Semidefinite programs for completely bounded norms

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June 4, 2009

Abstract

The completely bounded trace and spectral norms in finite dimensions are shown to be expressible by semidefinite programs. This provides an efficient method by which these norms may be both calculated and verified, and gives alternate proofs of some known facts about them.

1 Introduction

Linear mappings from one space of operators (or matrices) to another play an important role in quantum information theory. Quantum channels in particular, which model general discrete-time changes in quantum systems, are represented by mappings of this sort.

It is natural to consider *distances* between quantum channels, so as to quantify the similarity with which they act on quantum states. One way to define such a notion is to define a suitable *norm* on the space of mappings in which channels of a given size are represented. Then, the distance between two channels is defined as the norm of their difference. A natural question that arises is: *what norms give rise to the most physically meaningful notions of distance?* As is argued in [GLN05], the answer to this question may depend on the problem at hand—but perhaps the most natural and widely applicable choice within quantum information theory is the *completely bounded trace norm*, also known as the *diamond norm*. This norm was first used in the setting of quantum information by Kitaev [Kit97], who used it mainly as a tool in studying quantum error correction and fault-tolerance. It is equivalent, up to taking the adjoint of a mapping, to its spectral norm variant, which is usually known simply as the *completely bounded norm*. The completely bounded norm, as well as variants that include the completely bounded trace norm, have been studied in operator theory for many years. (See [Pau02] for historical comments and further details.)

The definition of the completely bounded trace and spectral norms, which can be found in the section following this introduction, may seem unnecessarily complicated at first glance. It turns out, however, that they are quite natural and satisfy many remarkable properties. They are, in particular, much easier to reason about and to work with than the seemingly simpler norms induced by the trace norm and spectral norm, primarily because the completely bounded variants of these norms respect the structure of tensor products while the induced norms do not. The physical importance of this property within the setting of quantum information theory has been discussed in several sources [Kit97, AKN98, CPR00, DPP01, Ací01, GLN05, RW05, Sac05a, Sac05b, Ros08, Wat08, PW09]. Additional references that highlight the properties and uses of completely bounded norms in quantum information include [DJKR06, Jen06, PGWP$^+$08].

One obvious question that comes to mind about the completely bounded trace and spectral norms is: *can they be efficiently computed?* Unlike the norms of operators that are most typically encountered in quantum information theory, which are trivially computable from spectral or singular-value decompositions, the computation of completely bounded norms is not known to be straightforward. To the author's knowledge there were only two papers written prior to this one, namely [Zar06] and [JKP09], that presented methods to compute the completely bounded trace or spectral norm of a given mapping. Both papers describe iterative methods, and analyze the complexity of each iteration of these methods, but do not analyze their rates of convergence. So, although these papers may describe potentially efficient methods, they do not include complete proofs of their efficiency.

The purpose of this paper is to explain how the completely bounded trace norm of a given mapping (and therefore its completely bounded spectral norm as well) can be expressed as the optimal value of a semidefinite program whose size is polynomial in the dimension of the spaces on which the mapping acts. Using known algorithms for solving semidefinite programs, one obtains a deterministic polynomial-time algorithm for calculating these norms. This approach also has the obvious practical advantage that it is more easily implemented through the use of existing semidefinite programming optimization libraries, and allows one to take advantage of the extensive work that has been done to solve semidefinite programs efficiently and accurately in practice. Moreover, through semidefinite programming duality, one obtains a means by which a certificate of the value of the completely bounded trace or spectral norm of a given mapping may be quickly verified.

In a recent paper written independently from this one, Ben-Aroya and Ta-Shma [BATS09] have found a different (but related) way to efficiently compute the completely bounded trace norm using convex programming.

The essence of the semidefinite programming formulation of the completely bounded trace norm that is described in this paper appears, at least to some extent, in the paper [KW00]; although it was not made explicit or considered in full generality therein. The present paper aims to present this formulation explicitly and without any discussion of the *quantum interactive proof system* model of computation that was the primary focus of [KW00]. A second semidefinite programming formulation of the completely bounded trace norm is also presented, based on the *quantum games* framework of [GW07]. This formulation is slightly simpler, but is valid only for mappings that are the difference between two quantum channels—which happens to be an important special case in quantum information.

Semidefinite programming is useful not only as a computational tool, but as an analytic tool as well. The last section of this paper gives two examples along these lines that are derived from the more general semidefinite programming formulation of the completely bounded trace norm. The first example concerns an alternate characterization of the completely bounded trace norm and the second illustrates a precise sense in which two known characterizations of the fidelity function (given by Uhlmann's theorem and Alberti's theorem) are dual statements to one another.

2 Background

The two subsections that follow aim to provide the reader with a summary of the background knowledge assumed in the remainder of the paper. The first subsection discusses well-known concepts from linear algebra and (finite-dimensional) operator theory, and is mainly intended to make clear the notation and terminology that is used throughout the paper. It also includes the definitions of the completely bounded norms that are the main focus of this paper. The second subsection discusses semidefinite programming.

Linear algebra and operator theory

Throughout this paper the scripted letters $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, and $\mathcal{W}$ will denote vector spaces of the form $\mathbb{C}^n$ for $n \geq 1$, whose elements are identified with n-dimensional column vectors. On any such space $\mathcal{X} = \mathbb{C}^n$, an inner product is defined as

$$\langle u, v \rangle = \sum_{j=1}^{n} \overline{u_j} v_j,$$

and the Euclidean norm is defined as

$$\| u \| = \sqrt{\langle u, u \rangle}$$

for all $u \in \mathcal{X}$. The unit sphere in $\mathcal{X}$ is denoted

$$\mathcal{S}(\mathcal{X}) = \{ u \in \mathcal{X} \,:\, \| u \| = 1 \},$$

and the j-th elementary unit vector in $\mathcal{X}$ is denoted e_j.

For $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$, the complex vector space consisting of all linear mappings (or operators) of the form $A : \mathcal{X} \to \mathcal{Y}$ is denoted $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ and is identified with the set of $m \times n$ complex matrices in the usual way. For each $A \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$, we define $A^* \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$ to be the unique operator that satisfies $\langle v, Au \rangle = \langle A^* v, u \rangle$ for all $u \in \mathcal{X}$ and $v \in \mathcal{Y}$. In terms of matrices, A^* is simply the conjugate transform of A. By identifying a given vector $u \in \mathcal{X}$ with the linear mapping $\alpha \mapsto \alpha u$, the linear functional $u^* \in \mathrm{L}(\mathcal{X}, \mathbb{C})$ is also defined in this way. More explicitly, u^* satisfies $u^* v = \langle u, v \rangle$ for all $v \in \mathcal{X}$. An inner product on $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ is defined as

$$\langle A, B \rangle = \mathrm{Tr}(A^* B)$$

for $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$. For each pair of indices i, j we write

$$E_{i,j} = e_i e_j^*,$$

which is the operator whose matrix representation has a 1 in entry (i, j) and zeroes in all other entries. The notation $\mathrm{L}(\mathcal{X})$ is shorthand for $\mathrm{L}(\mathcal{X}, \mathcal{X})$, and the identity operator on $\mathcal{X}$, which is an element of $\mathrm{L}(\mathcal{X})$, is denoted $\mathbb{1}_{\mathcal{X}}$.

The following special types of operators are discussed throughout the paper.

1. An operator $X \in \mathrm{L}(\mathcal{X})$ is *Hermitian* if $X = X^*$. The set of such operators is denoted $\mathrm{Herm}(\mathcal{X})$.
2. An operator $P \in \mathrm{L}(\mathcal{X})$ is *positive semidefinite* if it is Hermitian and all of its eigenvalues are nonnegative. The set of such operators is denoted $\mathrm{Pos}(\mathcal{X})$. The notation $P \geq 0$ also indicates that P is positive semidefinite, and more generally the notations $X \leq Y$ and $Y \geq X$ indicate that $Y - X \geq 0$ for Hermitian operators X and Y.
3. An operator $P \in \mathrm{L}(\mathcal{X})$ is *positive definite* if it is both positive semidefinite and invertible. (Equivalently, P is positive definite if it is Hermitian and all of its eigenvalues are positive.) The set of such operators is denoted $\mathrm{Pd}(\mathcal{X})$. The notation $P > 0$ also indicates that P is positive definite, and the notations $X < Y$ and $Y > X$ indicate that $Y - X > 0$ for Hermitian operators X and Y.
4. An operator $\rho \in \mathrm{L}(\mathcal{X})$ is a *density operator* if it is both positive semidefinite and has trace equal to 1. The set of such operators is denoted $\mathrm{D}(\mathcal{X})$.

5. An operator $U \in \mathrm{L}(\mathcal{X})$ is *unitary* if $U^*U = \mathbb{1}_{\mathcal{X}}$. The set of such operators is denoted $\mathrm{U}(\mathcal{X})$.

A useful notion concerning positive semidefinite operators is that of a *purification*. For any positive semidefinite operator $P \in \mathrm{Pos}(\mathcal{X})$, and any space $\mathcal{Y}$ satisfying $\dim(\mathcal{Y}) \geq \mathrm{rank}(P)$, there must exist a vector of the form $u \in \mathcal{X} \otimes \mathcal{Y}$ that satisfies

$$P = \mathrm{Tr}_{\mathcal{Y}}(uu^*),$$

where $\mathrm{Tr}_{\mathcal{Y}}$ denotes the partial trace on $\mathcal{Y}$. The vector u is called a *purification* of P. Given any two purifications $u, v \in \mathcal{X} \otimes \mathcal{Y}$ of a given positive semidefinite operator $P \in \mathrm{Pos}(\mathcal{X})$, it necessarily holds that there exists a unitary operator $U \in \mathrm{U}(\mathcal{Y})$ such that $v = (\mathbb{1}_{\mathcal{X}} \otimes U)u$. This fact is often referred to as the *unitary equivalence of purifications*.

Three operator norms are discussed in this paper: the *trace norm*, *Frobenius norm*, and *spectral norm*, defined as

$$\|A\|_1 = \mathrm{Tr}\sqrt{A^*A}, \qquad \|A\|_2 = \sqrt{\langle A, A\rangle}, \qquad \text{and} \qquad \|A\|_\infty = \max_{u \in \mathcal{S}(\mathcal{X})} \|Au\|,$$

respectively, for each $A \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$. These norms correspond precisely to the 1-norm, 2-norm, and ∞-norm of the vector of singular values of A, which explains the notation used to denote them.

Some specific properties of the above norms that will be needed in this paper will now be summarized. (Readers interested in a more comprehensive discussion are referred to [Bha97].) First, for every operator A it holds that

$$\|A\|_\infty \leq \|A\|_2 \leq \|A\|_1,$$

which is clear from the description of these norms in terms of the singular values of A. Next, it holds that trace and spectral norms are dual, which means that

$$\|A\|_1 = \max\{|\langle B, A\rangle| \,:\, \|B\|_\infty \leq 1\}$$
$$\|A\|_\infty = \max\{|\langle B, A\rangle| \,:\, \|B\|_1 \leq 1\}$$

for all $A \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$, and with B ranging over operators within the same space. By convexity it is sufficient to restrict the maximizations in these equations to those choices of B that are extreme points in the (compact and convex) sets being maximized over. In particular, for operators of the form $X \in \mathrm{L}(\mathcal{X})$ it holds that

$$\|X\|_1 = \max_{U \in \mathrm{U}(\mathcal{X})} |\langle U, X\rangle|.$$

Various other properties follow from the duality of the trace and spectral norms, including the inequality

$$|\langle A, B\rangle| \leq \|A\|_1 \|B\|_\infty, \tag{1}$$

along with the monotonicity of the trace norm: for every operator of the form $A \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$ it holds that

$$\|A\|_1 \geq \|\mathrm{Tr}_{\mathcal{Y}}(A)\|_1. \tag{2}$$

Finally, we note that

$$\|A^*A\|_\infty = \|AA^*\|_\infty = \|A\|_\infty^2 \tag{3}$$

for every choice of an operator A.

Linear mappings from one space of operators to another are very important in the theory of quantum information. The space of all linear mapping of the form $\Phi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{Y})$ is denoted $\mathrm{T}(\mathcal{X}, \mathcal{Y})$. The adjoint mapping to $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is the unique mapping $\Phi^* \in \mathrm{T}(\mathcal{Y}, \mathcal{X})$ for which

$$\langle Y, \Phi(X)\rangle = \langle \Phi^*(Y), X\rangle$$

for all $X \in \mathrm{L}(\mathcal{X})$ and $Y \in \mathrm{L}(\mathcal{Y})$. The identity mapping from $\mathrm{L}(\mathcal{X})$ to itself is denoted $\mathbb{1}_{\mathrm{L}(\mathcal{X})}$.

The following special types of mappings will be referred to later in the paper:

1. $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is *Hermiticity-preserving* if $\Phi(X) \in \mathrm{Herm}(\mathcal{Y})$ for every $X \in \mathrm{Herm}(\mathcal{X})$.
2. $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is *completely positive* if it holds that

$$(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{W})})(P) \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{W})$$

   for every choice of $\mathcal{W} = \mathbb{C}^k$ and $P \in \mathrm{Pos}(\mathcal{X} \otimes \mathcal{W})$.
3. $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is *trace-preserving* if $\mathrm{Tr}(\Phi(X)) = \mathrm{Tr}(X)$ for every $X \in \mathrm{L}(\mathcal{X})$.
4. $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is a *quantum channel* if it is both completely positive and trace-preserving.

The *Choi-Jamiołkowski representation* $J(\Phi) \in \mathrm{L}(\mathcal{Y} \otimes \mathcal{X})$ of a mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is the operator defined as

$$J(\Phi) = \sum_{1 \leq i,j \leq n} \Phi(E_{i,j}) \otimes E_{i,j}$$

(where this expression assumes $\mathcal{X} = \mathbb{C}^n$). The mapping J is a linear bijection from $\mathrm{T}(\mathcal{X}, \mathcal{Y})$ to $\mathrm{L}(\mathcal{Y} \otimes \mathcal{X})$. The operator $J(\Phi)$, written as an $nm \times nm$ matrix, represents one convenient way that a mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ may be expressed in concrete terms. It holds that

1. $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is Hermiticity-preserving if and only if $J(\Phi)$ is Hermitian [dP67],
2. $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is completely positive if and only if $J(\Phi)$ is positive semidefinite [Jam72, Cho75], and
3. $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is trace-preserving if and only if $\mathrm{Tr}_{\mathcal{Y}}(J(\Phi)) = \mathbb{1}_{\mathcal{X}}$.

A pair of operators (A, B) in $\mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ is a *Stinespring pair* for $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ if it holds that

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(AXB^*) \tag{4}$$

for all $X \in \mathrm{L}(\mathcal{X})$, and an expression of the form (4) is called a *Stinespring representation* of Φ. The minimal dimension of $\mathcal{Z}$ for which a Stinespring representation of Φ exists is equal to $\mathrm{rank}(J(\Phi))$. It is straightforward to compute such a Stinespring pair (A, B) from the Choi-Jamiołkowski representation of Φ—for any expression

$$J(\Phi) = \sum_{l=1}^{r} u_l v_l^*,$$

it holds that

$$A = \sum_{l=1}^{r} \sum_{i=1}^{m} \sum_{j=1}^{n} \langle e_i \otimes e_j, u_l \rangle \, E_{i,j} \otimes e_l \qquad \text{and} \qquad B = \sum_{l=1}^{r} \sum_{i=1}^{m} \sum_{j=1}^{n} \langle e_i \otimes e_j, v_l \rangle \, E_{i,j} \otimes e_l$$

forms a Stinespring pair of Φ (assuming $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$).

It will be helpful later to observe that if $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is a given mapping, $\mathcal{Z}$ is taken to have the minimal dimension $\dim(\mathcal{Z}) = \mathrm{rank}(J(\Phi))$ to admit a Stinespring pair of Φ, and if (A, B) is such a pair, then it must hold that $\mathrm{Tr}_{\mathcal{Y}}(AA^*)$ and $\mathrm{Tr}_{\mathcal{Y}}(BB^*)$ are positive definite.

Now we will discuss norms of operator mappings. For each $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$, one defines the *induced* norms:

$$\|\Phi\|_1 = \max \{ \|\Phi(X)\|_1 \,:\, X \in \mathrm{L}(\mathcal{X}),\ \|X\|_1 \leq 1 \},$$

$$\|\Phi\|_\infty = \max \{ \|\Phi(X)\|_\infty \,:\, X \in \mathrm{L}(\mathcal{X}),\ \|X\|_\infty \leq 1 \},$$

as well as *completely bounded* variants of these norms:

$$|||\Phi|||_1 = \sup_{k\geq 1} \left\| \Phi \otimes \mathbb{1}_{\mathrm{L}(\mathbb{C}^k)} \right\|_1 \qquad \text{and} \qquad |||\Phi|||_\infty = \sup_{k\geq 1} \left\| \Phi \otimes \mathbb{1}_{\mathrm{L}(\mathbb{C}^k)} \right\|_\infty .$$

As was done in the introduction, we will refer to $|||\Phi|||_1$ as the *completely bounded trace norm* and to $|||\Phi|||_\infty$ as the *completely bounded spectral norm*. It follows from the duality of the trace and spectral norms that $|||\Phi|||_1 = |||\Phi^*|||_\infty$ for every $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$.

It is common that $|||\Phi|||_1$ is denoted $\|\Phi\|_\diamond$ and called the *diamond norm*, and that $|||\Phi|||_\infty$ is denoted $\|\Phi\|_{\mathrm{cb}}$ and called simply the *completely bounded norm*. We will not follow this notation (or terminology) in this paper in the interest of making the connections of these norms to the trace and spectral norms clear, as well as to stress the close relationship they share through the duality of the trace and spectral norms.

For every mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ it holds that

$$|||\Phi|||_1 = \left\| \Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})} \right\|_1 \qquad \text{and} \qquad |||\Phi|||_\infty = \left\| \Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Y})} \right\|_\infty .$$

These two equalities are equivalent (through the duality of the trace and spectral norms), and can be proved in multiple ways. The second equality was evidently first proved by Smith [Smi83], who considered the general situation where $\mathcal{X}$ need not be finite-dimensional. For the finite-dimensional case, simple proofs of the first equality (and therefore the second by duality) can be found in [GLN05] and [Wat05].

The completely bounded trace and spectral norms are both multiplicative with respect to tensor products, meaning that

$$|||\Phi \otimes \Psi|||_1 = |||\Phi|||_1 \, |||\Psi|||_1 \qquad \text{and} \qquad |||\Phi \otimes \Psi|||_\infty = |||\Phi|||_\infty \, |||\Psi|||_\infty$$

for any choice of mappings Φ and Ψ. A proof of this fact can be found in [KSV02].

Next, let us note for convenience that for any mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ it holds that

$$|||\Phi|||_1 = \max \left\{ \left\| (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(uv^*) \right\|_1 \, : \, u, v \in \mathcal{S}(\mathcal{X} \otimes \mathcal{X}) \right\} . \tag{5}$$

This is a simple consequence of the convexity of norms along with the fact that the extreme points of the unit ball with respect to the trace norm take the form uv^* for unit vectors u and v. For a Hermiticity-preserving mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$, the expression (5) can be further simplified [GLN05, RW05] as

$$|||\Phi|||_1 = \max \left\{ \left\| (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(uu^*) \right\|_1 \, : \, u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{X}) \right\} .$$

This equation is generally not valid for non-Hermiticity-preserving mappings.

Finally, we note that small perturbations in the Stinespring representations of mappings cause small perturbations in the completely bounded trace norm. In particular, if $\Phi_0, \Phi_1 \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ satisfy

$$\Phi_0(X) = \mathrm{Tr}_{\mathcal{Z}}(A_0 X B_0^*) \qquad \text{and} \qquad \Phi_1(X) = \mathrm{Tr}_{\mathcal{Z}}(A_1 X B_1^*),$$

then it holds that

$$|||\Phi_0 - \Phi_1|||_1 \leq \|A_0\|_\infty \|B_0 - B_1\|_\infty + \|B_1\|_\infty \|A_0 - A_1\|_\infty . \tag{6}$$

One may also exchange A_0 with A_1 and B_0 with B_1 in this inequality. The inequality (6) follows easily from the inequalities (2) and (1), along with the triangle inequality. (This inequality was proved in [KSW08] for the special case that $A_0 = B_0$ and $A_1 = B_1$, and the general case follows by identical reasoning.)

Semidefinite programming

This section gives a brief overview of semidefinite programming, which is discussed in greater detail in several sources (including [Ali95, VB96, dK02, Lov03, BV04], for instance).

A *semidefinite program* over $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$ is specified by a triple (Ψ, C, D), where

1. $\Psi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is a Hermiticity-preserving mapping, and
2. $C \in \mathrm{Herm}(\mathcal{X})$ and $D \in \mathrm{Herm}(\mathcal{Y})$ are Hermitian operators.

The following two optimization problems are associated with such a semidefinite program:

Primal problem		Dual problem	
maximize:	$\langle C, X \rangle$	minimize:	$\langle D, Y \rangle$
subject to:	$\Psi(X) \leq D,$	subject to:	$\Psi^*(Y) \geq C,$
	$X \in \mathrm{Pos}(\mathcal{X}).$		$Y \in \mathrm{Pos}(\mathcal{Y}).$

Readers familiar with semidefinite programming will note that the above form of a semidefinite program is different from the well-known *standard form*, but it is equivalent and better suited for this paper's needs. The form given above is, in essence, the one that is typically followed for general conic programming [BV04].

With the above optimization problems in mind, one defines the *primal feasible* set $\mathcal{P}$ and the *dual feasible* set $\mathcal{D}$ as

$$\mathcal{P} = \{X \in \mathrm{Pos}(\mathcal{X}) \,:\, \Psi(X) \leq D\},$$
$$\mathcal{D} = \{Y \in \mathrm{Pos}(\mathcal{Y}) \,:\, \Psi^*(Y) \geq C\}.$$

Operators $X \in \mathcal{P}$ and $Y \in \mathcal{D}$ are also said to be *primal feasible* and *dual feasible*, respectively. The functions $X \mapsto \langle C, X \rangle$ and $Y \mapsto \langle D, Y \rangle$ are called the primal and dual *objective functions*, and the *optimal values* associated with the primal and dual problems are defined as follows:

$$\alpha = \sup_{X \in \mathcal{P}} \langle C, X \rangle \qquad \text{and} \qquad \beta = \inf_{Y \in \mathcal{D}} \langle D, Y \rangle.$$

(If it is the case that $\mathcal{P} = \varnothing$ or $\mathcal{D} = \varnothing$, the above definitions are to be interpreted as $\alpha = -\infty$ and $\beta = \infty$, respectively.) The supremum and infimum cannot always be replaced by the maximum and minimum—in some cases even finite values α and β may not be achieved for any choice of $X \in \mathcal{P}$ and $Y \in \mathcal{D}$.

Semidefinite programs have associated with them a powerful theory of *duality*, which refers to the special relationship between the primal and dual problems. The property of *weak duality*, which holds for all semidefinite programs, is stated in the following theorem.

Theorem 1 (Weak duality). *For every semidefinite program (Ψ, C, D) as defined above, it holds that $\alpha \leq \beta$.*

This property implies that every dual feasible operator $Y \in \mathcal{D}$ provides an upper bound of $\langle D, Y \rangle$ on the value $\langle C, X \rangle$ that is achievable over all choices of a primal feasible $X \in \mathcal{P}$, and likewise every primal feasible operator $X \in \mathcal{P}$ provides a lower bound of $\langle C, X \rangle$ on the value $\langle D, Y \rangle$ that is achievable over all choices of a dual feasible $Y \in \mathcal{D}$.

It is not always the case that $\alpha = \beta$ for a given semidefinite program (Ψ, C, D), even when α and β are finite. For most semidefinite programs that arise in practice, however, it is the case that $\alpha = \beta$, which is a situation known as *strong duality*. There are different conditions under which this property is guaranteed, one of which is given by the following theorem.

Theorem 2 (Slater-type condition for strong duality)**.** *The following two implications hold for every semidefinite program* (Ψ, C, D) *as defined above.*

1. *Strict primal feasibility: If* β *is finite and there exists an operator* $X > 0$ *such that* $\Psi(X) < D$*, then* $\alpha = \beta$ *and there exists* $Y \in \mathcal{D}$ *such that* $\langle D, Y \rangle = \beta$*.*
2. *Strict dual feasibility: If* α *is finite and there exists an operator* $Y > 0$ *such that* $\Psi^*(Y) > C$*, then* $\alpha = \beta$ *and there exists* $X \in \mathcal{P}$ *such that* $\langle C, X \rangle = \alpha$*.*

One may consider a general computational problem that asks for the optimal primal and dual values of a given semidefinite program, possibly up to some specified accuracy. The specific formulation of this problem that this paper follows requires that we first define, for each choice of $\varepsilon > 0$, the following sets:

$$\mathcal{P}_\varepsilon = \{X \in \mathrm{Pos}(\mathcal{X}) \,:\, X + H \in \mathcal{P} \text{ for all } H \in \mathrm{Herm}(\mathcal{X}) \text{ satisfying } \|H\|_2 \leq \varepsilon\},$$
$$\mathcal{D}_\varepsilon = \{Y \in \mathrm{Pos}(\mathcal{Y}) \,:\, Y + H \in \mathcal{D} \text{ for all } H \in \mathrm{Herm}(\mathcal{Y}) \text{ satisfying } \|H\|_2 \leq \varepsilon\}.$$

Intuitively speaking, $\mathcal{P}_\varepsilon$ contains primal feasible operators that are not too close to the boundary of the primal feasible set, and likewise for $\mathcal{D}_\varepsilon$. Having defined the sets $\mathcal{P}_\varepsilon$ and $\mathcal{D}_\varepsilon$ in this way, we now phrase the problem of approximating the optimal value of a semidefinite program as a *promise problem* [ESY84] as follows:

Problem 3. The *semidefinite programming approximation* problem is as follows.

Input: A semidefinite program (Ψ, C, D) over $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$, an accuracy parameter $\varepsilon > 0$, and a positive integer R.

Promise: The set $\mathcal{P}_\varepsilon$ is non-empty, and for every $X \in \mathcal{P}$ it holds that $\|X\|_2 \leq R$. (In the terminology of [GLS93], the primal feasible region $\mathcal{P}$ of (Ψ, C, D) is *well-bounded*, with parameters ε and R.)

Output: A real number γ such that $|\gamma - \alpha| < \varepsilon$, where α is the optimal value of the primal problem associated with (Ψ, C, D).

The description of this problem does not explicitly state how the mapping Ψ is to be represented, but we will assume it is specified by the matrix representation of $J(\Psi)$. Other forms, including Stinespring representations and Kraus representations, are easily converted to this form. It is also assumed that the entries of $J(\Psi)$, C, and D have rational real and imaginary parts.

The computational problem stated above can be solved in polynomial time using the ellipsoid method [GLS93], as the following theorem states.

Theorem 4. *There exists an algorithm that solves the semidefinite programming approximation problem stated above that runs in time polynomial in* n*,* m*,* $\log(R)$*,* $\log(1/\varepsilon)$*, and the maximum bit-length of the entries of* $J(\Psi)$*,* C*, and* D*.*

Here, the *bit-length* of a complex number $z = (a/b) + i(c/d)$ is the number of bits needed to represent the 4-tuple (a, b, c, d), where a, b, c, and d are integers written in binary notation. It should be noted that one would typically not use the ellipsoid method to solve semidefinite programming problems in practice, given that *interior point methods* [Ali95, dK02] are considered to be more practical.

Note that the above problem asks only for an approximation to the optimal primal value, but the simple transformation $(\Psi, C, D) \to (-\Psi^*, -D, -C)$ shows that any algorithm for it also allows one to approximate the optimal dual value. (Alternately, the ellipsoid method can be applied directly to the dual problem.)

It is possible to approximate more general classes of semidefinite programs efficiently. For instance, the bound $\|X\|_2 \leq R$ need not hold for every primal feasible X, provided certain assumptions are known about the size of the optimal solution. These generalizations are not important for this paper, however, and the above problem has the advantage of being easily fit to the general presentation of [GLS93] (which is described for the specific setting of semidefinite programming in [Lov03]).

3 A semidefinite program for the completely bounded trace norm

We will now describe and analyze a semidefinite program whose optimal (primal and dual) value is $|||\Phi|||_1^2$, where $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is an arbitrary mapping given by a Stinespring representation

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(AXB^*)$$

for $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$. It is assumed $\mathcal{Z}$ has the minimal dimension $\dim(\mathcal{Z}) = \mathrm{rank}(J(\Phi))$ for which such a Stinespring representation exists, although this assumption is really only necessary for the discussion of computational efficiency below.

The primal and dual problems associated with the semidefinite program we will consider may be stated as follows:

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \langle BB^*, W\rangle \\ \text{subject to:} \quad & \mathrm{Tr}_{\mathcal{Y}}(W) = \mathrm{Tr}_{\mathcal{Y}}(A\rho A^*), \\ & \rho \in \mathrm{D}(\mathcal{X}), \\ & W \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{Z}). \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \|A^*(\mathbb{1}_{\mathcal{Y}} \otimes Z)A\|_\infty \\ \text{subject to:} \quad & \mathbb{1}_{\mathcal{Y}} \otimes Z \geq BB^*, \\ & Z \in \mathrm{Pos}(\mathcal{Z}). \end{aligned}$$

These problems are associated with the semidefinite program that is more formally specified in the following way. We define a Hermiticity-preserving mapping

$$\Psi : \mathrm{L}(\mathcal{X} \oplus (\mathcal{Y} \otimes \mathcal{Z})) \rightarrow \mathrm{L}(\mathbb{C} \oplus \mathcal{Z})$$

as

$$\Psi \begin{pmatrix} X & \cdot \\ \cdot & W \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}(X) & 0 \\ 0 & \mathrm{Tr}_{\mathcal{Y}}(W - AXA^*) \end{pmatrix}.$$

The adjoint mapping

$$\Psi^* : \mathrm{L}(\mathbb{C} \oplus \mathcal{Z}) \rightarrow \mathrm{L}(\mathcal{X} \oplus (\mathcal{Y} \otimes \mathcal{Z}))$$

is given by

$$\Psi^* \begin{pmatrix} \lambda & \cdot \\ \cdot & Z \end{pmatrix} = \begin{pmatrix} \lambda \mathbb{1}_{\mathcal{X}} - A^*(\mathbb{1}_{\mathcal{Y}} \otimes Z)A & 0 \\ 0 & \mathbb{1}_{\mathcal{Y}} \otimes Z \end{pmatrix}.$$

(In these expressions of Ψ and Ψ^*, the symbol $\cdot$ denotes an operator or vector of the appropriate dimensions upon which the output of these mappings does not depend.) We also define $C \in \mathrm{Herm}(\mathcal{X} \oplus (\mathcal{Y} \otimes \mathcal{Z}))$ and $D \in \mathrm{Herm}(\mathbb{C} \oplus \mathcal{Z})$ as

$$C = \begin{pmatrix} 0 & 0 \\ 0 & BB^* \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, the primal and dual problem associated with (Ψ, C, D) may be expressed as follows:

Primal problem

$$\begin{aligned}\text{maximize:}\quad & \langle BB^*, W\rangle \\ \text{subject to:}\quad & \operatorname{Tr}_{\mathcal{Y}}(W) \leq \operatorname{Tr}_{\mathcal{Y}}(AXA^*), \\ & \operatorname{Tr}(X) \leq 1, \\ & X \in \operatorname{Pos}(\mathcal{X}), \\ & W \in \operatorname{Pos}(\mathcal{Y} \otimes \mathcal{Z}).\end{aligned}$$

Dual problem

$$\begin{aligned}\text{minimize:}\quad & \lambda \\ \text{subject to:}\quad & \lambda \mathbb{1}_{\mathcal{X}} \geq A^*(\mathbb{1}_{\mathcal{Y}} \otimes Z)A \\ & \mathbb{1}_{\mathcal{Y}} \otimes Z \geq BB^*, \\ & \lambda \geq 0, \\ & Z \in \operatorname{Pos}(\mathcal{Z}).\end{aligned}$$

Notice that for any choice of a primal feasible operator

$$\begin{pmatrix} X & M \\ M^* & W \end{pmatrix}, \tag{7}$$

there exist operators $P \in \operatorname{Pos}(\mathcal{X})$ and $Q \in \operatorname{Pos}(\mathcal{Y} \otimes \mathcal{Z})$ such that $\operatorname{Tr}(X + P) = 1$ and

$$\operatorname{Tr}_{\mathcal{Y}}(W + Q) = \operatorname{Tr}_{\mathcal{Y}}(A(X + P)A^*).$$

The operator

$$\begin{pmatrix} X + P & M \\ M^* & W + Q \end{pmatrix}$$

is therefore primal feasible, and obtains at least the value achieved by (7) (by virtue of the fact that BB^* is positive semidefinite). This explains the equivalence of the two different statements for the primal problem above, where the equality constraints in the first are replaced by inequality constraints in the second. The two statements of the dual problems are obviously equivalent, because $A^*(\mathbb{1}_{\mathcal{Y}} \otimes Z)A$ is positive semidefinite for any positive semidefinite operator Z, and therefore

$$\min\{\lambda \geq 0 : \lambda \mathbb{1}_{\mathcal{X}} \geq A^*(\mathbb{1}_{\mathcal{Y}} \otimes Z)A\} = \|A^*(\mathbb{1}_{\mathcal{Y}} \otimes Z)A\|_\infty.$$

Strong duality

We will now verify that strong duality holds for the above semidefinite program, using Theorem 2. To begin, we will prove that the optimal primal value α is necessarily finite. By equations (1) and (3), it holds that

$$\langle BB^*, W\rangle \leq \|BB^*\|_\infty \|W\|_1 = \|B\|_\infty^2 \operatorname{Tr}(W)$$

for every positive semidefinite operator W. Whenever W is primal feasible it holds that

$$\operatorname{Tr}(W) \leq \operatorname{Tr}(AXA^*) = \langle A^*A, X\rangle \leq \|A^*A\|_\infty \|X\|_1 \leq \|A\|_\infty^2$$

for some $X \in \operatorname{Pos}(\mathcal{X})$ with $\operatorname{Tr}(X) \leq 1$. Thus, the optimal primal value satisfies

$$\alpha \leq \|A\|_\infty^2 \|B\|_\infty^2,$$

which shows that α is finite.

Now, to verify strict dual feasibility, suppose that μ and λ are positive real numbers such that $\mu > \|B\|_\infty^2$ and $\lambda > \mu \|A\|_\infty^2$. Then it holds that

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \mathbb{1}_{\mathcal{Z}} \end{pmatrix} > 0$$

and

$$\Psi^*\begin{pmatrix}\lambda & 0\\ 0 & \mu\mathbb{1}_{\mathcal{Z}}\end{pmatrix} = \begin{pmatrix}\lambda\mathbb{1}_{\mathcal{X}} - \mu A^*A & 0\\ 0 & \mu\mathbb{1}_{\mathcal{Y}}\otimes\mathbb{1}_{\mathcal{Z}}\end{pmatrix} > \begin{pmatrix}0 & 0\\ 0 & BB^*\end{pmatrix},$$

which illustrates strict dual feasibility. Thus, by Theorem 2, the optimal value α associated with the primal problem is equal to the optimal dual value β, and is achieved for some choice of a primal feasible operator.

Having already established strong duality, it is not really essential for this paper's needs that strict primal feasibility is proved. One may wonder, however, whether strict primal feasibility holds for the semidefinite program above. Indeed it does, but it relies on the assumption

$$\dim(\mathcal{Z}) = \operatorname{rank}(J(\Phi)).$$

This observation, which happens to imply that the optimal dual value is achieved for some dual feasible operator, will follow from the discussion of computational efficiency below.

Optimal value

Now let us verify that the optimal value $\alpha = \beta$ of our semidefinite program is equal to $|||\Phi|||_1^2$. To do this we first define $\mathcal{W} = \mathbb{C}^k$ for $k = \max\{\dim(\mathcal{X}), \dim(\mathcal{Y}\otimes\mathcal{Z})\}$. Given that $\dim(\mathcal{W}) \geq \dim(\mathcal{X})$, it holds that

$$\begin{aligned}
|||\Phi|||_1^2 &= \max_{\substack{u,v\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})\\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})}} \left|\langle U, \operatorname{Tr}_{\mathcal{Z}}\left((A\otimes\mathbb{1}_{\mathcal{W}})uv^*(B^*\otimes\mathbb{1}_{\mathcal{W}})\right)\rangle\right|^2\\
&= \max_{\substack{u\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})\\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})}} \left\|(B^*\otimes\mathbb{1}_{\mathcal{W}})(U^*\otimes\mathbb{1}_{\mathcal{Z}})(A\otimes\mathbb{1}_{\mathcal{W}})u\right\|^2\\
&= \max_{\substack{u\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})\\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})}} u^*(A^*\otimes\mathbb{1}_{\mathcal{W}})(U\otimes\mathbb{1}_{\mathcal{Z}})(BB^*\otimes\mathbb{1}_{\mathcal{W}})(U^*\otimes\mathbb{1}_{\mathcal{Z}})(A\otimes\mathbb{1}_{\mathcal{W}})u\\
&= \max_{\substack{u\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W})\\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})}} \langle BB^*, \operatorname{Tr}_{\mathcal{W}}\left[(U^*\otimes\mathbb{1}_{\mathcal{Z}})(A\otimes\mathbb{1}_{\mathcal{W}})uu^*(A^*\otimes\mathbb{1}_{\mathcal{W}})(U\otimes\mathbb{1}_{\mathcal{Z}})\right]\rangle.
\end{aligned}$$

Now define two sets $\mathcal{Q},\mathcal{R}\subseteq \operatorname{Pos}(\mathcal{Y}\otimes\mathcal{Z})$ as

$$\begin{aligned}
\mathcal{Q} &= \{W\in\operatorname{Pos}(\mathcal{Y}\otimes\mathcal{Z}) \,:\, \operatorname{Tr}_{\mathcal{Y}}(W) = \operatorname{Tr}_{\mathcal{Y}}(A\rho A^*) \text{ for some choice of } \rho\in\mathrm{D}(\mathcal{X})\},\\
\mathcal{R} &= \{\operatorname{Tr}_{\mathcal{W}}\left[(U^*\otimes\mathbb{1}_{\mathcal{Z}})(A\otimes\mathbb{1}_{\mathcal{W}})uu^*(A^*\otimes\mathbb{1}_{\mathcal{W}})(U\otimes\mathbb{1}_{\mathcal{Z}})\right] \,:\, u\in\mathcal{S}(\mathcal{X}\otimes\mathcal{W}),\ U\in\mathrm{U}(\mathcal{Y}\otimes\mathcal{W})\}.
\end{aligned}$$

Our interest in the set $\mathcal{R}$ is clear, for the equation above has established that

$$|||\Phi|||_1^2 = \max_{W\in\mathcal{R}}\langle BB^*, W\rangle.$$

The set $\mathcal{Q}$, on the other hand, is of interest because the optimal value α of the primal problem for the semidefinite program defined above is given by

$$\alpha = \max_{W\in\mathcal{Q}}\langle BB^*, W\rangle.$$

To establish that $\alpha = |||\Phi|||_1^2$, it therefore suffices to prove that $\mathcal{Q} = \mathcal{R}$, which can be done as follows.

First consider an arbitrary choice of $u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{W})$ and $U \in \mathrm{U}(\mathcal{Y} \otimes \mathcal{W})$, and let

$$W = \mathrm{Tr}_{\mathcal{W}}\left[(U^* \otimes \mathbb{1}_{\mathcal{Z}})(A \otimes \mathbb{1}_{\mathcal{W}})uu^*(A^* \otimes \mathbb{1}_{\mathcal{W}})(U \otimes \mathbb{1}_{\mathcal{Z}})\right].$$

Then $\mathrm{Tr}_{\mathcal{Y}}(W) = \mathrm{Tr}_{\mathcal{Y}}\left(A\,\mathrm{Tr}_{\mathcal{W}}(uu^*)A^*\right)$, and so it holds that $W \in \mathcal{Q}$, which proves $\mathcal{R} \subseteq \mathcal{Q}$.

Now consider an arbitrary element $W \in \mathcal{Q}$, and let $\rho \in \mathrm{D}(\mathcal{X})$ be a density operator satisfying $\mathrm{Tr}_{\mathcal{Y}}(W) = \mathrm{Tr}_{\mathcal{Y}}(A\rho A^*)$. Given that we have chosen $\mathcal{W}$ to have dimension at least as large as that of both $\mathcal{X}$ and $\mathcal{Y} \otimes \mathcal{Z}$, there must exist vectors $u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{W})$ and $w \in \mathcal{Y} \otimes \mathcal{Z} \otimes \mathcal{W}$ that purify ρ and W, respectively, meaning that $\rho = \mathrm{Tr}_{\mathcal{W}}(uu^*)$ and $W = \mathrm{Tr}_{\mathcal{W}}(ww^*)$. This implies that

$$\mathrm{Tr}_{\mathcal{Y}\otimes\mathcal{W}}(ww^*) = \mathrm{Tr}_{\mathcal{Y}\otimes\mathcal{W}}\left((A \otimes \mathbb{1}_{\mathcal{W}})uu^*(A^* \otimes \mathbb{1}_{\mathcal{W}})\right),$$

so by the unitary equivalence of purifications there must exist a unitary operator $U \in \mathrm{U}(\mathcal{Y} \otimes \mathcal{W})$ such that $(U^* \otimes \mathbb{1}_{\mathcal{Z}})(A \otimes \mathbb{1}_{\mathcal{W}})u = w$. Therefore

$$W = \mathrm{Tr}_{\mathcal{W}}(ww^*) = \mathrm{Tr}_{\mathcal{W}}\left[(U^* \otimes \mathbb{1}_{\mathcal{Z}})(A \otimes \mathbb{1}_{\mathcal{W}})uu^*(A^* \otimes \mathbb{1}_{\mathcal{W}})(U \otimes \mathbb{1}_{\mathcal{Z}})\right],$$

which proves that $W \in \mathcal{R}$, so that $\mathcal{Q} \subseteq \mathcal{R}$ as required.

Computational efficiency

Now let us verify that the optimal value $|||\Phi|||_1^2$ of the semidefinite program described above can be approximated by an efficient computation. By Theorem 4 our task is to argue that suitable parameters R and ε for the promise in Problem 3 can be determined.

For the sake of clarity, let us summarize our notation: we have $\mathcal{X} = \mathbb{C}^n$, $\mathcal{Y} = \mathbb{C}^m$, $\mathcal{Z} = \mathbb{C}^r$, and $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ is the mapping given by

$$\Phi(X) = \mathrm{Tr}_{\mathcal{Z}}(AXB^*)$$

for which we wish to approximate $|||\Phi|||_1^2$. The semidefinite program that represents this quantity is represented by the Hermiticity-preserving mapping $\Psi \in \mathrm{T}(\mathcal{X} \oplus (\mathcal{Y} \otimes \mathcal{Z}), \mathbb{C} \oplus \mathcal{Z})$ and Hermitian operators $C \in \mathrm{Herm}(\mathcal{X} \oplus (\mathcal{Y} \otimes \mathcal{Z}))$ and $D \in \mathrm{Herm}(\mathbb{C} \oplus \mathcal{Z})$ as described above.

At this point it is important to stress that we must make an assumption on the operators A and B, which is that they are represented by matrices having rational real and imaginary parts. This is a natural assumption to make within a computational setting—and for an arbitrary choice of A and B we note that one may simply approximate the entries of A and B by numbers having rational real and imaginary parts, and then apply (6) to bound the error that results from this approximation.

Hereafter we will write N to refer to the total bit-length of the semidefinite program (Ψ, C, D), which is polynomially related to n, m and the maximum bit-length of the entries of A and B.

First, it is clear that every primal feasible operator has trace bounded by $1 + \|A\|_\infty^2$. Given that the Frobenius norm is upper-bounded by the trace for positive semidefinite operators, it therefore suffices to choose $R = 1 + \|A\|_\infty^2$, which is obviously bounded by 2^{cN} for some positive integer constant c.

The specification of ε is slightly more complicated. Consider first the operator $\mathrm{Tr}_{\mathcal{Y}}(AA^*)$. We have chosen $\mathcal{Z}$ to have minimal dimension to admit a Stinespring representation of Φ, and from this assumption it follows that $\mathrm{Tr}_{\mathcal{Y}}(AA^*)$ is positive definite. Using the assumption that the real and imaginary parts of the entries of A are rational, along with the fact that nonzero roots of integer polynomials cannot be too close to zero (see, for instance, Theorem 2.9 of [Bug04]), one may derive a lower-bound on the smallest eigenvalue of $\operatorname{Tr}_{\mathcal{Y}}(AA^*)$. For the purposes of this analysis, it suffices to note that there exists an integer constant $d_0 \geq 1$ such that for $\delta = 2^{-d_0 N}$ we have that the smallest eigenvalue of $\operatorname{Tr}_{\mathcal{Y}}(AA^*)$ is at least δ, and therefore $\delta\,\mathbb{1}_{\mathcal{Z}} \leq \operatorname{Tr}_{\mathcal{Y}}(AA^*)$.

Now consider the operator

$$P = \begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix}$$

where

$$X = \frac{3}{4n}\mathbb{1}_{\mathcal{X}} \qquad \text{and} \qquad W = \frac{3}{8nm}\mathbb{1}_{\mathcal{Y}} \otimes \operatorname{Tr}_{\mathcal{Y}}(AA^*),$$

along with any choice of a real number $\varepsilon > 0$ that satisfies

$$\varepsilon \leq \frac{\delta}{8nm}.$$

Let us note, in particular, that this bound holds for $\varepsilon = 2^{-dN}$ for some choice of a positive integer constant d. It is our goal to show that every Hermitian operator whose distance from P is at most ε (with respect to the Frobenius norm) lies within the primal feasible set $\mathcal{P}$, and therefore that $\mathcal{P}_\varepsilon$ is nonempty. In other words, for any choice of operators $H \in \operatorname{Herm}(\mathcal{X})$, $K \in \operatorname{Herm}(\mathcal{Y} \otimes \mathcal{Z})$, and $M \in \operatorname{L}(\mathcal{Y} \otimes \mathcal{Z}, \mathcal{X})$ satisfying

$$\left\| \begin{pmatrix} H & M \\ M^* & K \end{pmatrix} \right\|_2 < \varepsilon,$$

we wish to prove that

$$\begin{pmatrix} X + H & M \\ M^* & W + K \end{pmatrix} \tag{8}$$

is primal feasible.

It is clear that $\varepsilon\mathbb{1} < P$, and therefore (8) is positive semidefinite. As $\|K\|_\infty < \varepsilon$ it follows that

$$W + K \leq W + \varepsilon\,\mathbb{1}_{\mathcal{Y}\otimes\mathcal{Z}} \leq \frac{1}{2nm}\mathbb{1}_{\mathcal{Y}} \otimes \operatorname{Tr}_{\mathcal{Y}}(AA^*)$$

and therefore

$$\operatorname{Tr}_{\mathcal{Y}}(W + K) \leq \frac{1}{2n}\operatorname{Tr}_{\mathcal{Y}}(AA^*).$$

As $\|H\|_\infty \leq \varepsilon$ it holds that

$$\frac{1}{2n}\mathbb{1}_{\mathcal{X}} \leq X - \varepsilon\mathbb{1}_{\mathcal{X}} \leq X + H$$

and therefore

$$\frac{1}{2n}\operatorname{Tr}_{\mathcal{Y}}(AA^*) \leq \operatorname{Tr}_{\mathcal{Y}}(A(X+H)A^*).$$

It follows that $\operatorname{Tr}_{\mathcal{Y}}(W+K) \leq \operatorname{Tr}_{\mathcal{Y}}(A(X+H)A^*)$. Finally, it is clear that $\operatorname{Tr}(X+H) \leq 1$, and therefore the above operator (8) is primal feasible as required.

We have shown that the requirements of the promise in Problem 3 are met for $R = 2^{cN}$ and $\varepsilon = 2^{-dN}$ for some positive integer constants c and d. By Theorem 4 the value $|||\Phi|||_1^2$ may therefore be approximated to within error ε in time polynomial in n, m and the bit-length of the entries of A and B. (It is possible of course to choose a smaller error, $\varepsilon = 2^{-p(N)}$ for any polynomial p for instance, if this is desired.)

4 A simpler semidefinite program for quantum channel distance

A somewhat simpler semidefinite program exists for the completely bounded trace norm of the difference between two quantum channels, which is a special case that is relevant to quantum information. This case was discussed in [GLN05], and shown to reduce to a convex optimization problem. The discussion that follows is somewhat different, and is derived from the *quantum games* framework of [GW07].

Suppose hereafter in this section that $\Phi = \Phi_0 - \Phi_1$ for quantum channels $\Phi_0, \Phi_1 \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$, and consider the semidefinite program whose primal and dual problems are as follows:

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \langle J(\Phi), W \rangle \\ \text{subject to:} \quad & W \leq \mathbb{1}_{\mathcal{Y}} \otimes \rho, \\ & W \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X}), \\ & \rho \in \mathrm{D}(\mathcal{X}). \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \|\mathrm{Tr}_{\mathcal{Y}}(Z)\|_\infty \\ \text{subject to:} \quad & Z \geq J(\Phi), \\ & Z \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X}). \end{aligned}$$

As in the previous section, these problems can be matched to the formal description of a semidefinite program (Ψ, C, D), for which strong duality is easily proved. Our goal will be to prove that the optimal value of this semidefinite program is given by $\frac{1}{2} |||\Phi|||_1$.

Given that Φ is a Hermiticity-preserving mapping, it holds that

$$|||\Phi|||_1 = \max_{u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{X})} \left\| (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(uu^*) \right\|_1 .$$

For every $u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{X})$ it holds that the operator $(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(uu^*)$ is the difference between two density operators, and therefore

$$|||\Phi|||_1 = 2 \max \left\{ \left\langle P, (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(uu^*) \right\rangle \,:\, u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{X}),\; P \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X}),\; P \leq \mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}} \right\}.$$

Now, for every unit vector $u \in \mathcal{S}(\mathcal{X} \otimes \mathcal{X})$ there is a corresponding operator $B \in \mathrm{L}(\mathcal{X})$ with $\|B\|_2 = 1$ such that

$$u = \sum_{1 \leq i,j \leq n} \langle E_{i,j}, B \rangle \, e_i \otimes e_j.$$

For this choice of B we have

$$(\mathbb{1}_{\mathcal{Y}} \otimes B) J(\Phi) (\mathbb{1}_{\mathcal{Y}} \otimes B^*) = (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{X})})(uu^*).$$

It follows that

$$|||\Phi|||_1 = 2 \max_{B,P} \langle (\mathbb{1} \otimes B^*) P (\mathbb{1} \otimes B), J(\Phi) \rangle$$

where the maximum is over all $B \in \mathrm{L}(\mathcal{X})$ with $\|B\|_2 = 1$ and $P \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X})$ with $P \leq \mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}}$.

Now define sets $\mathcal{Q}$ and $\mathcal{R}$ as follows:

$$\begin{aligned} \mathcal{Q} &= \{ R \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X}) \,:\, R \leq \mathbb{1}_{\mathcal{Y}} \otimes \rho \text{ for some } \rho \in \mathrm{D}(\mathcal{X}) \}, \\ \mathcal{R} &= \{ (\mathbb{1}_{\mathcal{Y}} \otimes B^*) P (\mathbb{1}_{\mathcal{Y}} \otimes B) \,:\, B \in \mathrm{L}(\mathcal{X}),\; P \in \mathrm{Pos}(\mathcal{Y} \otimes \mathcal{X}),\; \|B\|_2 = 1,\; P \leq \mathbb{1}_{\mathcal{Y} \otimes \mathcal{X}} \}. \end{aligned}$$

It holds that

$$|||\Phi|||_1 = 2 \sup_{X \in \mathcal{R}} \langle J(\Phi), X \rangle$$

while the optimal value of the semidefinite program above is

$$\alpha = \sup_{X \in \mathcal{Q}} \langle J(\Phi), X \rangle .$$

The fact that $\alpha = \frac{1}{2} |||\Phi|||_1$ therefore follows from the equality $\mathcal{Q} = \mathcal{R}$, which is easily proved by selecting ρ or B so that $\rho = B^* B$.

5 Connections with known results

This final section of the paper describes two interesting connections between the semidefinite programming formulation from Section 3 and known results, the first being directly about completely bounded norms, and the second concerning the fidelity function.

Spectral norms of Stinespring representations

The following theorem gives an alternate characterization of the completely bounded trace norm. Proofs can be found in Kitaev, Shen and Vyalyi [KSV02] and Paulsen [Pau02]. The two proofs use rather different techniques, and here the theorem is proved in a third way using semidefinite programming duality.

Theorem 5. *For every mapping $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$, it holds that*

$$|||\Phi|||_1 = \inf_{(A,B)} \|A\|_\infty \|B\|_\infty , \tag{9}$$

where the infimum is over all Stinespring pairs (A, B) for Φ.

Proof. For any Stinespring pair (A, B) of Φ, where $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$, and for any choice of $\mathcal{W} = \mathbb{C}^k$, it holds that

$$\begin{aligned} \left\| (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{W})})(X) \right\|_1 &= \| \mathrm{Tr}_{\mathcal{Z}} \left[(A \otimes \mathbb{1}_{\mathcal{W}}) X (B^* \otimes \mathbb{1}_{\mathcal{W}}) \right] \|_1 \\ &\leq \| (A \otimes \mathbb{1}_{\mathcal{W}}) X (B^* \otimes \mathbb{1}_{\mathcal{W}}) \|_1 \\ &\leq \|A\|_\infty \|X\|_1 \|B\|_\infty \end{aligned}$$

for all $X \in \mathrm{L}(\mathcal{X} \otimes \mathcal{W})$, where the inequalities follow from (2) and (1), respectively. It follows that $|||\Phi|||_1 \leq \|A\|_\infty \|B\|_\infty$.

To prove that the infimum is no larger than $|||\Phi|||_1$, first choose an arbitrary Stinespring pair (A, B) of Φ, where $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$. The optimal value for the dual problem stated in Section 3 does not change if Z is restricted to be positive definite, provided we accept that an optimal solution may not be achieved. We therefore have

$$|||\Phi|||_1^2 = \inf \{ \|A^* (\mathbb{1}_{\mathcal{Y}} \otimes Z) A\|_\infty \, : \, \mathbb{1}_{\mathcal{Y}} \otimes Z \geq BB^*, \; Z \in \mathrm{Pd}(\mathcal{Z}) \}.$$

Thus, for a given $\varepsilon > 0$, we may choose $Z \in \mathrm{Pd}(\mathcal{Z})$ such that

$$\|A^* (\mathbb{1}_{\mathcal{Y}} \otimes Z) A\|_\infty \leq (|||\Phi|||_1 + \varepsilon)^2$$

and $\mathbb{1}_{\mathcal{Y}} \otimes Z \geq BB^*$. This second inequality is equivalent to

$$\left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2} \right) BB^* \left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2} \right) \right\|_\infty \leq 1.$$

So now we have that

$$\left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{1/2}\right) A \right\|_{\infty} \left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2}\right) B \right\|_{\infty} \leq |||\Phi|||_1 + \varepsilon,$$

and it holds that

$$\left(\left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{1/2}\right) A, \left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2}\right) B \right)$$

is a Stinespring pair for Φ. This establishes that the infimum equals $|||\Phi|||_1$ in the expression (9), which completes the proof. □

Connection with fidelity

Consider the semidefinite program from Section 3, for the special case where $\mathcal{X} = \mathbb{C}$. Replacing A and B with vectors $u, v \in \mathcal{Y} \otimes \mathcal{Z}$, and making simplifications, the problems become as follows:

<u>Primal problem</u>

$$\begin{aligned} \text{maximize:} \quad & \langle vv^*, W \rangle \\ \text{subject to:} \quad & \operatorname{Tr}_{\mathcal{Y}}(W) \leq \operatorname{Tr}_{\mathcal{Y}}(uu^*), \\ & W \in \operatorname{Pos}(\mathcal{Y} \otimes \mathcal{Z}). \end{aligned}$$

<u>Dual problem</u>

$$\begin{aligned} \text{minimize:} \quad & \langle \operatorname{Tr}_{\mathcal{Y}}(uu^*), Z \rangle \\ \text{subject to:} \quad & \mathbb{1}_{\mathcal{Y}} \otimes Z \geq vv^*, \\ & Z \in \operatorname{Pos}(\mathcal{Z}). \end{aligned}$$

As will be explained shortly, the quantity that is represented by the optimal value of these problems is given by the *fidelity* function, which is defined as

$$\mathrm{F}(P, Q) = \left\| \sqrt{P}\sqrt{Q} \right\|_1 = \operatorname{Tr} \sqrt{\sqrt{P} Q \sqrt{P}}$$

for positive semidefinite operators P and Q. In particular, the optimal value (for the primal and dual problems) is

$$\mathrm{F}\left(\operatorname{Tr}_{\mathcal{Y}}(uu^*), \operatorname{Tr}_{\mathcal{Y}}(vv^*)\right)^2. \tag{10}$$

The fact that the optimal value of the primal problem coincides with the fidelity function as just described follows from Uhlmann's theorem [Uhl76], which is as follows.

Theorem 6 (Uhlmann's theorem)**.** *Let $\mathcal{Y}$ and $\mathcal{Z}$ be finite-dimensional complex vector spaces, and let $P, Q \in \operatorname{Pos}(\mathcal{Z})$ be positive semidefinite operators, both having rank at most $\dim(\mathcal{Y})$. Then for any choice of $v \in \mathcal{Y} \otimes \mathcal{Z}$ satisfying $\operatorname{Tr}_{\mathcal{Y}}(vv^*) = Q$, it holds that*

$$\mathrm{F}(P, Q) = \max\left\{ |\langle u, v \rangle| \,:\, u \in \mathcal{Y} \otimes \mathcal{Z},\ \operatorname{Tr}_{\mathcal{Y}}(uu^*) = P \right\}.$$

This theorem provides an alternate characterization of the fidelity function that is useful for establishing many interesting properties of the fidelity. Among these is the fact that the fidelity is monotonic, meaning

$$\mathrm{F}(P, Q) \leq \mathrm{F}(\operatorname{Tr}_{\mathcal{Y}}(P), \operatorname{Tr}_{\mathcal{Y}}(Q))$$

for every choice of $P, Q \in \operatorname{Pos}(\mathcal{X} \otimes \mathcal{Y})$. It is straightforward to apply this fact back to Uhlmann's theorem itself to obtain the following corollary, which is precisely the statement that the optimal primal value of our semidefinite program is given by the fidelity.

Corollary 7. *Assume $u, v \in \mathcal{Y} \otimes \mathcal{Z}$ are vectors, and let $P = \operatorname{Tr}_{\mathcal{Y}}(uu^*)$ and $Q = \operatorname{Tr}_{\mathcal{Y}}(vv^*)$. Then*

$$\mathrm{F}(P, Q)^2 = \max\left\{ \langle vv^*, W \rangle \,:\, W \in \operatorname{Pos}(\mathcal{Y} \otimes \mathcal{Z}),\ \operatorname{Tr}_{\mathcal{Y}}(W) \leq P \right\}.$$

The optimal value of the dual problem is, of course, equal to (10) by strong duality. A different way to evaluate the optimal dual value begins with the following simple proposition.

Proposition 8. *For any vector* $v \in \mathcal{Y} \otimes \mathcal{Z}$ *and any positive definite operator* $Z \in \mathrm{Pd}(\mathcal{Z})$ *it holds that* $\mathbb{1}_{\mathcal{Y}} \otimes Z \geq vv^*$ *if and only if* $\left\langle \mathrm{Tr}_{\mathcal{Y}}(vv^*), Z^{-1} \right\rangle \leq 1$.

Proof. It holds that $\mathbb{1}_{\mathcal{Y}} \otimes Z \geq vv^*$ if and only if

$$\left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2}\right) vv^* \left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2}\right) \leq \mathbb{1}_{\mathcal{Y} \otimes \mathcal{Z}}. \tag{11}$$

Given that the operator on the left-hand-side of (11) is positive semidefinite and has rank equal to 1, we have that (11) is equivalent to

$$\left\| \left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2}\right) v \right\| \leq 1,$$

which in turn is equivalent to

$$\mathrm{Tr}\left(\left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2}\right) vv^* \left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2}\right)\right) \leq 1.$$

As

$$\mathrm{Tr}\left(\left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2}\right) vv^* \left(\mathbb{1}_{\mathcal{Y}} \otimes Z^{-1/2}\right)\right) = \left\langle \mathrm{Tr}_{\mathcal{Y}}(vv^*), Z^{-1} \right\rangle,$$

the proof is complete. □

We have that the optimal value of the dual problem does not change if Z is optimized over only positive definite rather than positive semidefinite operators (again accepting that the optimal value may not be achieved for such an operator). Combined with the proposition just proved, we find that the optimal dual value is given by

$$\beta = \inf\left\{ \langle \mathrm{Tr}_{\mathcal{Y}}(uu^*), Z\rangle \,:\, Z \in \mathrm{Pd}(\mathcal{Z}),\ \langle \mathrm{Tr}_{\mathcal{Y}}(vv^*), Z^{-1}\rangle \leq 1 \right\}.$$

That this value is given by (10) follows from a different characterization of the fidelity due to Alberti [Alb83].

Theorem 9 (Alberti's theorem)**.** *Let* $P, Q \in \mathrm{Pos}(\mathcal{Z})$ *be positive semidefinite operators. Then*

$$(\mathrm{F}(P,Q))^2 = \inf_{Z \in \mathrm{Pd}(\mathcal{Z})} \langle P, Z\rangle \langle Q, Z^{-1}\rangle.$$

We have therefore established a simple and precise sense in which Uhlmann's theorem and Alberti's theorem are dual statements in finite dimensions, each implying the other.

Acknowledgments

Many of the key ideas presented in this paper are contained in the papers [KW00] and [GW07]. I thank Alexei Kitaev and Gus Gutoski for discussions during and after the collaborations that produced these papers, which helped to clarify these ideas. I also thank Bill Rosgen, Mary Beth Ruskai, and the anonymous referees for helpful comments. This research was supported by Canada's NSERC and the Canadian Institute for Advanced Research (CIFAR).

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