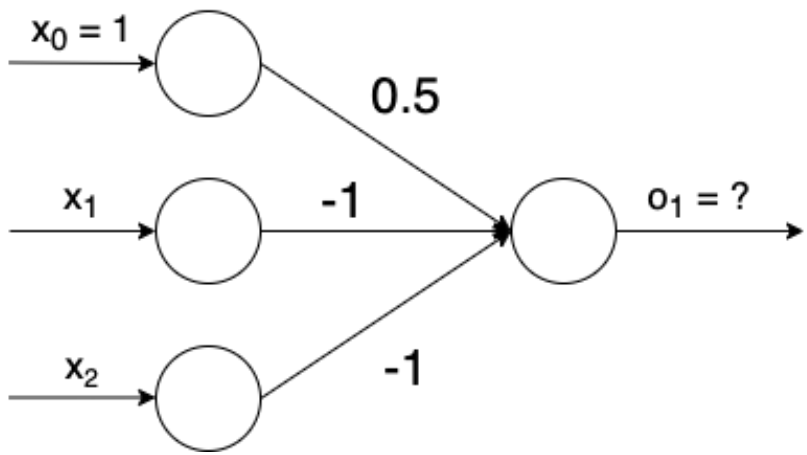
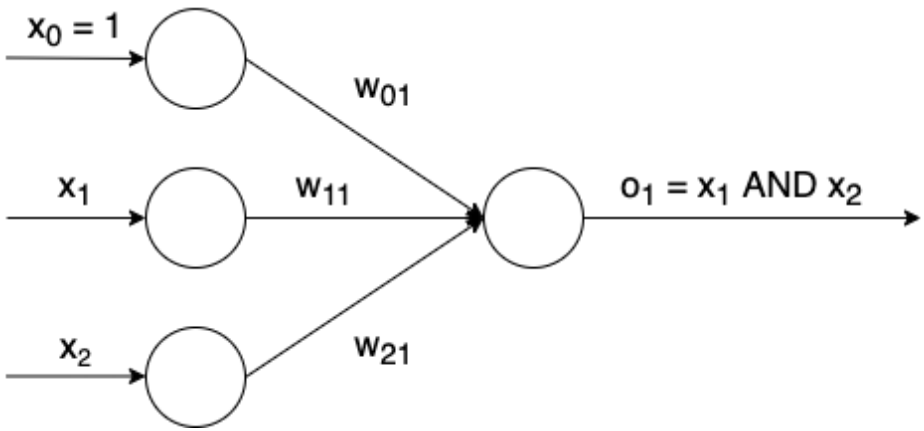


→ (D) is correct. This perceptron computes a NOR gate.

Q: Learning a perceptron for the AND function

Q #2: Consider the perceptron below where the activation function is the step function ($g(x) = 1$ if $x > 0$. $g(x) = 0$ if $x \leq 0$.). What should the weights w_{01}, w_{11} and w_{21} be such that the perceptron represents an AND function?



x_1	x_2	o_1
0	0	0
0	1	0
1	0	0
1	1	1

(A) $w_{01} = -1.\ w_{11} = 0.5.\ w_{21} = 0.5$
(B) $w_{01} = 0.5.\ w_{11} = -1.\ w_{21} = 1$
(C) $w_{01} = 1.5.\ w_{11} = -1.\ w_{21} = -1$
(D) $w_{01} = -1.5.\ w_{11} = 1.\ w_{21} = 1$

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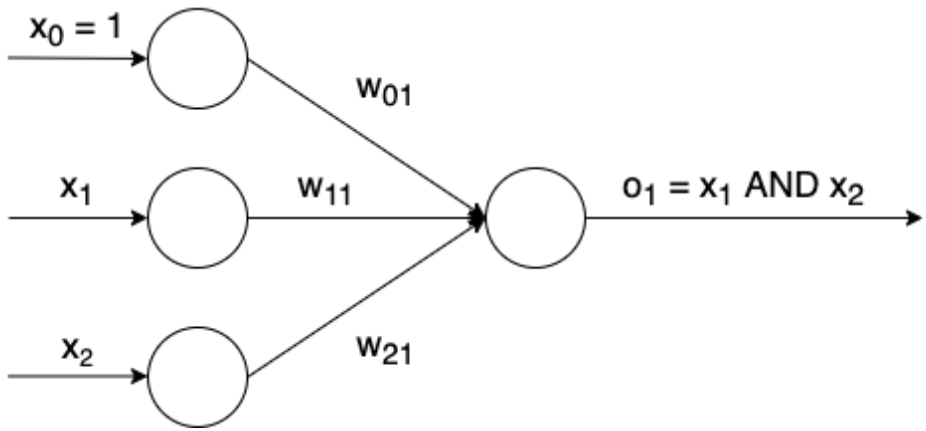

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$\rightarrow$ (D) is correct. $(-1.5)x_0 + 1x_1 + 1x_2 \geq 0.$

Q: What does this perceptron compute?

Q #3: What does h_1 compute?

$$h_1 = g(x_1 + x_2 - 0.5)$$

where $g(x) = 1$ if $x > 0$ and $g(x) = 0$ if $x \leq 0$.

(A) $(x_1 \vee x_2)$

(B) $(x_1 \wedge x_2)$

(C) $(\neg(x_1 \vee x_2))$

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(D) $(\neg(x_1 \wedge x_2))$

$\rightarrow$ (A) is the correct answer. This perceptron computes an OR gate.

Q: What does this perceptron compute?

Q #4: What does h_2 compute?

$$h_2 = g(-x_1 - x_2 + 1.5)$$

where $g(x) = 1$ if $x > 0$ and $g(x) = 0$ if $x \leq 0$.

(A) $(x_1 \vee x_2)$

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(C) $(\neg(x_1 \vee x_2))$

(D) $(\neg(x_1 \wedge x_2))$

$\rightarrow$ (D) is the correct answer. This perceptron computes a NAND gate.

Limitations of perceptrons

- Perceptrons: An introduction to computational geometry. Minsky and Papert. MIT Press. Cambridge MA 1969.

  $\rightarrow$ Marvin Minsky (founder of MIT AI lab)

  Seymour Papert (director of the lab).

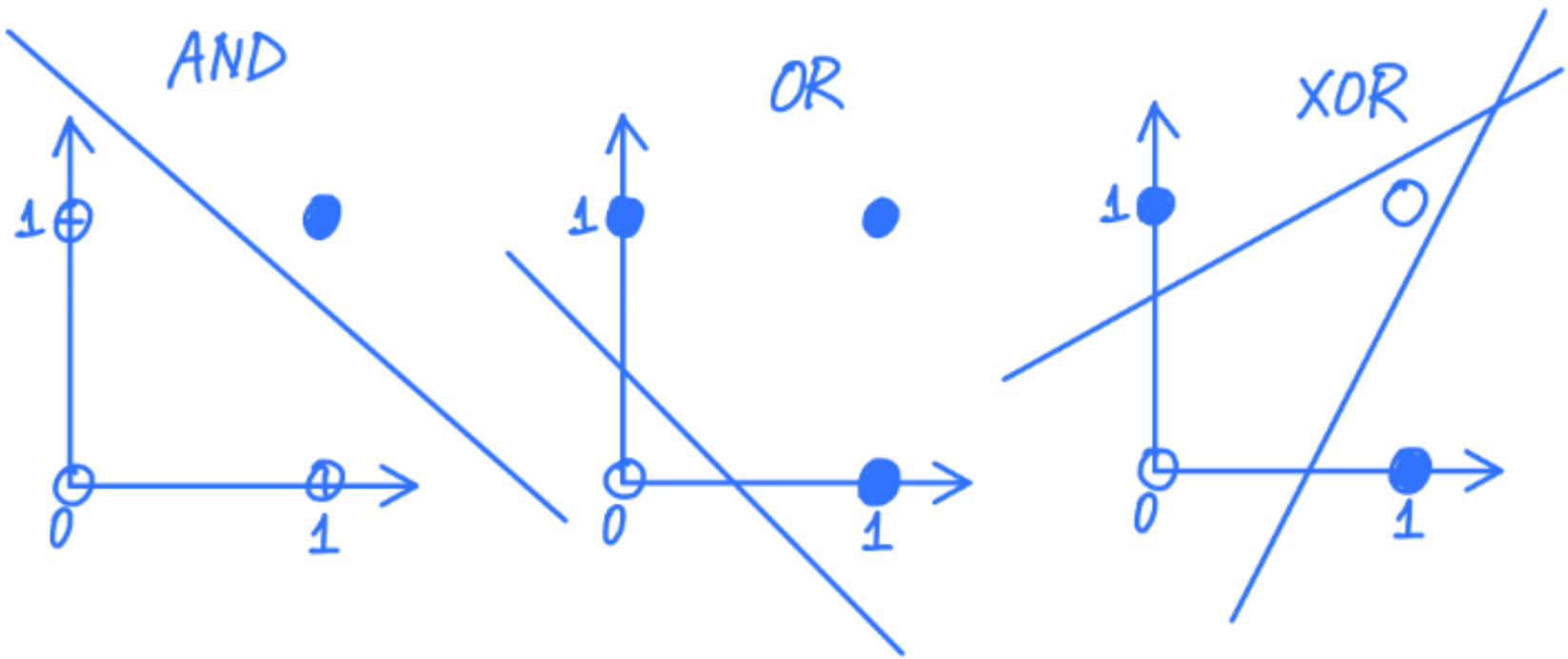
- Results:
  - XOR cannot be represented using perceptrons. We need a deeper network.
  - No one knew how to train deeper networks.
- Led to the first AI winter.

  $\rightarrow$ This approach was a dead end. A freeze to funding and publications.

Why can't a perceptron represent XOR?

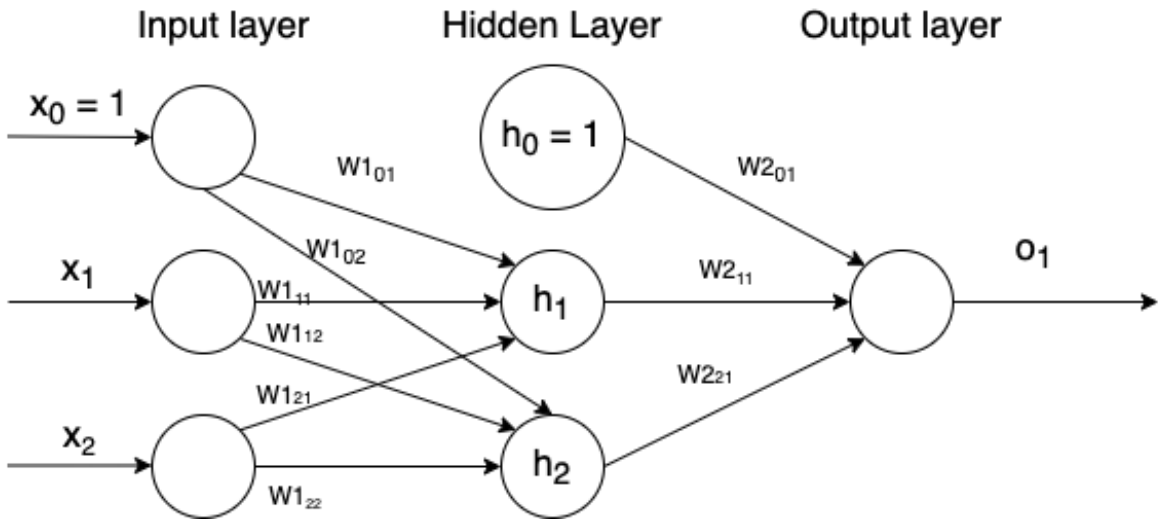
Why can't a perceptron represent XOR?

→ Intuition: a perceptron is a linear classifier. XOR is not linearly separable.



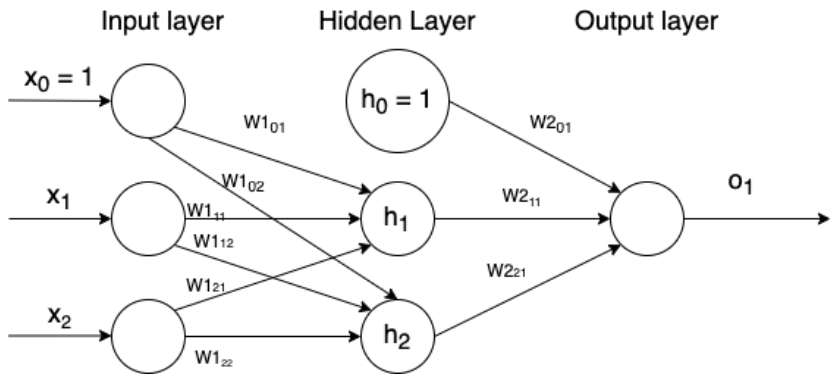
XOR as a 2-Layer Neural Network

Can you come up with the weights such that the following network represents the XOR function?



→ Hint: Start by converting XOR to logical formula composed of logical operations that can be represented by a perceptron:
$o_1 = (x_1 \vee x_2) \wedge (\neg(x_1 \wedge x_2)) = h_1 \wedge h_2$

XOR as a 2-Layer Neural Network



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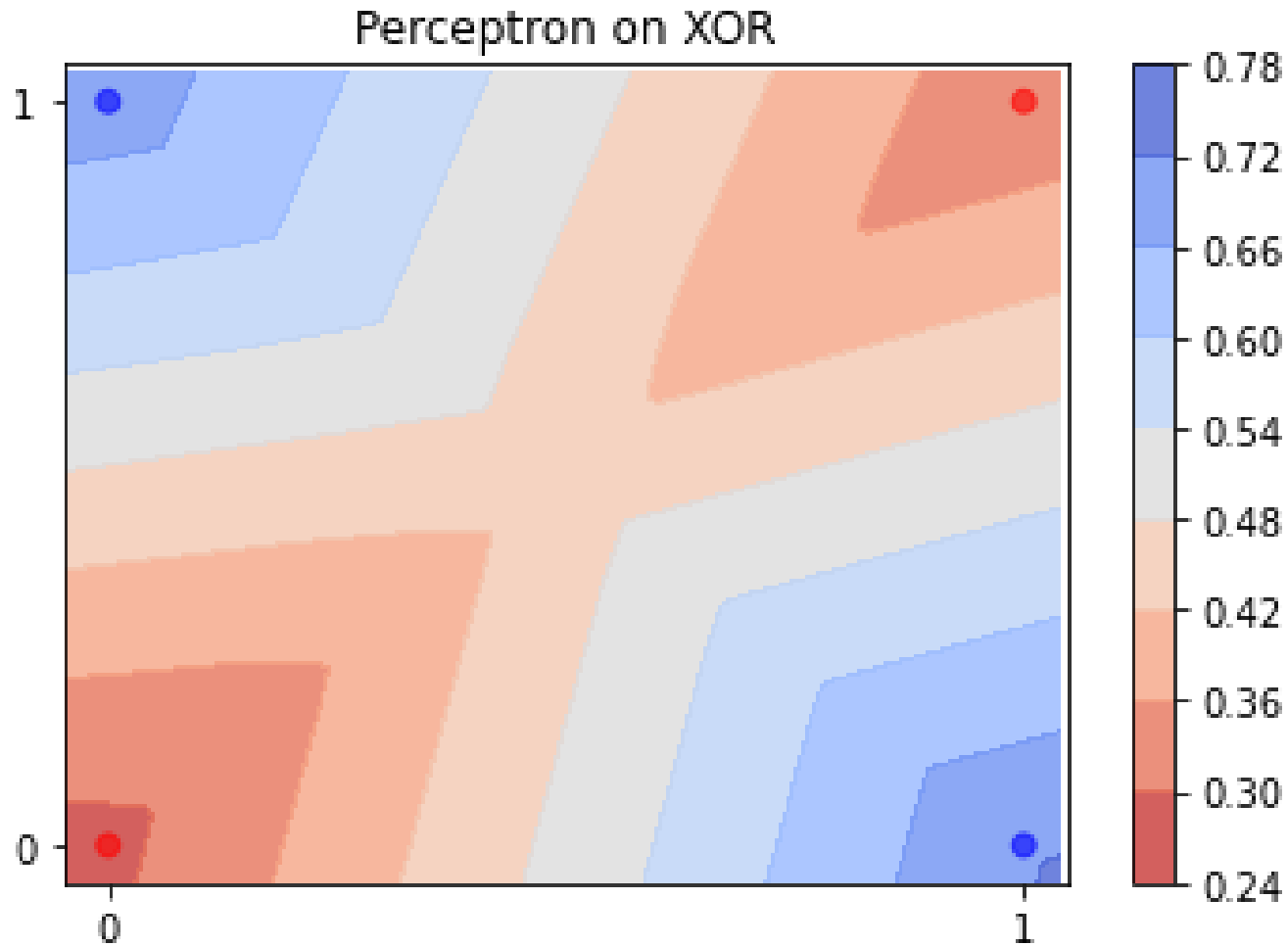
XOR function

The decision boundary of XOR network:

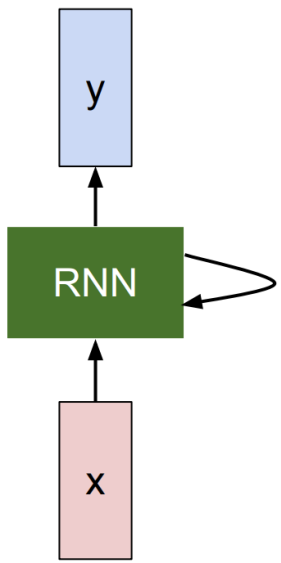


Recurrent Neural Network

We can process a sequence of vectors x by applying a recurrence formula at every time step:

We need to maintain a state variable h_t to keep track of the history of all the seen information.

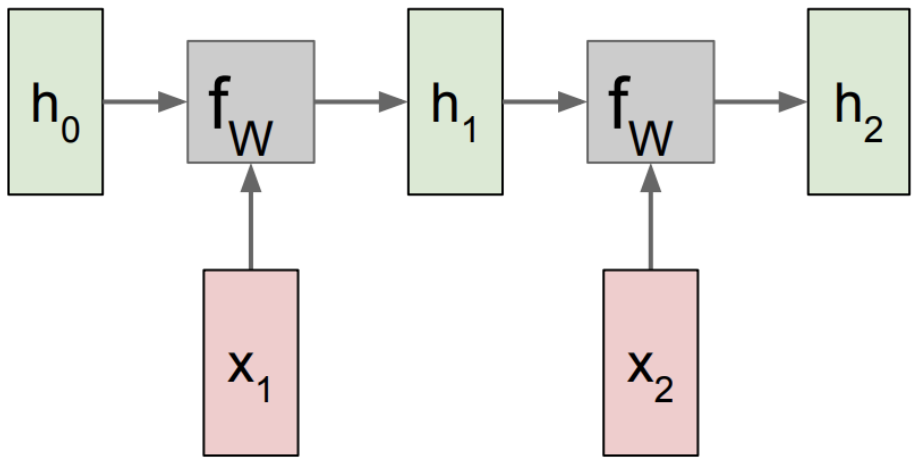
$$h_t = f_W(h_{t-1}, x_t) \quad y_t = f_Y(h_t)$$



Vanilla Recurrent Neural Network

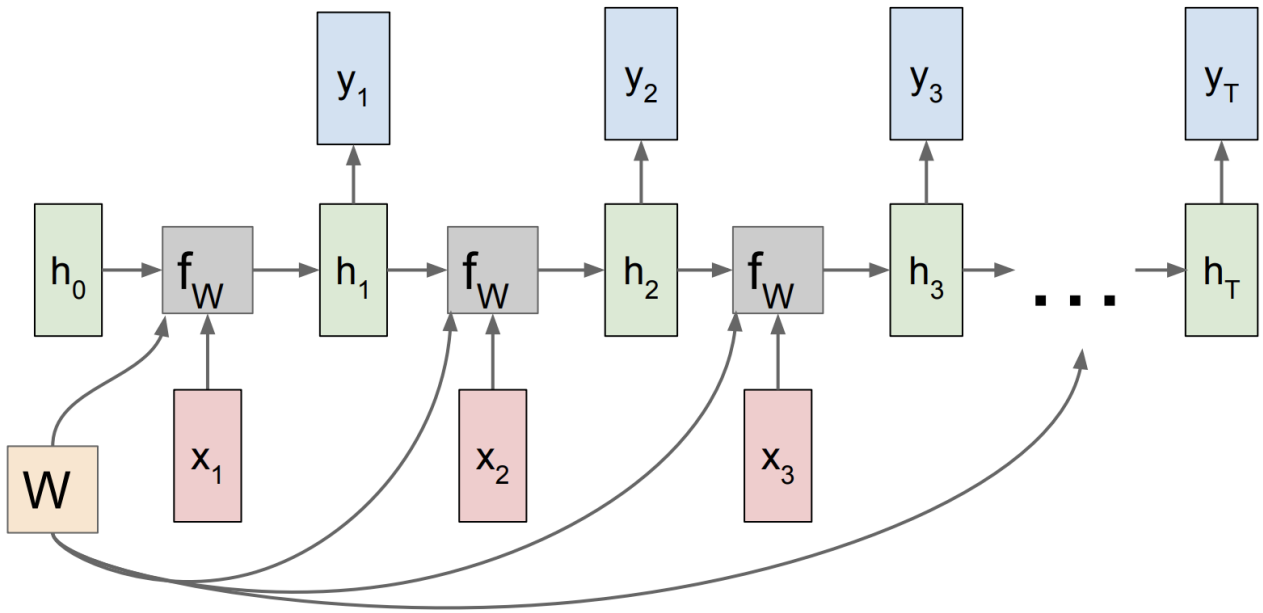
$$h_t = f_W(h_{t-1}, x_t)$$

$$h_t = tanh(W_{hh}h_{t-1} + W_{xh}x_t); \quad y_t = W_{hy}h_t$$



Vanilla Recurrent Neural Network

Proceed from $x_1 \cdots x_n$, the weights of the transition is being shared among all the transitions.

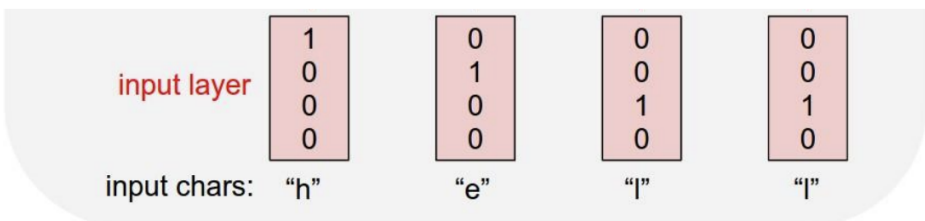


Modeling Language

Example:
Character-level
Language Model

Vocabulary:
[h,e,l,o]

Example training sequence:
“hello”



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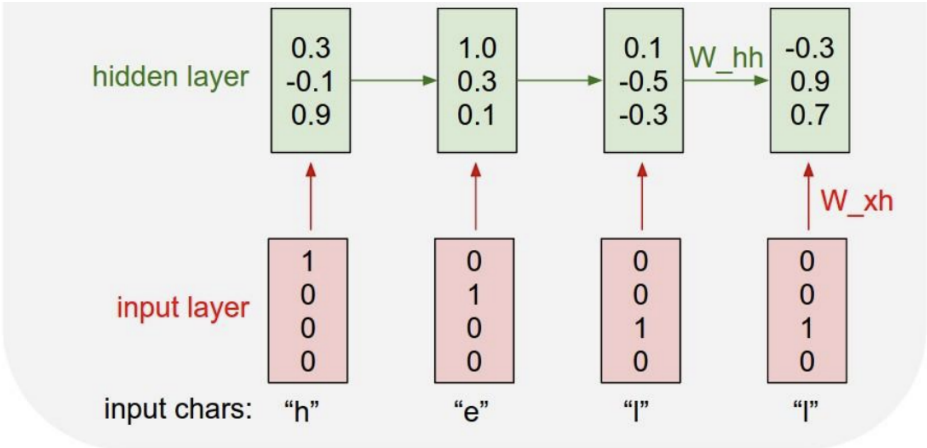

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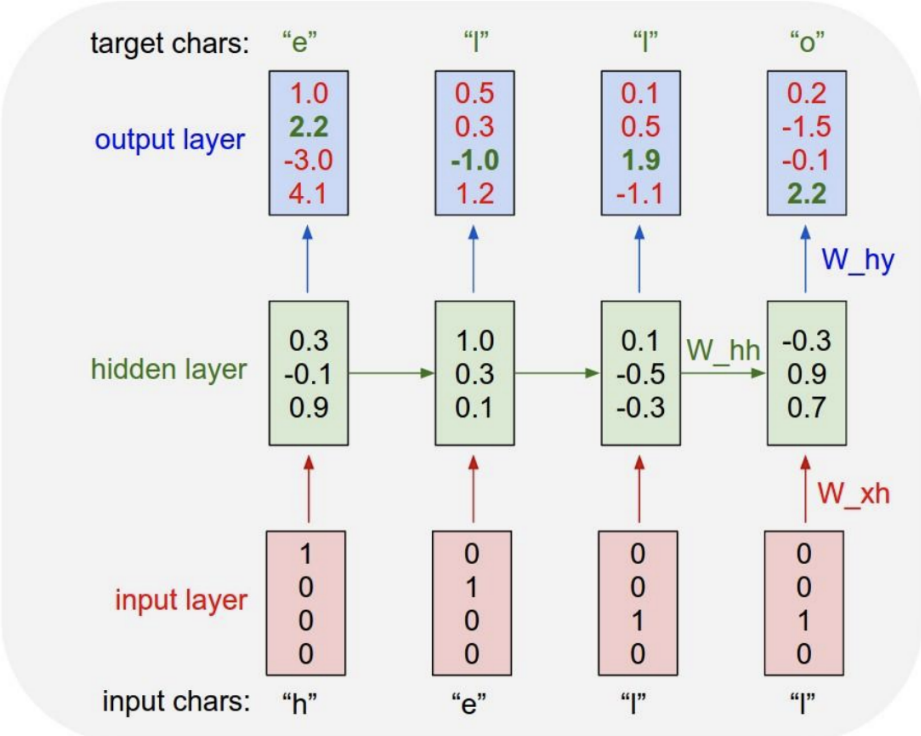

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Example: Character-level Language Model Sampling

Vocabulary:
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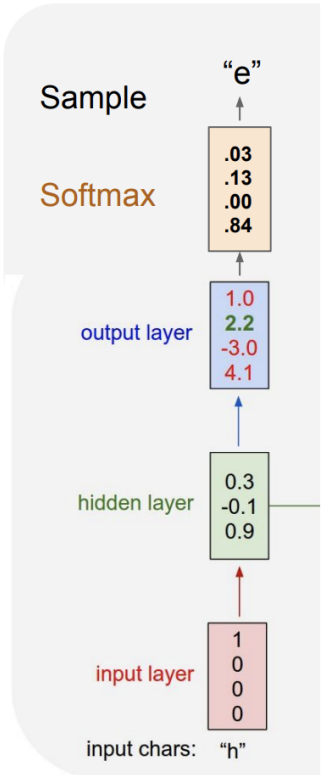
At test-time sample characters one at a time, feed back to model



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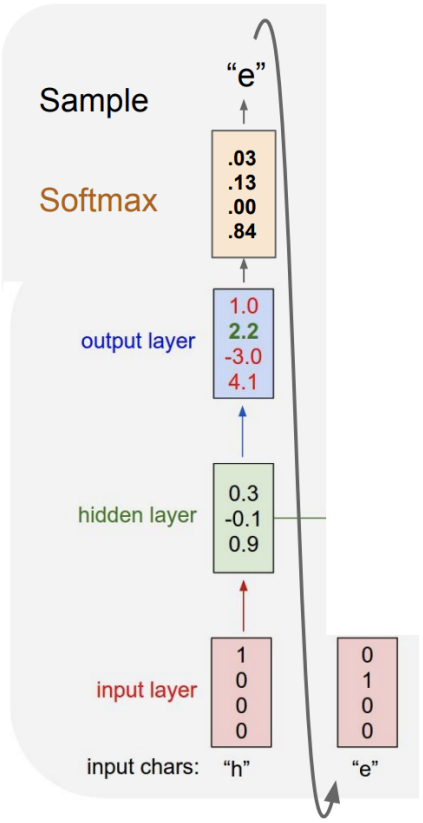

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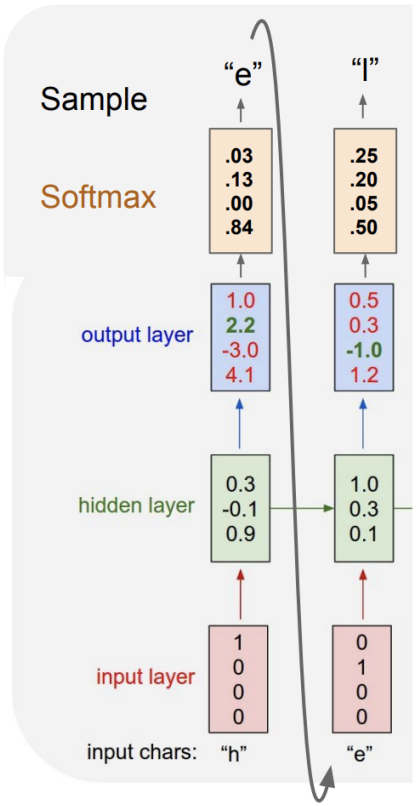

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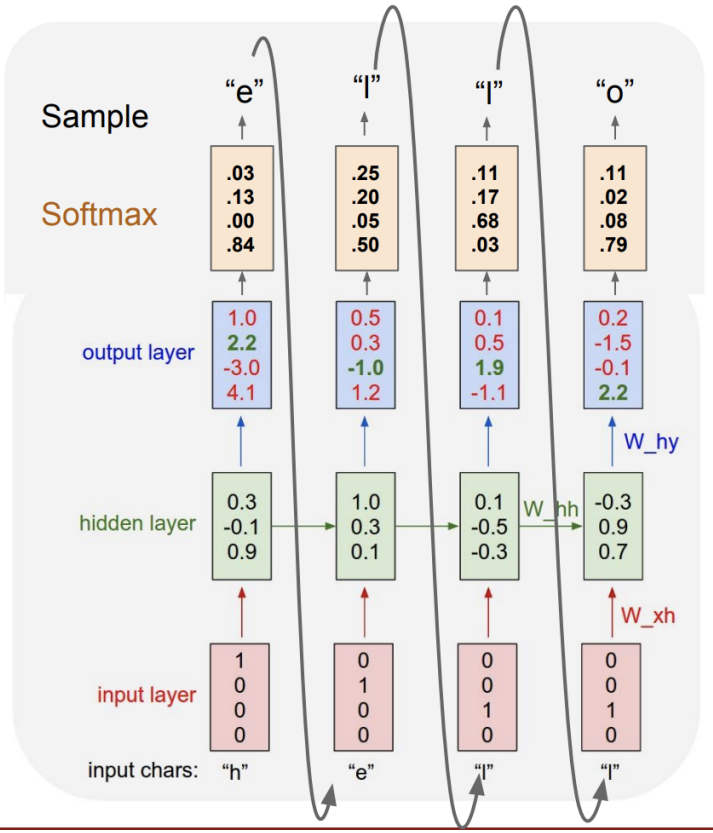

Revisiting Learning Goals

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