Lecture 21: Alternate characterizations of the completely bounded trace norm

In the previous lecture we discussed the completely bounded trace norm, its connection to the problem of distinguishing channels, and some of its basic properties. In this lecture we will discuss a few alternate ways in which this norm may be characterized, including a semidefinite programming formulation that allows for an efficient calculation of the norm.

21.1 Maximum output fidelity characterization

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are complex Euclidean spaces and $\Phi_0, \Phi_1 \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ are completely positive (but not necessarily trace-preserving) maps. Let us define the *maximum output fidelity* of Φ_0 and Φ_1 as

$$\mathrm{F}_{\max}(\Phi_0, \Phi_1) = \max \{ \mathrm{F}(\Phi_0(\rho_0), \Phi_1(\rho_1)) \,:\, \rho_0, \rho_1 \in \mathrm{D}(\mathcal{X}) \}.$$

In other words, this is the maximum fidelity between an output of Φ_0 and an output of Φ_1, ranging over all pairs of density operator inputs.

Our first alternate characterization of the completely bounded trace norm is based on the maximum output fidelity, and is given by the following theorem.

Theorem 21.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ be an arbitrary mapping. Suppose further that $\mathcal{Z}$ is a complex Euclidean space and $A_0, A_1 \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ satisfy*

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(A_0 X A_1^*)$$

for all $X \in \mathrm{L}(\mathcal{X})$. For completely positive mappings $\Psi_0, \Psi_1 \in \mathrm{T}(\mathcal{X}, \mathcal{Z})$ defined as

$$\Psi_0(X) = \mathrm{Tr}_{\mathcal{Y}}(A_0 X A_0^*),$$
$$\Psi_1(X) = \mathrm{Tr}_{\mathcal{Y}}(A_1 X A_1^*),$$

for all $X \in \mathrm{L}(\mathcal{X})$, we have $|||\Phi|||_1 = \mathrm{F}_{\max}(\Psi_0, \Psi_1)$.

Remark 21.2. Note that it is the space $\mathcal{Y}$ that is traced-out in the definition of Ψ_0 and Ψ_1, rather than the space $\mathcal{Z}$.

To prove this theorem, we will begin with the following lemma that establishes a simple relationship between the fidelity and the trace norm. (This appeared as a problem on problem set 1.)

Lemma 21.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $u, v \in \mathcal{X} \otimes \mathcal{Y}$. It holds that*

$$\mathrm{F}(\mathrm{Tr}_{\mathcal{Y}}(uu^*), \mathrm{Tr}_{\mathcal{Y}}(vv^*)) = \| \mathrm{Tr}_{\mathcal{X}}(uv^*) \|_1.$$

Proof. It is the case that $u \in \mathcal{X} \otimes \mathcal{Y}$ is a purification of $\operatorname{Tr}_{\mathcal{Y}}(uu^*)$ and $v \in \mathcal{X} \otimes \mathcal{Y}$ is a purification of $\operatorname{Tr}_{\mathcal{Y}}(vv^*)$. By the unitary equivalence of purifications (Theorem 4.3 in the lecture notes), it holds that every purification of $\operatorname{Tr}_{\mathcal{Y}}(uu^*)$ in $\mathcal{X} \otimes \mathcal{Y}$ takes the form $(\mathbb{1}_{\mathcal{X}} \otimes U)u$ for some choice of a unitary operator $U \in \mathrm{U}(\mathcal{Y})$. Consequently, by Uhlmann's theorem we have

$$\mathrm{F}(\operatorname{Tr}_{\mathcal{Y}}(uu^*), \operatorname{Tr}_{\mathcal{Y}}(vv^*)) = \mathrm{F}(\operatorname{Tr}_{\mathcal{Y}}(vv^*), \operatorname{Tr}_{\mathcal{Y}}(uu^*)) = \max\{|\langle v, (\mathbb{1}_{\mathcal{X}} \otimes U)u\rangle| \,:\, U \in \mathrm{U}(\mathcal{Y})\}.$$

For any unitary operator U it holds that

$$\langle v, (\mathbb{1}_{\mathcal{X}} \otimes U)u\rangle = \operatorname{Tr}((\mathbb{1}_{\mathcal{X}} \otimes U)uv^*) = \operatorname{Tr}(U \operatorname{Tr}_{\mathcal{X}}(uv^*)),$$

and therefore

$$\max\{|\langle v, (\mathbb{1}_{\mathcal{X}} \otimes U)u\rangle| \,:\, U \in \mathrm{U}(\mathcal{Y})\} = \max\{|\operatorname{Tr}(U \operatorname{Tr}_{\mathcal{X}}(uv^*))| \,:\, U \in \mathrm{U}(\mathcal{Y})\} = \|\operatorname{Tr}_{\mathcal{X}}(uv^*)\|_1$$

as required. □

Proof of Theorem 21.1. Let us take $\mathcal{W}$ to be a complex Euclidean space with the same dimension as $\mathcal{X}$, so that

$$\begin{aligned}
|||\Phi|||_1 &= \max\left\{\left\|(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{W})})(uv^*)\right\|_1 \,:\, u, v \in \mathcal{S}(\mathcal{X} \otimes \mathcal{W})\right\} \\
&= \max\left\{\|\operatorname{Tr}_{\mathcal{Z}}\left[(A_0 \otimes \mathbb{1}_{\mathcal{W}})\, uv^*\, (A_1^* \otimes \mathbb{1}_{\mathcal{W}})\right]\|_1 \,:\, u, v \in \mathcal{S}(\mathcal{X} \otimes \mathcal{W})\right\}.
\end{aligned}$$

For any choice of vectors $u, v \in \mathcal{X} \otimes \mathcal{W}$ we have

$$\begin{aligned}
\operatorname{Tr}_{\mathcal{Y} \otimes \mathcal{W}}\left[(A_0 \otimes \mathbb{1}_{\mathcal{W}})uu^*(A_0^* \otimes \mathbb{1}_{\mathcal{W}})\right] &= \Psi_0(\operatorname{Tr}_{\mathcal{W}}(uu^*)), \\
\operatorname{Tr}_{\mathcal{Y} \otimes \mathcal{W}}\left[(A_1 \otimes \mathbb{1}_{\mathcal{W}})vv^*(A_1^* \otimes \mathbb{1}_{\mathcal{W}})\right] &= \Psi_1(\operatorname{Tr}_{\mathcal{W}}(vv^*)),
\end{aligned}$$

and therefore by Lemma 21.3 it follows that

$$\|\operatorname{Tr}_{\mathcal{Z}}\left[(A_0 \otimes \mathbb{1}_{\mathcal{W}})uv^*(A_1^* \otimes \mathbb{1}_{\mathcal{W}})\right]\|_1 = \mathrm{F}(\Psi_0(\operatorname{Tr}_{\mathcal{W}}(uu^*)), \Psi_1(\operatorname{Tr}_{\mathcal{W}}(vv^*))).$$

Consequently

$$\begin{aligned}
|||\Phi|||_1 &= \max\left\{\mathrm{F}(\Psi_0(\operatorname{Tr}_{\mathcal{W}}(uu^*)), \Psi_1(\operatorname{Tr}_{\mathcal{W}}(vv^*))) \,:\, u, v \in \mathcal{S}(\mathcal{X} \otimes \mathcal{W})\right\} \\
&= \max\left\{\mathrm{F}(\Psi_0(\rho_0), \Psi_1(\rho_1)) \,:\, \rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})\right\} \\
&= \mathrm{F}_{\max}(\Psi_0, \Psi_1)
\end{aligned}$$

as required. □

The following corollary follows immediately from this characterization along with the fact that the completely bounded trace norm is multiplicative with respect to tensor products.

Corollary 21.4. *Let $\Phi_1, \Psi_1 \in \mathrm{T}(\mathcal{X}_1, \mathcal{Y}_1)$ and $\Phi_2, \Psi_2 \in \mathrm{T}(\mathcal{X}_2, \mathcal{Y}_2)$ be completely positive. It holds that*

$$\mathrm{F}_{\max}(\Phi_1 \otimes \Phi_2, \Psi_1 \otimes \Psi_2) = \mathrm{F}_{\max}(\Phi_1, \Psi_1) \cdot \mathrm{F}_{\max}(\Phi_2, \Psi_2).$$

This is a simple but not obvious fact: it says that the maximum fidelity between the outputs of any two completely positive product mappings is achieved for product state inputs. In contrast, several other quantities of interest based on quantum channels fail to respect tensor products in this way.

21.2 A semidefinite program for the completely bounded trace norm (squared)

The square of the completely bounded trace norm of an arbitrary mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ can be expressed as the optimal value of a semidefinite program, as we will now verify. This provides a means to efficiently approximate the completely bounded trace norm of a given mapping—because there exist efficient algorithms to approximate the optimal value of very general classes of semidefinite programs (which includes our particular semidefinite program) to high precision.

Let us begin by describing the semidefinite program, starting first with its associated primal and dual problems. After doing this we will verify that its value corresponds to the square of the completely bounded trace norm. Throughout this discussion we assume that a Stinespring representation

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(A_0 X A_1^*)$$

of an arbitrary mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ has been fixed.

21.2.1 Description of the semidefinite program

The primal and dual problems for the semidefinite program we wish to consider are as follows:

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \langle A_1 A_1^*, X \rangle \\ \text{subject to:} \quad & \mathrm{Tr}_{\mathcal{Y}}(X) = \mathrm{Tr}_{\mathcal{Y}}\left(A_0 \rho A_0^*\right), \\ & \rho \in \mathrm{Pos}(\mathcal{X}), \\ & X \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Z}). \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \| A_0^* (\mathbb{1}_{\mathcal{Y}} \otimes Y) A_0 \| \\ \text{subject to:} \quad & \mathbb{1}_{\mathcal{Y}} \otimes Y \geq A_1 A_1^*, \\ & Y \in \mathrm{Pos}(\mathcal{Z}). \end{aligned}$$

This pair of problems may be expressed more formally as a semidefinite program in the following way. Define $\Xi \in \mathrm{T}((\mathcal{Y} \otimes \mathcal{Z}) \oplus \mathcal{X}, \mathbb{C} \oplus \mathcal{Z})$ as follows:

$$\Xi \begin{pmatrix} X & \cdot \\ \cdot & \rho \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}(\rho) & 0 \\ 0 & \mathrm{Tr}_{\mathcal{Y}}(X) - \mathrm{Tr}_{\mathcal{Y}}(A_0 \rho A_0^*) \end{pmatrix}.$$

(The submatrices indicated by $\cdot$ are ones we do not care about and do not bother to assign a name.) We see that the primal problem above asks for the maximum (or supremum) value of

$$\left\langle \begin{pmatrix} A_1 A_1^* & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X & \cdot \\ \cdot & \rho \end{pmatrix} \right\rangle$$

subject to the constraints

$$\Xi \begin{pmatrix} X & \cdot \\ \cdot & \rho \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} X & \cdot \\ \cdot & \rho \end{pmatrix} \in \mathrm{Pos}((\mathcal{Y} \otimes \mathcal{Z}) \oplus \mathcal{X}).$$

The dual problem is therefore to minimize the inner product

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & \cdot \\ \cdot & Y \end{pmatrix} \right\rangle,$$

for $\lambda \geq 0$ and $Y \in \mathrm{Pos}(\mathcal{Z})$, subject to the constraint

$$\Xi^* \begin{pmatrix} \lambda & \cdot \\ \cdot & Y \end{pmatrix} \geq \begin{pmatrix} A_1 A_1^* & 0 \\ 0 & 0 \end{pmatrix}.$$

One may verify that

$$\Xi^*\begin{pmatrix}\lambda & \cdot\\ \cdot & Y\end{pmatrix} = \begin{pmatrix}\mathbb{1}_{\mathcal{Y}}\otimes Y & 0\\ 0 & \lambda\mathbb{1}_{\mathcal{X}} - A_0^*(\mathbb{1}_{\mathcal{Y}}\otimes Y)A_0\end{pmatrix}.$$

Given that Y is positive semidefinite, the minimum value of λ for which $\lambda\mathbb{1}_{\mathcal{X}} - A_0^*(\mathbb{1}_{\mathcal{Y}}\otimes Y)A_0 \geq 0$ is equal to $\|A_0^*(\mathbb{1}_{\mathcal{Y}}\otimes Y)A_0\|$, and so we have obtained the dual problem as it is originally stated.

21.2.2 Analysis of the semidefinite program

We will now analyze the semidefinite program given above. Before we discuss its relationship to the completely bounded trace norm, let us verify that it satisfies strong duality. The dual problem is strictly feasible, for we may choose

$$Y = (\|A_1A_1^*\| + 1)\mathbb{1}_{\mathcal{Z}} \qquad \text{and} \qquad \lambda = \|A_1A_1^*\|\,\|A_0A_0^*\| + 1$$

to obtain a strictly feasible solution. The primal problem is of course feasible, for we may choose $\rho \in \mathrm{D}(\mathcal{X})$ arbitrarily and take $X = A_0\rho A_0^*$ to obtain a primal feasible operator. Thus, by Slater's theorem, strong duality holds for our semidefinite program, and we also have that the optimal primal value is obtained by a primal feasible operator.

Now let us verify that the optimal value associated with this semidefinite program corresponds to $|||\Phi|||_1^2$. Let us define a set

$$\mathcal{A} = \{X \in \mathrm{Pos}(\mathcal{Y}\otimes\mathcal{Z}) : \mathrm{Tr}_{\mathcal{Y}}(X) = \mathrm{Tr}_{\mathcal{Y}}(A_0\rho A_0^*) \text{ for some } \rho \in \mathrm{D}(\mathcal{X})\}.$$

It holds that the optimal primal value α of the semidefinite program is given by

$$\alpha = \max_{X\in\mathcal{A}} \langle A_1A_1^*, X\rangle.$$

For any choice of a complex Euclidean space $\mathcal{W}$ for which $\dim(\mathcal{W}) \geq \dim(\mathcal{X})$, we have

$$\begin{aligned}
|||\Phi|||_1^2 &= \max_{u,v\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})} \|\mathrm{Tr}_{\mathcal{Z}}[(A_0\otimes\mathbb{1}_{\mathcal{W}})uv^*(A_1\otimes\mathbb{1}_{\mathcal{W}})^*]\|_1^2\\
&= \max_{\substack{u,v\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})\\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})}} |\mathrm{Tr}[(U\otimes\mathbb{1}_{\mathcal{Z}})(A_0\otimes\mathbb{1}_{\mathcal{W}})uv^*(A_1\otimes\mathbb{1}_{\mathcal{W}})^*]|^2\\
&= \max_{\substack{u,v\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})\\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})}} |v^*(A_1\otimes\mathbb{1}_{\mathcal{W}})^*(U\otimes\mathbb{1}_{\mathcal{Z}})(A_0\otimes\mathbb{1}_{\mathcal{W}})u|^2\\
&= \max_{\substack{u\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})\\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})}} \|(A_1\otimes\mathbb{1}_{\mathcal{W}})^*(U\otimes\mathbb{1}_{\mathcal{Z}})(A_0\otimes\mathbb{1}_{\mathcal{W}})u\|^2\\
&= \max_{\substack{u\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})\\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})}} \mathrm{Tr}[(A_1A_1^*\otimes\mathbb{1}_{\mathcal{W}})(U\otimes\mathbb{1}_{\mathcal{Z}})(A_0\otimes\mathbb{1}_{\mathcal{W}})uu^*(A_0\otimes\mathbb{1}_{\mathcal{W}})^*(U\otimes\mathbb{1}_{\mathcal{Z}})^*]\\
&= \max_{\substack{u\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})\\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})}} \langle A_1A_1^*, \mathrm{Tr}_{\mathcal{W}}[(U\otimes\mathbb{1}_{\mathcal{Z}})(A_0\otimes\mathbb{1}_{\mathcal{W}})uu^*(A_0\otimes\mathbb{1}_{\mathcal{W}})^*(U\otimes\mathbb{1}_{\mathcal{Z}})^*]\rangle.
\end{aligned}$$

It now remains to prove that

$$\mathcal{A} = \{\mathrm{Tr}_{\mathcal{W}}[(U\otimes\mathbb{1}_{\mathcal{Z}})(A_0\otimes\mathbb{1}_{\mathcal{W}})uu^*(A_0\otimes\mathbb{1}_{\mathcal{W}})^*(U\otimes\mathbb{1}_{\mathcal{Z}})^*] : u \in \mathcal{S}(\mathcal{X}\otimes\mathcal{W}),\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})\}$$

for some choice of $\mathcal{W}$ with $\dim(\mathcal{W}) \geq \dim(\mathcal{X})$. We will choose $\mathcal{W}$ such that

$$\dim(\mathcal{W}) = \max\{\dim(\mathcal{X}), \dim(\mathcal{Y} \otimes \mathcal{Z})\}.$$

First consider an arbitrary choice of $u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{W})$ and $U \in \mathrm{U}(\mathcal{Y} \otimes \mathcal{W})$, and let

$$X = \operatorname{Tr}_{\mathcal{W}}\left[(U \otimes \mathbb{1}_{\mathcal{Z}})(A_0 \otimes \mathbb{1}_{\mathcal{W}})uu^*(A_0 \otimes \mathbb{1}_{\mathcal{W}})^*(U \otimes \mathbb{1}_{\mathcal{Z}})^*\right].$$

It follows that $\operatorname{Tr}_{\mathcal{Y}}(X) = \operatorname{Tr}_{\mathcal{Y}}\left(A_0 \operatorname{Tr}_{\mathcal{W}}(uu^*)A_0^*\right)$, and so $X \in \mathcal{A}$. Now consider an arbitrary element $X \in \mathcal{A}$, and let $\rho \in \mathrm{D}(\mathcal{X})$ satisfy $\operatorname{Tr}_{\mathcal{Y}}(X) = \operatorname{Tr}_{\mathcal{Y}}(A_0 \rho A_0^*)$. Let $u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{W})$ purify ρ and let $x \in \mathcal{Y} \otimes \mathcal{Z} \otimes \mathcal{W}$ purify X. We have

$$\operatorname{Tr}_{\mathcal{Y} \otimes \mathcal{W}}(xx^*) = \operatorname{Tr}_{\mathcal{Y} \otimes \mathcal{W}}\left((A_0 \otimes \mathbb{1}_{\mathcal{W}})uu^*(A_0 \otimes \mathbb{1}_{\mathcal{W}})^*\right),$$

so there exists $U \in \mathrm{U}(\mathcal{Y} \otimes \mathcal{W})$ such that $(U \otimes \mathbb{1}_{\mathcal{Z}})(A_0 \otimes \mathbb{1}_{\mathcal{W}})u = x$, and therefore

$$X = \operatorname{Tr}_{\mathcal{W}}(xx^*) = \operatorname{Tr}_{\mathcal{W}}\left[(U \otimes \mathbb{1}_{\mathcal{Z}})(A_0 \otimes \mathbb{1}_{\mathcal{W}})uu^*(A_0 \otimes \mathbb{1}_{\mathcal{W}})^*(U \otimes \mathbb{1}_{\mathcal{Z}})^*\right].$$

We have therefore proved that

$$\mathcal{A} = \left\{\operatorname{Tr}_{\mathcal{W}}\left[(U \otimes \mathbb{1}_{\mathcal{Z}})(A_0 \otimes \mathbb{1}_{\mathcal{W}})uu^*(A_0 \otimes \mathbb{1}_{\mathcal{W}})^*(U \otimes \mathbb{1}_{\mathcal{Z}})^*\right] : u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{W}),\, U \in \mathrm{U}(\mathcal{Y} \otimes \mathcal{W})\right\},$$

and so we have that the optimal primal value of our semidefinite program is $\alpha = |||\Phi|||_1^2$ as claimed.

21.3 Spectral norm characterization of the completely bounded trace norm

We will now use the semidefinite program from the previous section to obtain a different characterization of the completely bounded trace norm. Let us begin with a definition, followed by a theorem that states the characterization precisely.

Consider any mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$, for complex Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$. For a given choice of a complex Euclidean space $\mathcal{Z}$, we have that there exists a Stinespring representation

$$\Phi(X) = \operatorname{Tr}_{\mathcal{Z}}\left(A_0 X A_1^*\right),$$

for some choice of $A_0, A_1 \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ if and only if $\dim(\mathcal{Z}) \geq \operatorname{rank}(J(\Phi))$. Under the assumption that $\dim(\mathcal{Z}) \geq \operatorname{rank}(J(\Phi))$, we may therefore consider the non-empty set of pairs (A_0, A_1) that represent Φ in this way:

$$\mathcal{S}_\Phi = \left\{(A_0, A_1) : A_0, A_1 \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z}),\, \Phi(X) = \operatorname{Tr}_{\mathcal{Z}}\left(A_0 X A_1^*\right) \text{ for all } X \in \mathrm{L}(\mathcal{X})\right\}.$$

The characterization of the completely bounded trace norm that is established in this section concerns the spectral norm of the operators in this set, and is given by the following theorem.

Theorem 21.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$, and let $\mathcal{Z}$ be a complex Euclidean space with dimension at least* $\operatorname{rank}(J(\Phi))$. *It holds that*

$$|||\Phi|||_1 = \inf\left\{\|A_0\|\,\|A_1\| : (A_0, A_1) \in \mathcal{S}_\Phi\right\}.$$

Proof. For any choice of operators $A_0, A_1 \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ and unit vectors $u, v \in \mathcal{X} \otimes \mathcal{W}$, we have

$$\begin{aligned}\| \mathrm{Tr}_{\mathcal{Z}} [(A_0 \otimes \mathbb{1}_{\mathcal{W}}) u v^* (A_1^* \otimes \mathbb{1}_{\mathcal{W}})] \|_1 &\leq \| (A_0 \otimes \mathbb{1}_{\mathcal{W}}) u v^* (A_1^* \otimes \mathbb{1}_{\mathcal{W}}) \|_1 \\ &\leq \| A_0 \otimes \mathbb{1}_{\mathcal{W}} \| \, \| u v^* \|_1 \, \| A_1 \otimes \mathbb{1}_{\mathcal{W}} \| = \| A_0 \| \, \| A_1 \| ,\end{aligned}$$

which implies that $|||\Phi|||_1 \leq \| A_0 \| \, \| A_1 \|$ for all $(A_0, A_1) \in \mathcal{S}_\Phi$, and consequently

$$|||\Phi||| \leq \inf \{ \| A_0 \| \, \| A_1 \| \, : \, (A_0, A_1) \in \mathcal{S}_\Phi \} .$$

It remains to establish the reverse inequality. Let $(B_0, B_1) \in \mathcal{S}_\Phi$ be an arbitrary pair of operators in $\mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ giving a Stinespring representation for Φ. Given the description of $|||\Phi|||_1^2$ by the semidefinite program from the previous section, along with the fact that strong duality holds for that semidefinite program, we have that $|||\Phi|||_1^2$ is equal to the infimum value of $\| B_0^* (\mathbb{1}_{\mathcal{Y}} \otimes Y) B_0 \|$ over all choices of $Y \in \mathrm{Pos}(\mathcal{Z})$ for which $\mathbb{1}_{\mathcal{Y}} \otimes Y \geq B_1 B_1^*$. This infimum value does not change if we restrict Y to be positive definite, so that

$$|||\Phi|||_1^2 = \inf \{ \| B_0^* (\mathbb{1}_{\mathcal{Y}} \otimes Y) B_0 \| \, : \, \mathbb{1}_{\mathcal{Y}} \otimes Y \geq B_1 B_1^*, \; Y \in \mathrm{Pd}(\mathcal{Z}) \}.$$

For any $\varepsilon > 0$ we may therefore choose $Y \in \mathrm{Pd}(\mathcal{Z})$ such that $\mathbb{1}_{\mathcal{Y}} \otimes Y \geq B_1 B_1^*$ and

$$\left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes Y^{1/2} \right) B_0 \right\|^2 = \| B_0^* (\mathbb{1}_{\mathcal{Y}} \otimes Y) B_0 \| \leq (|||\Phi|||_1 + \varepsilon)^2 .$$

Note that the inequality $\mathbb{1}_{\mathcal{Y}} \otimes Y \geq B_1 B_1^*$ is equivalent to

$$\left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes Y^{-1/2} \right) B_1 \right\|^2 = \left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes Y^{-1/2} \right) B_1 B_1^* \left(\mathbb{1}_{\mathcal{Y}} \otimes Y^{-1/2} \right) \right\| \leq 1.$$

We therefore have that

$$\left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes Y^{1/2} \right) B_0 \right\| \, \left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes Y^{-1/2} \right) B_1 \right\| \leq |||\Phi|||_1 + \varepsilon.$$

It holds that

$$\left(\left(\mathbb{1}_{\mathcal{Y}} \otimes Y^{1/2} \right) B_0, \, \left(\mathbb{1}_{\mathcal{Y}} \otimes Y^{-1/2} \right) B_1 \right) \in \mathcal{S}_\Phi,$$

so

$$\inf \{ \| A_0 \| \, \| A_1 \| \, : \, (A_0, A_1) \in \mathcal{S}_\Phi \} \leq |||\Phi|||_1 + \varepsilon.$$

This inequality holds for all $\varepsilon > 0$, and therefore

$$\inf \{ \| A_0 \| \, \| A_1 \| \, : \, (A_0, A_1) \in \mathcal{S}_\Phi \} \leq |||\Phi|||_1$$

as required. □

21.4 A different semidefinite program for the completely bounded trace norm

There are alternate ways to express the completely bounded trace norm as a semidefinite program from the one described previously. Here is one alternative based on the maximum output fidelity characterization from the start of the lecture.

As before, let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ be an arbitrary mapping. Suppose further that $\mathcal{Z}$ is a complex Euclidean space and $A_0, A_1 \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ satisfy

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(A_0 X A_1^*)$$

for all $X \in \mathrm{L}(\mathcal{X})$. Define completely positive mappings $\Psi_0, \Psi_1 \in \mathrm{T}(\mathcal{X}, \mathcal{Z})$ as

$$\Psi_0(X) = \mathrm{Tr}_{\mathcal{Y}}\left(A_0 X A_0^*\right),$$
$$\Psi_1(X) = \mathrm{Tr}_{\mathcal{Y}}\left(A_1 X A_1^*\right),$$

for all $X \in \mathrm{L}(\mathcal{X})$, and consider the following semidefinite program:

Primal problem

$$\begin{aligned}
\text{maximize:}\quad & \frac{1}{2}\mathrm{Tr}(Y) + \frac{1}{2}\mathrm{Tr}(Y^*)\\
\text{subject to:}\quad & \begin{pmatrix} \Psi_0(\rho_0) & Y \\ Y^* & \Psi_1(\rho_1) \end{pmatrix} \geq 0\\
& \rho_0, \rho_1 \in \mathrm{D}(\mathcal{X})\\
& Y \in \mathrm{L}(\mathcal{Z}).
\end{aligned}$$

Dual problem

$$\begin{aligned}
\text{minimize:}\quad & \frac{1}{2}\left\|\Psi_0^*(Z_0)\right\| + \frac{1}{2}\left\|\Psi_1^*(Z_1)\right\|\\
\text{subject to:}\quad & \begin{pmatrix} Z_0 & -\mathbb{1}_{\mathcal{Z}} \\ -\mathbb{1}_{\mathcal{Z}} & Z_1 \end{pmatrix} \geq 0\\
& Z_0, Z_1 \in \mathrm{Pos}(\mathcal{Z}).
\end{aligned}$$

I will leave it to you to translate this semidefinite program into the formal definition we have been using, and to verify that the dual problem is as stated. Note that the discussion of the semidefinite program for the fidelity function from Lecture 8 is helpful for this task. In light of that discussion, it is not difficult to see that the optimal primal value equals $\mathrm{F}_{\max}(\Psi_0, \Psi_1) = |||\Phi|||_1$. It may also be proved that strong duality holds, leading to an alternate proof of Theorem 21.5.