

Quantum game theory and the complexity of approximating quantum Nash equilibria

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Abstract

This paper is concerned with complexity theoretic aspects of a general formulation of quantum game theory that models strategic interactions among rational agents that process and exchange quantum information. In particular, we prove that the computational problem of finding an approximate Nash equilibrium in a broad class of quantum games is, like the analogous problem for classical games, included in (and therefore complete for) the complexity class PPAD. Our main technical contribution, which facilitates this inclusion, is an extension of prior methods in computational game theory to strategy spaces that are characterized by semidefinite programs.

1. Introduction

Game theory is a fascinating topic of study with connections to computer science, economics, and the social sciences, among other subjects. This paper focuses on complexity theoretic aspects of game theory within the context of quantum information and computation.

Quantum game theory began with the work of David Meyer [Mey99] and Jens Eisert, Martin Wilkens, and Maciej Lewenstein [EWL99] in 1999.¹ These works investigated games involving quantum information, highlighting examples in which quantum players

¹ Some authors argue that the origins of quantum game theory go back further. Here, however, we are referring to the specific line of work that self-identifies as being concerned with a quantum information theoretic variant of game theory in the tradition of von Neumann and Morgenstern [vNM53] and Nash [Nas50a, Nas50b], as opposed to quantum information and computation research that can be associated with game theory as a broad umbrella term.

have advantages over classical players. Many other examples of quantum games, primarily based on the frameworks proposed by Meyer and Eisert, Wilkens, and Lewenstein, were subsequently analyzed. (See, for instance, the survey [GZK08] for summaries and references.)

Aspects of this line of work have been criticized for multiple reasons. A common point of criticism of many (but certainly not all) quantum game theory papers is their poorly motivated notion of classical behavior. In particular, classical players in quantum game theory papers are often limited to *coherent* permutations of standard basis states, or similarly restricted classes of unitary operations, while quantum players have access to a less restricted set of unitary operations, possibly all of them. This notion of classicality, which is a key ingredient in the original examples of Meyer and Eisert, Wilkens, and Lewenstein, essentially invites exploitation by quantum players. A more standard interpretation of classical behavior in quantum information theory assumes the complete decoherence of any quantum system a classical player manipulates.

Another point of criticism, raised by van Enk and Pike [vEP02], is that comparing quantum games with their classical namesakes within the specific frameworks typically adopted by quantum game theory papers is akin to comparing apples with oranges. Although one may argue that these games offer faithful representations of classical games when players' actions are restricted to permutations of standard basis states, their quantum reformulations are, simply put, different games. It is therefore not surprising that less restricted quantum players may find advantages, leading to new Nash equilibria and so on.

However, although it was not their primary focus, Meyer and Eisert, Wilkens, and Lewenstein did both clearly suggest more general definitions of quantum games in which a wide range of interactions could be considered, including ones in which the criticisms just raised no longer have relevance. In particular, Meyer mentions a convex form of his model of quantum games, in which classical players could be modeled by completely decohered operations. And, Eisert, Wilkens, and Lewenstein, in a footnote of their paper, describe a model in which players' actions correspond not just to unitary operations, but to arbitrary quantum channels (as modeled by completely positive and trace preserving linear maps). In either case, more general strategic interactions may be considered, and one need not restrict their attention to analogues of classical games or in identifying a "quantum advantage."

For example, quantum interactive proof systems of various sorts, as well as many quantum cryptographic scenarios and primitives, can be viewed as quantum games. Another example is quantum communication, which can be modeled as a game in which one player attempts to transmit a quantum state to another, while a third player representing an adversarial noise model attempts to disrupt the transmission. We do not offer any specific suggestions in this paper, but it is not unreasonable to imagine that quantum games having social or economic applications could be discovered.

We will now summarize the definition of quantum games we adopt, beginning with the comparatively simple non-interactive setting and then moving on to the more general

interactive setting. For the sake of simplicity and exposition in this introduction, we will restrict our attention to games in which there are just two players: Alice and Bob. The definitions are easily extended to any finite number of players, as is done in the main text.

Non-interactive quantum games

In a (two-player) *non-interactive quantum game*, the players Alice and Bob each hold a quantum system, represented by a register of a predetermined size: Alice holds X and Bob holds Y . They must each *independently* prepare in the register they hold a quantum state: Alice prepares a quantum state represented by a density operator ρ and Bob prepares a state represented by σ . Just like in the standard *non-cooperative* setting of classical game theory, Alice and Bob are assumed to be unable to correlate their state preparations with one another.² The registers X and Y are sent to a referee, who performs a joint measurement on the pair (X, Y) . Here, when we refer to a measurement, we mean a general quantum measurement, often called a POVM (positive operator valued measure), having any finite and non-empty set of measurement outcomes. The outcome of the measurement is assumed to determine a real number payoff for each player. We note explicitly that Jinshan Wu [Wu04a, Wu04b] has proposed and analyzed an equivalent definition of non-interactive quantum games to this one.

In order to formally describe a non-interactive quantum game, one must specify the referee's measurement together with the payoff functions for each player. As will be explained later, when it suffices to specify each player's *expected* payoff, given any choice of states the players may select, the referee may be described by a collection of Hermitian matrices, one for each player.

One may observe that the standard notion of a classical game in normal form is easily represented within this framework by defining the referee so that it first measures the registers X and Y with respect to the standard bases of the associated spaces, and then assigns payoffs in a completely classical manner.

A non-interactive quantum game can, up to a discretization, also be viewed as a classical game, where the players send the referee classical descriptions of their chosen density operators and the referee performs the required calculation to determine their payoffs, but the normal form description of this new classical game will, naturally, be exponentially larger than the description of the original quantum game.

Interactive quantum games

Quantum games in which players can process and exchange quantum information with a referee over the course of multiple rounds of interaction may also be considered.

²It is interesting to consider meaningful ways in which this assumption may be relaxed or dropped, but the simplest and most direct quantum extension of classical game theory beings with this assumption of independence.

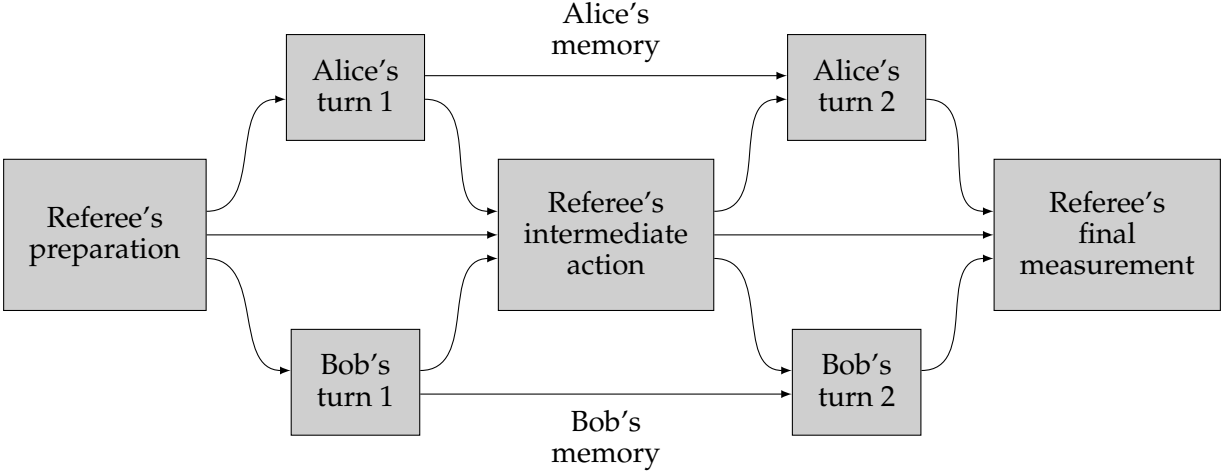


Figure 1: An illustration of a game between Alice and Bob, run by a referee, in which Alice and Bob each receive and transmit quantum information twice, potentially keeping a quantum memory between the two turns. Each arrow represents a quantum register, which could be of any fixed size (including the possibility of trivial registers, which are equivalent to nothing being transmitted). Games involving any finite number of rounds of interaction may be considered.

For example, the referee could prepare registers X and Y in a joint quantum state, send X to Alice and Y to Bob, allowing them to transform these registers as they choose, and then measure the pair (X, Y) upon receiving them back from Alice and Bob. In such a game, Alice and Bob therefore each play a quantum channel, with their payoffs again being determined by the outcome of the referee's measurement. The framework of Eisert, Wilkens, and Lewenstein falls into this category, provided that the players are permitted to play channels and not just unitary operations.

More generally, an *interactive quantum game* may involve an interaction between the referee and the players over the course of multiple rounds, as suggested by Figure 1. In this setting it is natural to assume that Alice and Bob each have their own private quantum memory, which they utilize if it is to their advantage.

The actions of the players in such a game may be represented through the framework alternatively known as the *quantum strategies* framework [GW07] and the *quantum combs* framework [CDP08, CDP09]. This framework, which will be described in greater detail in the next section, allows the actions of any one player over the course of multiple rounds of interaction, accounting for the possibility of a quantum memory, to be faithfully represented by a single positive semidefinite matrix that satisfies a finite collection of affine linear constraints. Thus, the sets of strategies available to the players are convex and compact, and one may efficiently optimize real-valued linear functions defined on these sets through the paradigm of semidefinite programming.

Similar to non-interactive quantum games, interactive quantum games are formally expressed by specifying the referee’s actions, including state preparations, channels, and measurements, along with payoff functions of the possible measurement outcomes corresponding to each player. Once again, when it is sufficient to describe the expected payoff for each player, given a specification of their strategies, a referee in an interactive quantum game may be specified by a list of Hermitian matrices, one for each player, as will be explained.

Our results and contributions

We prove, as our main technical result, that the problem of computing an approximate Nash equilibrium in any interactive game of the form described above, given an explicit matrix representation of the referee, is contained in the complexity class PPAD. As this problem includes non-interactive classical games as a special case, it follows that this problem is complete for PPAD [DGP09, CDT09].

There is a sense in which this result is not unexpected; prior work on the complexity of computing approximate Nash equilibria, and more generally on the complexity of computing fixed points of different classes of continuous maps, suggests that approximations of Nash equilibria in a wide variety of games should be contained in PPAD [Pap94, DGP09, CDT09, EY10]. The principal challenge that arises in the setting of interactive quantum games is that, although one may efficiently optimize over individual player’s strategies through semidefinite programming, closed form expressions of these optimizations are not known to exist.

To confront this challenge, we consider a fairly general convex form of Nash’s notion of a *gain function*—and then we fight fire with fire, so to speak, using semidefinite programming to approximate continuous functions that arise through this formulation. Possibly our methodology for handling this issue will be of independent interest.

Although it is not an essential aspect of our proof, we also make use of the elegant notion of a *discrete Wigner representation* of a quantum state, which is convenient within the proof. Although discrete Wigner representations have been investigated in the theory of quantum information (see, for instance, [GHW04] and [Gro06]), we are not aware that they have been used previously in quantum complexity theory, and we feel that they offer a convenient tool that might be useful in other contexts.

Beyond this main technical result, we hope that this paper may serve as a suggestion that quantum game theory is worthy of a second look. We believe that the general definition of quantum games we have reiterated is well motivated by the theory of quantum information, and can provide a basic foundation through which quantum game theory and its complexity theoretic aspects may be investigated. In the conclusion of this paper we mention several open problems and research directions concerning quantum game theory that may be of interest.

2. Technical preliminaries

This section summarizes technical concepts required later in the paper. The first three subsections that follow discuss aspects of the *barycentric subdivision* of a simplex, the *discrete Wigner representation* of quantum states, and the *quantum strategies/combs framework*, respectively. In the last subsection we present the standard definition of the complexity class PPAD and reference a theorem of Etessami and Yannakakis [EY10] that establishes the containment of a certain computational fixed-point problem in PPAD, to which we will reduce the problem of finding an approximate Nash equilibrium of a quantum game.

It will be assumed throughout the paper that the reader is familiar with basic notions of computational complexity [AB09] and quantum information [NC00, Wil17, Wat18]. Hereafter we will take $\Sigma = \{0, 1\}$ to denote the binary alphabet.

2.1. Barycentric subdivision of a simplex

Suppose that a positive integer n is given, and consider a simplex in an n -dimensional space having vertices $\{u_1, \dots, u_n\}$. The *barycentric subdivision* of such a simplex is a division of it into $n!$ simplices in the following way.

First, with each nonempty subset $A \subseteq \{1, \dots, n\}$ we define a point

$$v_A = \frac{1}{|A|} \sum_{k \in A} u_k, \quad (1)$$

which is the uniform convex combination, or *barycenter*, of the vertices of the original simplex labeled by elements of A . For example, $v_{\{k\}} = u_k$ for each $k \in \{1, \dots, n\}$, while $v_{\{1, \dots, n\}}$ is the true barycenter of the original simplex. Figure 2 illustrates the barycentric subdivision for a simplex when $n = 3$.

Next, by drawing an edge between v_A and v_B if and only if $A \subset B$ or $B \subset A$ (proper containments), we obtain a division of the original simplex into $n!$ new simplices, one for each possible chain

$$A_1 \subset \dots \subset A_n \quad (2)$$

of subsets of $\{1, \dots, n\}$. There are $n!$ such chains and they may be placed in correspondence with the symmetric group S_n . To be precise, for any fixed ordering (k_1, \dots, k_n) of the set $\{1, \dots, n\}$, we associate the chain (2) with the permutation $\pi \in S_n$ satisfying

$$A_j = \{k_{\pi(1)}, \dots, k_{\pi(j)}\} \quad (3)$$

for every $j \in \{1, \dots, n\}$. Thus, the simplices in the subdivision are identified with elements of S_n .

The barycentric subdivision may naturally be applied iteratively within the simplices constructed by the subdivision. For example, Figure 3 illustrates the barycentric subdivision of just the shaded simplex illustrated in Figure 2. Hereafter we shall assume that

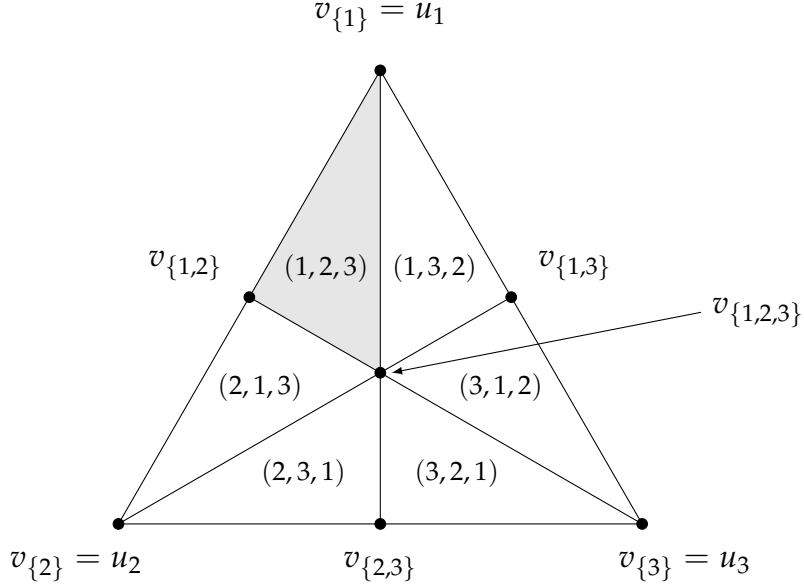


Figure 2: The barycentric subdivision of a simplex with three vertices u_1 , u_2 , and u_3 . The shaded region indicates one of the $3! = 6$ simplices formed by this subdivision: it is the one with vertices $\{v_{\{1\}}, v_{\{1,2\}}, v_{\{1,2,3\}}\}$, which is naturally identified with the identity permutation $\pi = (1, 2, 3)$.

the initial simplex is the standard simplex Δ_n , so that u_1, \dots, u_n are elementary unit vectors (or, equivalently, standard basis vectors). With this assumption in place, we define a sequence of finite subsets of the standard unit simplex

$$\mathcal{B}_n^0 \subset \mathcal{B}_n^1 \subset \mathcal{B}_n^2 \subset \dots \subset \Delta_n; \quad (4)$$

\mathcal{B}_n^0 contains the n vertices of the simplex Δ_n , \mathcal{B}_n^1 contains the $2^n - 1$ points corresponding to the nonempty subsets of $\{1, \dots, n\}$ when the barycentric subdivision has been applied a single time, and in general \mathcal{B}_n^r denotes the set of vertices after the r -th level subdivision has been performed.

For any function $f : \Delta_n \rightarrow \Delta_n$, and any nonnegative integer r , the r -th level barycentric approximation to f is the function $g_r : \Delta_n \rightarrow \Delta_n$ defined in the following way. Each point $v \in \Delta_n$ may be expressed uniquely as a convex combination

$$v = \lambda_1 v_1 + \dots + \lambda_m v_m \quad (5)$$

of distinct vertices $v_1, \dots, v_m \in \mathcal{B}_n^r$, all contained in the same r -th level simplex (which therefore implies $m \leq n$). One then defines

$$g_r(v) = \lambda_1 f(v_1) + \dots + \lambda_m f(v_m). \quad (6)$$

Various computations involving the r -th level barycentric subdivision may be performed efficiently and exactly through rational number computations for r being polynomial in the size of the problem being considered, but we will not have a need to explicitly

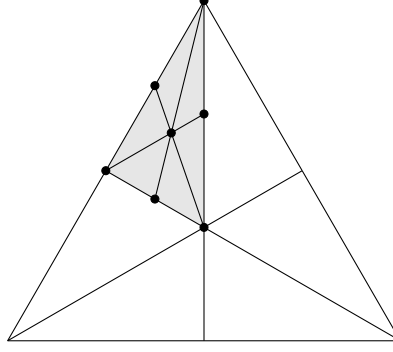


Figure 3: The barycentric subdivision applied to the simplex shaded gray in Figure 2.

refer to these computations. We will make use of the well known fact that any two points u and v contained in the same simplex constructed at the r -th level of the barycentric subdivision must satisfy

$$\|u - v\|_2 \leq \left(1 - \frac{1}{n+1}\right)^r, \quad (7)$$

with the norm being the Euclidean norm on \mathbb{R}^n . A proof of this fact may be found in many texts on algebraic topology, including in [Bre93] where it appears as Lemma 17.3.

2.2. Discrete Wigner representation

Throughout this subsection we will take n to be an odd positive integer, and let us rename the elements of the standard basis $\{|1\rangle, \dots, |n\rangle\}$ as $\{|a\rangle : a \in \mathbb{Z}_n\}$ by taking each index modulo n . In general, we shall interpret expressions inside of bras and kets as referring to modulo n arithmetic.

It is helpful to begin with the definition of the *discrete Weyl operators*. First define

$$X = \sum_{a \in \mathbb{Z}_n} |a+1\rangle\langle a| \quad \text{and} \quad Z = \sum_{a \in \mathbb{Z}_n} \omega_n^a |a\rangle\langle a| \quad (8)$$

for $\omega_n = \exp(2\pi i/n)$ denoting the first principal n -th root of unity. The discrete Weyl operators

$$\{W_{a,b} : a, b \in \mathbb{Z}_n\} \subset \mathbf{U}(\mathbb{C}^n), \quad (9)$$

as we will define them, are then given by

$$W_{a,b} = X^a Z^b \quad (10)$$

for every $a, b \in \mathbb{Z}_n$. The discrete Weyl operators form an orthogonal basis for the vector space $L(\mathbb{C}^n)$.

Next, define an operator $T \in L(\mathbb{C}^n)$ as

$$T = \sum_{a \in \mathbb{Z}_n} |-a\rangle\langle a|. \quad (11)$$

For example, in dimension $n = 5$ this operator may be expressed in matrix form as

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (12)$$

We then define

$$V_{a,b} = W_{a,b}TW_{a,b}^* \quad (13)$$

for every $a, b \in \mathbb{Z}_n$, and consider the collection

$$\{V_{a,b} : a, b \in \mathbb{Z}_n\}. \quad (14)$$

One may observe that T is unitary, Hermitian, and, by the assumption that n is odd, has unit trace, and therefore the same is true for every operator in the collection (14).

Next let us verify that the collection (14) is orthogonal, with respect to the usual (Hilbert–Schmidt) inner product on operators. For any choice of $a, b, c, d \in \mathbb{Z}_n$, one may verify directly that

$$\langle V_{a,b}, V_{c,d} \rangle = \langle T, V_{c-a, d-b} \rangle. \quad (15)$$

Also observe the following expression for the diagonal entries of the operator $TV_{a,b}$:

$$\langle c|TV_{a,b}|c \rangle = \langle c|TW_{a,b}TW_{a,b}^*|c \rangle = \begin{cases} 0 & a \neq 0 \\ \omega_n^{-2bc} & a = 0. \end{cases} \quad (16)$$

Noting the expression

$$\sum_{c \in \mathbb{Z}_n} \omega_n^{-2bc} = \begin{cases} n & b = 0 \\ 0 & b \neq 0, \end{cases} \quad (17)$$

where again we have used the assumption that n is odd, we conclude that

$$\langle T, V_{a,b} \rangle = \begin{cases} n & (a, b) = (0, 0) \\ 0 & (a, b) \neq (0, 0). \end{cases} \quad (18)$$

The collection (14) is therefore orthogonal.

At this point we have no further need to refer to modulo n arithmetic, so let us assume that the elements of the collection (14) have been renamed as $\{V_1, \dots, V_{n^2}\}$, with respect to any sensible way of doing this. The key property of this collection is that each V_k is unitary, Hermitian, and has trace equal to 1, and that the collection is orthogonal.

Finally, define an affine linear map of the form $\psi : \text{Herm}(\mathbb{C}^n) \rightarrow \mathbb{R}^{n^2}$ as

$$\psi(H) = \frac{1}{n(n+1)} \sum_{k=1}^{n^2} (\langle V_k, H \rangle + 1) |k\rangle \quad (19)$$

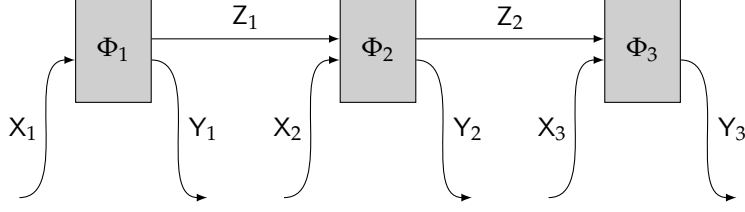


Figure 4: The actions of an agent, or a *strategy*, in a three-round interaction may be described by a network of three channels. Time goes from left to right: first the register X_1 is received and fed into the channel Φ_1 , which produces Y_1 and Z_1 as output, with Y_1 being sent to another agent and Z_1 representing a memory register that is retained by the agent being described. Then X_2 is received, both X_2 and the memory register Z_1 are fed into the second channel Φ_2 , and so on.

for every $H \in \text{Herm}(\mathbb{C}^n)$. The inverse of this mapping is given by

$$\psi^{-1}(v) = (n+1) \sum_{k=1}^{n^2} v_k V_k - \mathbb{1}_n \quad (20)$$

for every $v \in \mathbb{R}^{n^2}$. This mapping defines a *discrete Wigner representation* of quantum states; each density operator $\rho \in D(\mathbb{C}^n)$ is represented by the vector

$$v = \psi(\rho) \in \Delta_{n^2}. \quad (21)$$

The inclusion of the vector $v = \psi(\rho)$ in the unit simplex follows from two observations, the first being that $\text{Tr}(\psi^{-1}(v)) = 1$ if and only if $v_1 + \dots + v_{n^2} = 1$, and the second being that $\langle V_k, \rho \rangle \in [-1, 1]$ by virtue of the fact that V_k is unitary and Hermitian, for each $k \in \{1, \dots, n^2\}$.

It should be noted that, except in the trivial case $n = 1$, the inclusion $\psi(D(\mathbb{C}^n)) \subset \Delta_{n^2}$ is proper; only a subset of the vectors in the standard simplex represent a valid density operator, others represent unit-trace Hermitian operators having negative eigenvalues.

2.3. The quantum strategies framework

We now summarize the aspects of the quantum strategies/combs framework [GW07, CDP08, CDP09], hereafter be referred to as the *quantum strategies framework* in this paper, that are required for our main result. This framework provides a convenient way of describing and characterizing the actions of agents that interact and exchange quantum information with one another over the course of multiple rounds.

Consider an agent that engages in an interaction involving the exchange of quantum information with one or more other agents. Let us suppose, in particular, that the agent being considered first receives a register X_1 , then sends a register Y_1 , then receives X_2 , then sends Y_2 , and so on, with its role in the hypothetical interaction concluding after

it receives X_r and then sends Y_r . It is to be assumed that the agent may store quantum information between the rounds of interaction. Figure 4 depicts the actions of an agent of this sort in the case $r = 3$. Hereafter we will refer to a network of this form as a *strategy* for the agent being described.

In the quantum strategies framework, strategies of this sort are represented by the *Choi representation* of the network. To be more precise, the network is considered as a single quantum channel

$$\Phi \in \mathcal{C}(X_1 \otimes \cdots \otimes X_r, Y_1 \otimes \cdots \otimes Y_r), \quad (22)$$

with (X_1, \dots, X_r) collectively forming the input to this channel and (Y_1, \dots, Y_r) forming the output, and the Choi representation $J(\Phi)$ is taken as a representation of the strategy. In general, the Choi representation of a channel taking the form $\Phi \in \mathcal{C}(X, Y)$ is given by

$$J(\Phi) = \sum_{a,b} \Phi(|a\rangle\langle b|) \otimes |a\rangle\langle b|, \quad (23)$$

where a and b range over all classical states (or, equivalently, standard basis elements) of the input space X , and therefore for Φ taking the form (22) we have

$$J(\Phi) \in \mathcal{L}(Y_1 \otimes \cdots \otimes Y_r \otimes X_1 \otimes \cdots \otimes X_r). \quad (24)$$

Not every channel of the form (22) can be obtained by composing channels Φ_1, \dots, Φ_r in the manner just described; a given channel might not respect the “time ordering” in which each register Y_k is produced prior to the registers X_{k+1}, \dots, X_r being received. A necessary and sufficient condition for a channel of the form (22) to decompose into a network of channels Φ_1, \dots, Φ_r is that its Choi representation is positive semidefinite,

$$J(\Phi) \in \text{Pos}(Y_1 \otimes \cdots \otimes Y_r \otimes X_1 \otimes \cdots \otimes X_r), \quad (25)$$

and satisfies a collection of affine linear constraints:

$$\begin{aligned} \text{Tr}_{Y_r}(J(\Phi)) &= X_{r-1} \otimes \mathbb{1}_{X_r} \\ \text{Tr}_{Y_{r-1}}(X_{r-1}) &= X_{r-2} \otimes \mathbb{1}_{X_{r-1}} \\ &\vdots \\ \text{Tr}_{Y_2}(X_2) &= X_1 \otimes \mathbb{1}_{X_2} \\ \text{Tr}_{Y_1}(X_1) &= \mathbb{1}_{X_1} \end{aligned} \quad (26)$$

for X_1, \dots, X_{r-1} being operators having sizes required by the equalities. By the assumption that $J(\Phi)$ is positive semidefinite, the operators X_1, \dots, X_{r-1} (if they exist) must be positive semidefinite, and so we may write

$$\begin{aligned} X_{r-1} &\in \text{Pos}(Y_1 \otimes \cdots \otimes Y_{r-1} \otimes X_1 \otimes \cdots \otimes X_{r-1}) \\ &\vdots \\ X_1 &\in \text{Pos}(Y_1 \otimes X_1). \end{aligned} \quad (27)$$

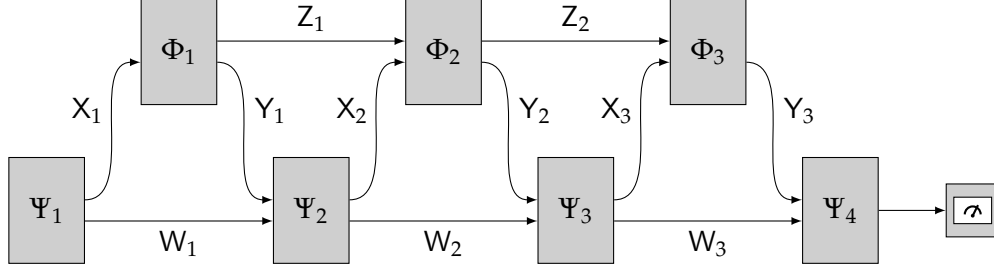


Figure 5: A strategy of the form depicted in Figure 4 may be interfaced with the actions of one or more other agents. In the interaction pictured, the second agent produces a measurement outcome at the conclusion of the interaction.

(The operators X_{r-1}, \dots, X_1 happen to be the Choi representations of the strategies obtained by “truncating” the strategy described by the channels Φ_1, \dots, Φ_r , assuming the final memory register is tracing out in each case.) It may be noted that, in the case $r = 1$, the usual necessary and sufficient conditions for a map to describe a quantum channel are recovered.

Now suppose that an agent of the form depicted in Figure 4 interacts with another agent, who performs a measurement at the conclusion of the interaction, as is suggested by Figure 5. For the sake of clarity, and to connect the framework to the setting of games, the agent depicted in Figure 4 will be called the *player* and the new agent will be called the *referee*. Through the quantum strategies framework, for each possible outcome a that may be produced by the referee’s measurement, one may compute a positive semidefinite operator

$$P_a \in \text{Pos}(\mathcal{Y}_1 \otimes \dots \otimes \mathcal{Y}_r \otimes \mathcal{X}_1 \otimes \dots \otimes \mathcal{X}_r) \quad (28)$$

with the property that, when the referee and player interact, the probability for each measurement outcome to appear is given by

$$\Pr(\text{measurement outcome equals } a) = \langle P_a, Q \rangle, \quad (29)$$

for $Q = J(\Phi)$ being the representation of the player’s strategy. It is not necessary for the purposes of this paper to explain precisely how each operator P_a is obtained, except to say that this operator may be computed efficiently given descriptions of the channels $\Phi_1, \dots, \Phi_{r+1}$ and the final measurement.³

Finally, the quantum strategies framework extends to interactions involving multiple agents in a fairly straightforward way. In the context of quantum games, we are interested in interactions in which a referee, who produces a final measurement outcome at the conclusion of the interaction, interacts not just with a single player, but with multiple players.

³The process for obtaining these operators is again based on the Choi representation, but one must account for the measurement, the spaces must be ordered in a way that matches with the representation $J(\Xi)$, and an entry-wise complex conjugation is required to ensure that the expression $\langle P_a, Q \rangle$ correctly represents the probability associated with the outcome a .

For example, Figure 6 depicts the situation in which a referee, represented by the channels Ψ_1, \dots, Ψ_4 along with the box suggesting a measurement, interacts with two players, each designated by a superscript 1 or 2 on their respective channels and the registers they touch. In general, such an interaction may involve any number of players m . Following the standard assumption in non-cooperative game theory, the m players are assumed to not directly interact with one another: all interactions are between a player and the referee. (The referee could choose to pass information from one player to another, but such an action must be understood as being in accordance with the referee’s specification.)

Also, although Figure 6 might suggest a symmetry between the players, this is not required—the registers being exchanged can have arbitrary size, including the possibility of trivial (dimension 1) registers that effectively represent the absence of information being sent or received. Equivalently, the referee may interleave the messages exchanged with different players in an arbitrary way, and the number of exchanges may be different with different players.

In any case of this sort, similar to the single-player case just discussed, there will always exist an efficiently computable positive semidefinite operator P_a , for each possible outcome of the referee’s measurement, for which the probability associated with that measurement outcome is given by

$$\Pr(\text{measurement outcome equals } a) = \langle P_a, Q_1 \otimes \dots \otimes Q_m \rangle, \quad (30)$$

assuming that the m players play strategies represented by matrices Q_1, \dots, Q_m . Indeed, aside from a permutation of tensor factors, one need not see this as being an extension of the single-player case at all, for if the m players do not directly interact, they may be collectively viewed as a single player, whose representation (again, up to a permutation of tensor factors) is given by the tensor product $Q_1 \otimes \dots \otimes Q_m$.

2.4. PPAD and fixed-point problems

We now recall the definition of the complexity class PPAD, which was first defined by Papadimitriou [Pap94] to capture the complexity of certain total functions, including approximate fixed-point problems when a fixed point is guaranteed to exist. We also state a result due to Etessami and Yannakakis [EY10] concerning the containment of a specific fixed-point problem in PPAD to which we will later reduce the problem of computing approximate fixed points of functions defined on density operators.

Before proceeding to these definitions, let us remark that all computational problems in this paper involving real or complex scalars, vectors, matrices, and so on, are assumed to refer to rational and/or Gaussian rational inputs and outputs in which the number a/b is encoded as a pair $\langle a, b \rangle$ and $a/b + ic/d$ is encoded as a 4-tuple $\langle a, b, c, d \rangle$, for integers a, b, c , and d represented in signed binary notation. The length of any such number then refers to the length of the encoding.

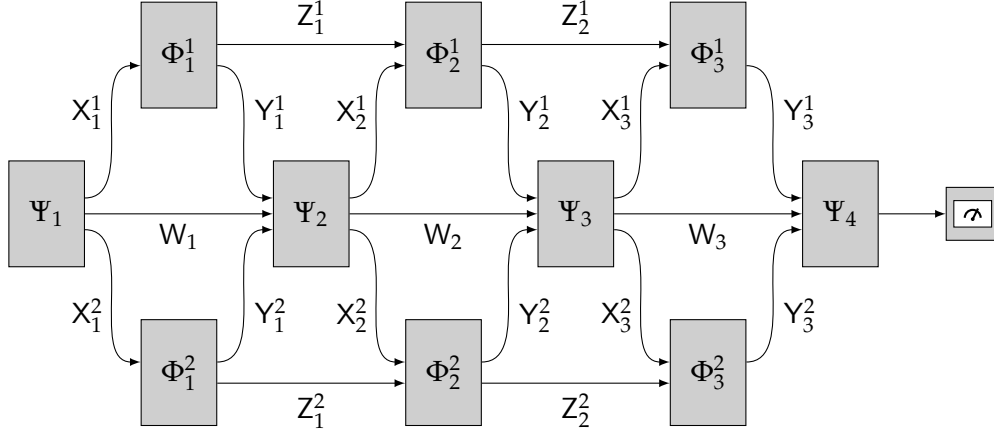


Figure 6: An interaction between a referee (represented by channels Ψ_1, \dots, Ψ_4 along with a measurement) and two players, both having a form similar to the strategy pictured in Figure 4.

Total search problems and the complexity class PPA

The complexity class PPA contains *total search problems*. In general, a total search problem in the complexity class TFNP is represented by a collection of sets $\{A_x : x \in \Sigma^*\}$, with $A_x \subseteq \Sigma^*$ for each $x \in \Sigma^*$, satisfying these properties:

1. There exists a polynomial p such that $|y| \leq p(|x|)$ for every $x \in \Sigma^*$ and $y \in A_x$.
2. There exists a polynomial-time computable predicate R such that $R(x, y) = 1$ if and only if $y \in A_x$, for every choice of $x, y \in \Sigma^*$.
3. For every $x \in \Sigma^*$, the set A_x is non-empty.

On a given input string $x \in \Sigma^*$, the goal of the associated problem is to find any string $y \in A_x$. Such search problems are deemed *total* because an acceptable solution is always guaranteed to exist.

In the context of total search problems in TFNP, it is said that a problem $\{A_x : x \in \Sigma^*\}$ is *polynomial-time reducible* to another problem $\{B_x : x \in \Sigma^*\}$ if there exist polynomial-time computable functions f and g with the property that for

$$y \in B_{f(x)} \Rightarrow g(y) \in A_x \quad (31)$$

for every string $x \in \Sigma^*$. In words, any input to the problem $A = \{A_x : x \in \Sigma^*\}$ can be transformed in polynomial time to an instance $f(x)$ of the problem $B = \{B_x : x \in \Sigma^*\}$ in such a way that any acceptable solution y to B on input $f(x)$ can be transformed in polynomial time back to an acceptable solution $g(y)$ to A on input x .

Next, to state the definition of the class PPA, which is contained in TFNP, we begin with one specific problem in this class called the *end-of-the-line problem*.

End-of-the-line problem

Input: Boolean circuits P and S , both having n input bits and n output bits, satisfying $P(0^n) = 0^n \neq S(0^n)$.

Output: Any string $z \in \Sigma^n$ such that $S(P(z)) \neq z \neq 0^n$ or $P(S(z)) \neq z$.

(Formally speaking, if one is given an input string that does not encode Boolean circuits P and S with the properties indicated, then the associated set of acceptable solutions is defined as the singleton set containing the empty string. Alternatively, one may modify the definition of TFNP so that problems may be defined only on a subset of the possible strings.)

The intuition behind this problem is that the circuits P and S allegedly represent *predecessor* and *successor* functions on the set Σ^n . We envision a graph having vertex set Σ^n with a directed edge from x to y , for distinct vertices x and y , if and only if both $y = S(x)$ and $x = P(y)$. The vertex 0^n must have in-degree 0 by the assumption $P(0^n) = 0^n$, meaning that it is a *source*. The goal is to find either a *sink*, meaning a vertex with out-degree 0, which must necessarily exist, or a source different from 0^n . If $P(S(z)) \neq z$, then z is a sink (which could include the possibility $z = 0^n$ if 0^n happens to have out-degree 0), while if $S(P(z)) \neq z \neq 0^n$ then z is a source different from 0^n . It is important that a source different from 0^n is an acceptable answer; the variant of this problem that demands a sink as an output might potentially be a more difficult computational problem.

Finally, PPAD is defined as the class of all total search problems that are polynomial-time reducible to the end-of-the-line problem.

Fixed points of barycentric approximations in PPAD

Suppose that $\{f_x : x \in \Sigma^*\}$ is a collection of functions having the form

$$f_x : \Delta_n \rightarrow \Delta_n, \quad (32)$$

for $n = n(x)$ being polynomially bounded and polynomial-time computable. We say that $\{f_x : x \in \Sigma^*\}$ is a *polynomial-time computable family* if there exists a polynomial-time computable function F so that

$$F(x, \langle v \rangle) = \langle f_x(v) \rangle \quad (33)$$

for every rational vector $v \in \Delta_n$, where angled brackets indicate the encoding of any rational element of Δ_n .

The following theorem follows from a more general result due to Etessami and Yannakakis [EY10].

Theorem 1. *Suppose that $\{f_x : x \in \Sigma^*\}$ is a polynomial-time computable family of functions having the form $f_x : \Delta_n \rightarrow \Delta_n$, and for each $x \in \Sigma^*$ and each positive integer r let*

$$g_{x,r} : \Delta_n \rightarrow \Delta_n \quad (34)$$

be the r -th order barycentric approximation to f_x . The problem of computing an exact fixed point of $g_{x,r}$ on the input $\langle x, 0^r \rangle$ is contained in the class PPAD.

3. Definitions of quantum games

We will now define a general class of quantum games and state the computational problem upon which the remainder of the paper focuses.

In the class of games we consider, a referee exchanges quantum registers with m players over the course of r rounds in a way that generalizes Figure 6 (in which $m = 2$ and $r = 3$). The referee's actions are described by channels $\Psi_1, \dots, \Psi_{r+1}$ along with a final measurement, with these objects taking the following forms. For $j \in \{2, \dots, r\}$, the channel Ψ_j takes input registers

$$(W_{j-1}, Y_{j-1}^1, \dots, Y_{j-1}^m) \quad (35)$$

and outputs registers

$$(W_j, X_j^1, \dots, X_j^m). \quad (36)$$

The channels Ψ_1 and Ψ_{r+1} have a similar form except that Ψ_1 takes no input and Ψ_{r+1} produces a single register W_{r+1} as output. We note that the registers need not all have the same size, and some may be trivial, effectively representing the absence of a message transmission. Finally, the measurement is performed on the register W_{r+1} and has set of outcomes Γ . In addition to the referee's actions, as just described, it is to be assumed that a payoff function $v_k : \Gamma \rightarrow \mathbb{R}$ has been selected for each player $k \in \{1, \dots, m\}$.

A referee of this form determines a non-cooperative game, in which an m -tuple of independent strategies for the players is interfaced with the referee in the most natural way, leading to a distribution over payoffs for the m players.

Suppose that, by means of the quantum strategies framework, a selection of the m players' strategies Q_1, \dots, Q_m has been made, with each being represented by an operator

$$Q_k \in \text{Pos}(y_1^k \otimes \dots \otimes y_r^k \otimes x_1^k \otimes \dots \otimes x_r^k), \quad (37)$$

and suppose moreover that the referee has been represented by a collection of operators $\{P_a : a \in \Gamma\}$, as was described in the previous section. We then have that each outcome $a \in \Gamma$ is produced by the referee with probability $\langle P_a, Q_1 \otimes \dots \otimes Q_m \rangle$, and the payoffs are then determined accordingly. The *expected payoff* for player k is therefore given by

$$\sum_{a \in \Gamma} v_k(a) \langle P_a, Q_1 \otimes \dots \otimes Q_m \rangle = \langle H_k, Q_1 \otimes \dots \otimes Q_m \rangle \quad (38)$$

for

$$H_k = \sum_{a \in \Gamma} v_k(a) P_a. \quad (39)$$

When it is convenient, we will refer to the operators H_1, \dots, H_m as *payoff operators*. We observe that the payoff operators of an interactive quantum game can be efficiently computed given the description of a referee's actions.

Hereafter let us write

$$\mathcal{S}_k \subset \text{Pos}(y_1^k \otimes \dots \otimes y_r^k \otimes x_1^k \otimes \dots \otimes x_r^k) \quad (40)$$

to denote the set of strategy representations available to player k , for each $k \in \{1, \dots, m\}$. Each of these sets is bounded and characterized by a finite collection of affine linear constraints on the positive semidefinite cone acting on the corresponding spaces. In particular, the sets $\mathcal{S}_1, \dots, \mathcal{S}_m$ are convex and compact.

A *Nash equilibrium* of a quantum game of the form being considered is an m -tuple $(Q_1, \dots, Q_m) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_m$ for which the equality

$$\langle H_k, Q_1 \otimes \dots \otimes Q_m \rangle = \sup_{R \in \mathcal{S}_k} \langle H_k, Q_1 \otimes \dots \otimes Q_{k-1} \otimes R \otimes Q_{k+1} \otimes \dots \otimes Q_m \rangle \quad (41)$$

holds for every $k \in \{1, \dots, m\}$. Thus, no player can increase their expected payoff by unilaterally deviating from a Nash equilibrium (Q_1, \dots, Q_m) . The existence of a Nash equilibrium in every interactive quantum game follows from Glicksberg's generalization of Nash's theorem [Gli52]. It is also straightforward to prove the existence of a Nash equilibrium in an interactive quantum game more directly, through the Kakutani fixed-point theorem (upon which Glicksberg's generalization is also based), following the same reasoning as in Nash's proof in [Nas50a] for the existence of an equilibrium point in classical games.

For any choice of $\varepsilon > 0$, an m -tuple of strategies $(Q_1, \dots, Q_m) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_m$ is an ε -approximate Nash equilibrium if it is the case that

$$\langle H_k, Q_1 \otimes \dots \otimes Q_m \rangle \geq \sup_{R \in \mathcal{S}_k} \langle H_k, Q_1 \otimes \dots \otimes Q_{k-1} \otimes R \otimes Q_{k+1} \otimes \dots \otimes Q_m \rangle - \varepsilon \quad (42)$$

for every $k \in \{1, \dots, m\}$. In words, no player can increase their expected payoff by more than ε by deviating from an ε -approximate Nash equilibrium (Q_1, \dots, Q_m) .

For the sake of efficiency, it is prudent at this point to introduce some additional notation. For each $k \in \{1, \dots, m\}$ we define

$$\mathcal{V}_k = \mathcal{Y}_1^k \otimes \dots \otimes \mathcal{Y}_r^k \otimes \mathcal{X}_1^k \otimes \dots \otimes \mathcal{X}_r^k, \quad (43)$$

so that $\mathcal{S}_k \subset \text{Pos}(\mathcal{V}_k)$, as well as

$$\mathcal{V}_{-k} = \mathcal{V}_1 \otimes \dots \otimes \mathcal{V}_{k-1} \otimes \mathcal{V}_{k+1} \otimes \dots \otimes \mathcal{V}_m. \quad (44)$$

For a given choice of strategies $(Q_1, \dots, Q_m) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_m$ we define

$$Q_{-k} = Q_1 \otimes \dots \otimes Q_{k-1} \otimes Q_{k+1} \otimes \dots \otimes Q_m, \quad (45)$$

so that $Q_{-k} \in \text{Pos}(\mathcal{V}_{-k})$. We stress that this is a tensor product—a similar notation is often used for Cartesian products.

Observe that, for each $k \in \{1, \dots, m\}$, there exists a Hermitian-preserving linear map taking the form $\Xi_k : L(\mathcal{V}_{-k}) \rightarrow L(\mathcal{V}_k)$ and having the property that

$$\langle H_k, Q_1 \otimes \dots \otimes Q_m \rangle = \langle \Xi_k(Q_{-k}), Q_k \rangle \quad (46)$$

for every choice of $(Q_1, \dots, Q_m) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_m$ (or indeed for any choice of Hermitian operators Q_1, \dots, Q_m , not just strategies). Explicitly,

$$\Xi_k(Q_{-k}) = \text{Tr}_{\mathcal{V}_{-k}}((Q_1 \otimes \dots \otimes Q_{k-1} \otimes \mathbf{1}_{\mathcal{V}_k} \otimes Q_{k+1} \otimes \dots \otimes Q_m)H_k). \quad (47)$$

An equivalent condition to (41) is then that

$$\langle \Xi_k(Q_{-k}), Q_k \rangle = \sup_{R \in \mathcal{S}_k} \langle \Xi_k(Q_{-k}), R \rangle, \quad (48)$$

while (42) is equivalent to

$$\langle \Xi_k(Q_{-k}), Q_k \rangle \geq \sup_{R \in \mathcal{S}_k} \langle \Xi_k(Q_{-k}), R \rangle - \varepsilon. \quad (49)$$

We may now define the computational problem of approximating a Nash equilibrium of a quantum game. We assume that the input to the problem consists of the payoff operators of a given game, along with positive real number ε , although as noted above one could alternatively describe a quantum game in terms of the referee's actions, from which the payoff operators may be computed.

Approximate quantum Nash equilibrium

- Input: Hermitian operators $H_1, \dots, H_m \in \text{Herm}(\mathcal{V}_1 \otimes \dots \otimes \mathcal{V}_m)$, for each \mathcal{V}_k taking the form (43), along with a positive real number ε .
- Output: An ε -approximate Nash equilibrium (Q_1, \dots, Q_m) of the interactive quantum game described by H_1, \dots, H_m .

The following theorem, which is proved in the next section, represents the main result of this paper.

Theorem 2. *The problem of computing an approximate quantum Nash equilibrium is contained in the complexity class PPAD.*

4. Approximate quantum Nash equilibria in PPAD

The purpose of this section is to prove Theorem 2. We shall begin with an overview of the proof, followed by three subsections that address specific aspects of it.

The proofs of the existence of Nash equilibria in interactive quantum games suggested above are both based on the Kakutani fixed-point theorem. Toward the goal of establishing that the problem of approximating Nash equilibria in quantum games is in the complexity class PPAD, however, it is instructive to consider a different path, based on an extension of Nash's 1951 proof [Nas51] of the existence of equilibria in classical games, which makes use of the Brouwer fixed-point theorem together with the notion of a gain function. This is a familiar path to analogous results in the classical setting [Pap94, DGP09, CDT09, EY10].

Our first step is to prove that the problem of approximating fixed points of a certain class of continuous functions defined on *density operators* is contained in PPAD. This is done by means of a reduction, based on the discrete Wigner representation defined in Section 2.2, to the fixed-point problem on the simplex established to be in PPAD by Theorem 1.

The second step is to consider an interactive quantum generalization of Nash’s gain function. Intuitively speaking, this is a function defined on m -tuples of strategies that improves each player’s strategy, relative to the other players’ strategies being considered, so that the fixed points of this function are equilibrium points. This allows for the reduction of the problem of finding an approximate Nash equilibrium in a quantum game to finding an approximate fixed point of this gain function, which may be expressed as a function on density operators.

The computations required by both of the steps just described cannot be performed exactly using rational number computations. To control the precision required by rational number approximations to these computations, we must bound the *Lipschitz moduli* of various functions that are composed to obtain the reduction. This includes functions expressible as semidefinite programs but not known to have closed form expressions.

The subsections that follow address these aspects of the proof. The first subsection is concerned entirely with the Lipschitz moduli of various function that will be needed in the remaining subsections, establishing bounds that allow the proof to go through. The second subsection establishes that the problem of computing approximate fixed points of continuous functions defined on density operators (or Cartesian products of density operators) is contained in PPAD. And finally, the third subsection reduces the problem of computing approximate Nash equilibria of interactive quantum games to the problem of computing fixed points of continuous functions on density operators.

4.1. Some useful functions and bounds on their Lipschitz moduli

This subsection simply lists several functions relevant to the proof together with bounds on their Lipschitz moduli.

Whenever we refer to the Lipschitz condition for any function, defined for vectors or operators, we will always use the 2-norm, meaning the standard Euclidean norm for \mathbb{R}^n and the Frobenius norm for the $n \times n$ complex Hermitian operators $\text{Herm}(\mathbb{C}^n)$. That is, a function f is K -Lipschitz if

$$\|f(u) - f(v)\|_2 \leq K\|u - v\|_2 \tag{50}$$

for all vectors u and v on which it is defined, and likewise for functions defined on operators rather than vectors. We refer to K as the *Lipschitz modulus* of f , as opposed to the more standard *Lipschitz constant*, as K will generally not be constant (as a function of the input length) for the functions we will encounter.

The discrete Wigner representation.

The function $\psi : \text{Herm}(\mathbb{C}^n) \rightarrow \mathbb{R}^{n^2}$ associated with the discrete Wigner representation we have defined is $(1/K)$ -Lipschitz, while ψ^{-1} is K -Lipschitz, for $K = \sqrt{n(n+1)}$. More precisely, by the orthogonality of the operators $\{V_1, \dots, V_{n^2}\}$, we have the equality conditions

$$\|\psi(H) - \psi(K)\|_2 = \frac{1}{\sqrt{n(n+1)}} \|H - K\|_2 \quad (51)$$

and

$$\|\psi^{-1}(u) - \psi^{-1}(v)\|_2 = \sqrt{n(n+1)} \|u - v\|_2. \quad (52)$$

These two moduli will cancel one another in the analysis to follow in the next subsection.

Tensor products of density operators.

The tensor product mapping

$$(\rho_1, \dots, \rho_m) \mapsto \rho_1 \otimes \dots \otimes \rho_m, \quad (53)$$

from $D(\mathbb{C}^{n_1}) \times \dots \times D(\mathbb{C}^{n_m})$ to $D(\mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_m})$, is \sqrt{m} -Lipschitz:

$$\begin{aligned} & \|\rho_1 \otimes \dots \otimes \rho_m - \sigma_1 \otimes \dots \otimes \sigma_m\|_2 \\ & \leq \|\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_m - \sigma_1 \otimes \rho_2 \otimes \dots \otimes \rho_m\|_2 \\ & \quad + \|\sigma_1 \otimes \rho_2 \otimes \dots \otimes \rho_m - \sigma_1 \otimes \sigma_2 \otimes \dots \otimes \sigma_m\|_2 \\ & \leq \|\rho_1 - \sigma_1\|_2 \|\rho_2 \otimes \dots \otimes \rho_m\|_2 \\ & \quad + \|\sigma_1\|_2 \|\rho_2 \otimes \dots \otimes \rho_m - \sigma_2 \otimes \dots \otimes \sigma_m\|_2 \\ & \leq \|\rho_1 - \sigma_1\|_2 + \|\rho_2 \otimes \dots \otimes \rho_m - \sigma_2 \otimes \dots \otimes \sigma_m\|_2, \end{aligned} \quad (54)$$

and by iterating,

$$\begin{aligned} & \|\rho_1 \otimes \dots \otimes \rho_m - \sigma_1 \otimes \dots \otimes \sigma_m\|_2 \\ & \leq \|\rho_1 - \sigma_1\|_2 + \dots + \|\rho_m - \sigma_m\|_2 \\ & \leq \sqrt{m} \|(\rho_1, \dots, \rho_m) - (\sigma_1, \dots, \sigma_m)\|_2. \end{aligned} \quad (55)$$

The maps Ξ_k .

Recall the maps $\Xi_k : L(\mathcal{V}_{-k}) \rightarrow L(\mathcal{V}_k)$ defined in the previous section, which satisfy

$$\langle H_k, Q_1 \otimes \dots \otimes Q_m \rangle = \langle \Xi_k(Q_{-k}), Q_k \rangle \quad (56)$$

for all Hermitian operators Q_1, \dots, Q_m , where H_k is considered to be fixed. Explicitly,

$$\Xi_k(Q_{-k}) = \text{Tr}_{\mathcal{V}_{-k}} ((Q_1 \otimes \dots \otimes Q_{k-1} \otimes \mathbb{1}_{\mathcal{V}_k} \otimes Q_{k+1} \otimes \dots \otimes Q_m) H_k). \quad (57)$$

Let us write $n_k = \dim(\mathcal{V}_k)$ and $n = n_1 \dots n_m$.

First, the mapping

$$Q_{-k} \mapsto Q_1 \otimes \cdots \otimes Q_{k-1} \otimes \mathbb{1}_{V_k} \otimes Q_{k+1} \otimes \cdots \otimes Q_m \quad (58)$$

is $\sqrt{n_k}$ -Lipschitz, while right-multiplication by H_k is $\|H_k\|$ -Lipschitz, $\|H_k\|$ denoting the spectral norm of H_k .

Next, every quantum channel $\Psi : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$, including the trace, is 1-Lipschitz with respect to the trace norm, and therefore

$$\|\Psi(X) - \Psi(Y)\|_2 \leq \|\Psi(X) - \Psi(Y)\|_1 \leq \|X - Y\|_1 \leq \sqrt{\dim(\mathcal{X})} \|X - Y\|_2. \quad (59)$$

That is, every channel is $\sqrt{\dim(\mathcal{X})}$ -Lipschitz with respect to the Frobenius norm, for \mathcal{X} being the input space of the channel. We may also note that tensoring any linear map (whether a channel or not) with the identity channel does not change its Lipschitz modulus. It follows that the partial trace over the space \mathcal{V}_{-k} has Lipschitz modulus $\sqrt{n/n_k}$.

Composing these functions, we find that the mapping Ξ_k is $(\|H_k\|\sqrt{n})$ -Lipschitz.

Projections onto closed and convex sets.

For any closed and convex set \mathcal{C} , we define $\text{proj}(X | \mathcal{C})$ to be the projection of X onto the set \mathcal{C} , meaning the unique point contained in \mathcal{C} that is closest to X with respect to the 2-norm (or Frobenius norm). The function $X \mapsto \text{proj}(X | \mathcal{C})$ is, as is well known, always 1-Lipschitz.

Normalizing positive semidefinite operators.

Next, define a function that *normalizes* any positive semidefinite operator $P \in \text{Pos}(\mathbb{C}^n)$ in the following way:

$$\text{normalize}(P) = \begin{cases} \frac{P}{\text{Tr}(P)} & \text{Tr}(P) \geq 1 \\ P + (1 - \text{Tr}(P))\frac{\mathbb{1}_n}{n} & \text{Tr}(P) < 1. \end{cases} \quad (60)$$

Strictly speaking this may not really be a normalization in the case that $\text{Tr}(P) < 1$, but this function serves our purposes nevertheless.

The function $\text{normalize} : \text{Pos}(\mathbb{C}^n) \rightarrow \text{D}(\mathbb{C}^n)$ is $(4n)$ -Lipschitz. This is perhaps easiest to prove by expressing the function as $\text{normalize} = g \circ f$ where f and g are defined as follows:

$$f(P) = \begin{cases} P & \text{Tr}(P) \geq 1 \\ P + (1 - \text{Tr}(P))\frac{\mathbb{1}_n}{n} & \text{Tr}(P) < 1, \end{cases} \quad (61)$$

$$g(P) = \begin{cases} \frac{P}{\text{Tr}(P)} & \text{Tr}(P) \geq 1 \\ P & \text{Tr}(P) < 1. \end{cases}$$

The function f is $(2\sqrt{n})$ -Lipschitz, which may be established by considering three cases. If $\text{Tr}(P) \geq 1$ and $\text{Tr}(Q) \geq 1$, then $\|f(P) - f(Q)\|_2 = \|P - Q\|_2$, trivially. If $\text{Tr}(P) \geq 1$ and $\text{Tr}(Q) < 1$, then

$$\begin{aligned} \|f(P) - f(Q)\|_1 &= \left\| P - Q - (1 - \text{Tr}(Q)) \frac{\mathbb{1}_n}{n} \right\|_1 \leq \|P - Q\|_1 + (1 - \text{Tr}(Q)) \\ &\leq \|P - Q\|_1 + (\text{Tr}(P) - \text{Tr}(Q)) \leq 2\|P - Q\|_1 \end{aligned} \quad (62)$$

and therefore

$$\|f(P) - f(Q)\|_2 \leq 2\|P - Q\|_1 \leq 2\sqrt{n}\|P - Q\|_2. \quad (63)$$

If $\text{Tr}(P) < 1$ and $\text{Tr}(Q) < 1$, then

$$\begin{aligned} \|f(P) - f(Q)\|_1 &= \left\| P - Q - (\text{Tr}(P) - \text{Tr}(Q)) \frac{\mathbb{1}_n}{n} \right\|_1 \\ &\leq \|P - Q\|_1 + |\text{Tr}(P) - \text{Tr}(Q)| \leq 2\|P - Q\|_1, \end{aligned} \quad (64)$$

and so again

$$\|f(P) - f(Q)\|_2 \leq 2\|P - Q\|_1 \leq 2\sqrt{n}\|P - Q\|_2. \quad (65)$$

On the set of positive semidefinite operators having trace at least one, the function g is $(1 + \sqrt{n})$ -Lipschitz; supposing that $P, Q \in \text{Pos}(\mathbb{C}^n)$ satisfy $\text{Tr}(Q) \geq 1$ and $\text{Tr}(P) \geq 1$, we find that

$$\begin{aligned} \left\| \frac{P}{\text{Tr}(P)} - \frac{Q}{\text{Tr}(Q)} \right\|_2 &\leq \left\| \frac{P}{\text{Tr}(P)} - \frac{Q}{\text{Tr}(P)} \right\|_2 + \left\| \frac{Q}{\text{Tr}(P)} - \frac{Q}{\text{Tr}(Q)} \right\|_2 \\ &= \frac{\|P - Q\|_2}{\text{Tr}(P)} + \left(\frac{\text{Tr}(Q) - \text{Tr}(P)}{\text{Tr}(P)\text{Tr}(Q)} \right) \|Q\|_2 \\ &\leq (1 + \sqrt{n})\|P - Q\|_2. \end{aligned} \quad (66)$$

Using $2\sqrt{n}(\sqrt{n} + 1) \leq 4n$ we obtain that normalize is $(4n)$ -Lipschitz.

4.2. Fixed points of functions on density operators

We now consider the computational problem of approximating fixed points of continuous functions defined on density operators, proving that this problem is in PPAD for functions having exponentially bounded Lipschitz moduli.

To state this fact more precisely, we require a few definitions. First, a density operator $\rho \in \text{D}(\mathbb{C}^n)$ is an ε -approximate fixed point of a function $f : \text{D}(\mathbb{C}^n) \rightarrow \text{D}(\mathbb{C}^n)$ provided that

$$\|f(\rho) - \rho\|_2 \leq \varepsilon. \quad (67)$$

Next, suppose that $\{f_x : x \in \Sigma^*\}$ is a collection of functions having the form

$$f_x : \text{D}(\mathbb{C}^n) \rightarrow \text{D}(\mathbb{C}^n) \quad (68)$$

for $n = n(x)$ being polynomially bounded. Mirroring a definition from Section 2.4 for functions defined on the unit simplex, we shall say that $\{f_x : x \in \Sigma^*\}$ is a *polynomial-time computable family* if there exists a polynomial-time computable function F so that

$$F(x, \langle \rho \rangle) = \langle f_x(\rho) \rangle \quad (69)$$

for every rational density operator $\rho \in D(\mathbb{C}^n)$, with angled brackets representing encodings of rational density operators. We must also define an approximate variant of this notion: $\{f_x : x \in \Sigma^*\}$ is a *polynomial-time approximable family* if there exists a polynomial-time computable family $\{g_{x,\varepsilon}\}$ satisfying

$$\|f_x - g_{x,\varepsilon}\|_2 \leq \varepsilon \quad (70)$$

for every $x \in \Sigma^*$ and every positive rational number ε .

Finally, the problem of computing ε -approximate fixed points of the family $\{f_x\}$ is to output the encoding of any ε -approximate fixed point of the function f_x on input (x, ε) .

Theorem 3. *Let $\{f_x\}$ be a polynomial-time approximable family of functions on density operators, let p be a polynomial, and assume that each function f_x is K_x -Lipschitz, for $K_x = 2^{p(|x|)}$. The problem of computing ε -approximate fixed points of the family $\{f_x\}$ is in PPAD.*

Proof. Let us first observe that there is no loss of generality in assuming that, for every input x , the dimension n is odd and at least 3. The case $n = 1$ is trivial, and if n is even, one may substitute the function f_x by $h_x : D(\mathbb{C}^{n+1}) \rightarrow D(\mathbb{C}^{n+1})$ defined as

$$h_x \begin{pmatrix} P & u \\ u^* & \lambda \end{pmatrix} = \begin{pmatrix} f_x(P + \lambda \frac{\mathbb{1}}{n}) & 0 \\ 0 & 0 \end{pmatrix}. \quad (71)$$

The Lipschitz modulus of h_x is at most $\sqrt{2}$ times that of f_x , and every fixed point of h_x takes the form

$$\sigma = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} \quad (72)$$

for ρ a fixed point of f_x . If

$$\begin{pmatrix} P & u \\ u^* & \lambda \end{pmatrix} \quad (73)$$

is an ε -approximate fixed point of h_x , then it follows that

$$2\|u\|^2 + \lambda^2 = \left\| \begin{pmatrix} 0 & u \\ u^* & \lambda \end{pmatrix} \right\|_2^2 \leq \left\| h_x \begin{pmatrix} P & u \\ u^* & \lambda \end{pmatrix} - \begin{pmatrix} P & u \\ u^* & \lambda \end{pmatrix} \right\|_2^2 \leq \varepsilon^2, \quad (74)$$

from which it follows that

$$\begin{aligned} & \left\| f_x \left(P + \frac{\lambda \mathbb{1}}{n} \right) - \left(P + \frac{\lambda \mathbb{1}}{n} \right) \right\|_2 \\ & \leq \left\| h_x \begin{pmatrix} P & u \\ u^* & \lambda \end{pmatrix} - \begin{pmatrix} P & u \\ u^* & \lambda \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} P & u \\ u^* & \lambda \end{pmatrix} - \begin{pmatrix} P + \frac{\lambda \mathbb{1}}{n} & 0 \\ 0 & 0 \end{pmatrix} \right\|_2 \leq (1 + \sqrt{2})\varepsilon. \end{aligned} \quad (75)$$

Thus, an ε -approximate fixed point of f_x is easily obtained from an $(\varepsilon/3)$ -approximate fixed point of h_x .

Assuming now that n is odd for each x , we define a function $g_x : \Delta_{n^2} \rightarrow \Delta_{n^2}$ as

$$g_x(v) = \psi(f_x(\text{proj}(\psi^{-1}(v) \mid D(\mathbb{C}^n)))) \quad (76)$$

Here, the projection function is as defined in the previous subsection and ψ is the mapping associated with the discrete Wigner representation defined in Section 2.2. Given that f_x is K_x -Lipschitz, it follows that g_x is K_x -Lipschitz as well, as the projection is 1-Lipschitz and the Lipschitz moduli of the discrete Wigner mappings cancel.

Given that f_x is polynomial-time approximable, it is possible to compute, in polynomial time, an approximation \tilde{g}_x to g_x satisfying

$$\|\tilde{g}_x(u) - g_x(u)\|_2 \leq \frac{\varepsilon}{16n^2K_x} \quad (77)$$

for every rational vector $u \in \Delta_{n^2}$. We note, in particular, that the projection onto $D(\mathbb{C}^n)$ may be approximated by first approximating a spectral decomposition of the operator $\psi^{-1}(v)$ and then projecting its eigenvalues onto the unit simplex Δ_n . Alternatively, this projection arises as a special case of one discussed in the next subsection, where the ellipsoid method provides a polynomial-time algorithm to approximate the projection.

Next, set

$$r = (n^2 + 1) \lceil \log(1/\varepsilon) + 2p(|x|) + 2 \log(n) + 4 \rceil. \quad (78)$$

This number is polynomial in $|x|$ and $\log(1/\varepsilon)$, and has been selected so that

$$\left(1 - \frac{1}{n^2 + 1}\right)^r < \exp(-\log(1/\varepsilon) - 2p(|x|) - 2 \log(n) - 4) < \frac{\varepsilon}{16n^2K_x^2}. \quad (79)$$

By Theorem 1, one may therefore compute an exact fixed point $v \in \Delta_{n^2}$ of the r -th level barycentric approximation to \tilde{g}_x in PPAD.

Supposing that such a fixed point v is expressed as a convex combination

$$v = q_1 v_1 + \cdots + q_{n^2} v_{n^2} \quad (80)$$

for $v_1, \dots, v_{n^2} \in \mathcal{B}_{n^2}^r$ denoting vertices in any one of the simplices constructed at the r -level of the barycentric subdivision, we find that

$$\begin{aligned} \|g_x(v) - v\|_2 &= \|g_x(v) - (q_1 \tilde{g}_x(v_1) + \cdots + q_{n^2} \tilde{g}_x(v_{n^2}))\|_2 \\ &\leq \sum_{j=1}^{n^2} q_j \|g_x(v) - \tilde{g}_x(v_j)\|_2 \\ &\leq \sum_{j=1}^{n^2} q_j \left(\|g_x(v) - g_x(v_j)\|_2 + \|g_x(v_j) - \tilde{g}_x(v_j)\|_2 \right) \\ &\leq K_x \left(1 - \frac{1}{n^2 + 1}\right)^r + \frac{\varepsilon}{16n^2K_x} \\ &\leq \frac{\varepsilon}{8n^2K_x}. \end{aligned} \quad (81)$$

Thus, v is an $(\varepsilon/(8n^2K_x))$ -approximate fixed point of g_x .

Now consider the Hermitian operator $\psi^{-1}(v)$. We have

$$\|\psi^{-1}(v) - \psi^{-1}(g(v))\|_2 = \sqrt{n}(n+1)\|v - g(v)\|_2 \leq \frac{\varepsilon}{4K_x}, \quad (82)$$

and given that $\psi^{-1}(g(v))$ is necessarily a density operator, the operator $\psi^{-1}(v)$ therefore has distance at most $\varepsilon/(4K_x)$ from the set of density operators. By computing a density operator ρ satisfying

$$\|\rho - \text{proj}(\psi^{-1}(v) | \mathbf{D}(\mathbb{C}^n))\|_2 \leq \frac{\varepsilon}{4K_x} \quad (83)$$

as suggested above, we therefore have

$$\|\rho - \psi^{-1}(v)\|_2 \leq \frac{\varepsilon}{2K_x}. \quad (84)$$

Consequently, noting that

$$f(\text{proj}(\psi^{-1}(v) | \mathbf{D}(\mathbb{C}^n))) = \psi^{-1}(g(v)), \quad (85)$$

we find, by the triangle inequality, that

$$\begin{aligned} \|f_x(\rho) - \rho\|_2 &\leq \|f(\rho) - f(\text{proj}(\psi^{-1}(v) | \mathbf{D}(\mathbb{C}^n)))\|_2 \\ &\quad + \|\psi^{-1}(g(v)) - \psi^{-1}(v)\|_2 + \|\psi^{-1}(v) - \rho\|_2 \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4K_x} + \frac{\varepsilon}{2K_x} \leq \varepsilon. \end{aligned} \quad (86)$$

Thus, ρ is an ε -approximate fixed point of f_x . As ρ has been computed in polynomial time from v , the theorem is proved. \square

Corollary 4. *Let $\{f_x\}$ be a polynomial-time approximable family of functions having the form*

$$f_x : \mathbf{D}(\mathbb{C}^{n_1}) \times \dots \times \mathbf{D}(\mathbb{C}^{n_m}) \rightarrow \mathbf{D}(\mathbb{C}^{n_1}) \times \dots \times \mathbf{D}(\mathbb{C}^{n_m}), \quad (87)$$

for positive integers n_1, \dots, n_m , let p be a polynomial, and assume that each function f_x is K_x -Lipschitz, for $K_x = 2^{p(|x|)}$. The problem of computing ε -approximate fixed points of the family $\{f_x\}$ is in PPAD.

Proof. Let $n = n_1 + \dots + n_m$ and define a mapping $h : \mathbf{D}(\mathbb{C}^n) \rightarrow \mathbf{D}(\mathbb{C}^n)$ as follows:

$$h \begin{pmatrix} X_{1,1} & \cdots & X_{1,m} \\ \vdots & \ddots & \vdots \\ X_{m,1} & \cdots & X_{m,m} \end{pmatrix} = \frac{1}{m} \begin{pmatrix} \text{normalize}(mX_{1,1}) & & 0 \\ & \ddots & \\ 0 & & \text{normalize}(mX_{m,m}) \end{pmatrix}, \quad (88)$$

where it is to be understood that each $X_{i,j}$ has n_i rows and n_j columns. The mapping h is $(4n)$ -Lipschitz and projects onto operators having the form

$$\frac{1}{m} \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_m \end{pmatrix}. \quad (89)$$

By composing f_x with h in the natural way, one obtains a function $g_x : D(\mathbb{C}^n) \rightarrow D(\mathbb{C}^n)$ such that

$$g_x \begin{pmatrix} X_{1,1} & \cdots & X_{1,m} \\ \vdots & \ddots & \vdots \\ X_{m,1} & \cdots & X_{m,m} \end{pmatrix} = \frac{1}{m} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_m \end{pmatrix} \quad (90)$$

for

$$(\sigma_1, \dots, \sigma_m) = f_x(\text{normalize}(mX_{1,1}), \dots, \text{normalize}(mX_{m,m})). \quad (91)$$

Finally, from any approximate fixed point of the family $\{g_x\}$, an ε -approximate fixed point for $\{f_x\}$ is obtained by applying to it the function h and reading off the diagonal operators. The problem of approximating fixed points of $\{f_x\}$ therefore reduces in polynomial time to that of $\{g_x\}$, which is in the class PPAD. \square

4.3. Nash equilibria as fixed points of functions

The final step of the proof of Theorem 2 is to reduce the problem of computing approximate Nash equilibria of interactive quantum games to the approximate fixed-point problem on Cartesian products of density operators established by Corollary 4 to be in PPAD. To do this, we will consider an extension of Nash's gain function to quantum strategies, as they are represented within the quantum strategies framework.

For a quantum game of the general form described in Section 3, the set of strategies available each player $k \in \{1, \dots, m\}$ is represented by the set

$$\mathcal{S}_k \subset \text{Pos}(y_1^k \otimes \cdots \otimes y_r^k \otimes x_1^k \otimes \cdots \otimes x_r^k), \quad (92)$$

and we observe that for every choice of $Q_k \in \mathcal{S}_k$ we have

$$\text{Tr}(Q_k) = d_k \stackrel{\text{def}}{=} \dim(x_1^k \otimes \cdots \otimes x_r^k). \quad (93)$$

Define the set

$$\mathcal{C}_k = \frac{1}{d_k} \mathcal{S}_k \subseteq D(\mathcal{V}_k), \quad (94)$$

as well as the cone

$$\mathcal{K}_k = \text{cone}(\mathcal{C}_k) = \{\lambda \rho : \lambda \geq 0, \rho \in \mathcal{C}_k\}. \quad (95)$$

Now, for a given m -tuple (ρ_1, \dots, ρ_m) of density operators, we define $G(\rho_1, \dots, \rho_m)$ in the following way. First, for each $k \in \{1, \dots, m\}$, define

$$\begin{aligned} \sigma_k &= \text{proj}(\rho_k | \mathcal{C}_k), \\ \alpha_k &= \langle \Xi_k(\sigma_{-k}), \sigma_k \rangle, \\ P_k &= \text{proj}(\Xi_k(\sigma_{-k}) - \alpha_k \mathbb{1}_{\mathcal{V}_k} | \mathcal{K}_k), \end{aligned} \quad (96)$$

and

$$G_k(\rho_1, \dots, \rho_m) = \text{normalize}(\sigma_k + P_k) = \frac{\sigma_k + P_k}{1 + \text{Tr}(P_k)}. \quad (97)$$

Then define

$$G(\rho_1, \dots, \rho_m) = (G_1(\rho_1, \dots, \rho_m), \dots, G_m(\rho_1, \dots, \rho_m)). \quad (98)$$

By combining the Lipschitz moduli for the functions from which G is formed, overestimating for the sake of a simple expression, we have that G is K -Lipschitz for

$$K = 4n^2mM, \quad M = \max\{\|H_1\|, \dots, \|H_m\|\} + 1, \quad \text{and} \quad n = n_1 \cdots n_m \quad (99)$$

for $n_k = \dim(\mathcal{V}_k)$. The following lemma establishes that G can be efficiently approximated.

Lemma 5. *There exists a deterministic, polynomial-time algorithm that, on input $H_1, \dots, H_m, \rho_1, \dots, \rho_m$, and $\delta > 0$, outputs $(\xi_1, \dots, \xi_m) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_m$ satisfying*

$$\|G(\rho_1, \dots, \rho_m) - (\xi_1, \dots, \xi_m)\|_2 < \delta. \quad (100)$$

Proof. Let us begin with the approximation of the projections $\sigma_k = \text{proj}(\rho_k | \mathcal{C}_k)$ for each $k \in \{1, \dots, m\}$. The projection σ_k is given by the unique optimal solution to the following semidefinite program:

$$\begin{aligned} & \text{minimize : } \text{Tr}(Z_k) \\ & \text{subject to : } \begin{pmatrix} Z_k & \rho_k - Y_k \\ \rho_k - Y_k & \mathbb{1}_{\mathcal{V}_k} \end{pmatrix} \geq 0 \\ & \quad Y_k \in \mathcal{C}_k \\ & \quad Z_k \in \text{Pos}(\mathcal{V}_k). \end{aligned} \quad (101)$$

Specifically, the unique optimal solution (Y_k, Z_k) to this semidefinite program satisfies $Y_k = \sigma_k = \text{proj}(\rho_k | \mathcal{C}_k)$ and $\text{Tr}(Z_k) = \|\rho_k - \sigma_k\|_2^2$.

Through the use of the ellipsoid method, as presented by [GLS88] for instance, one may compute in time polynomial in the input length and $\log(1/\eta)$, for any given positive real number η , a feasible solution (Z_k, Y_k) to this semidefinite program that is within η of its optimal value. That is, in polynomial time one may compute $\xi_k \in \mathcal{C}_k$ such that

$$\|\rho_k - \xi_k\|_2^2 \leq \|\rho_k - \sigma_k\|_2^2 + \eta. \quad (102)$$

This requires an examination of specific aspects of the semidefinite program that are reflected (up to a scalar multiple in the last constraint) by the equations (26) in Section 2.3 along with a recognition that the feasible region may be bounded. The analysis is straightforward and we omit it here.

Now, if it is the case that $\rho_k \in \mathcal{C}_k$, then $\sigma_k = \rho_k$, and we conclude immediately that

$$\|\xi_k - \sigma_k\|_2 \leq \sqrt{\eta}. \quad (103)$$

If $\rho_k \notin \mathcal{C}_k$, then it follows that $\langle \xi_k - \sigma_k, \rho_k - \sigma_k \rangle \leq 0$; that this inequality holds for every choice of $\xi_k \in \mathcal{C}_k$ is, in fact, a well known necessary and sufficient condition for σ_k to be the projection of ρ_k into \mathcal{C}_k . By the law of cosines we have

$$\|\rho_k - \xi_k\|_2^2 = \|\rho_k - \sigma_k\|_2^2 + \|\xi_k - \sigma_k\|_2^2 - 2\langle \xi_k - \sigma_k, \rho_k - \sigma_k \rangle, \quad (104)$$

and so we conclude that

$$\|\rho_k - \tilde{\xi}_k\|_2^2 \geq \|\rho_k - \sigma_k\|_2^2 + \|\tilde{\xi}_k - \sigma_k\|_2^2, \quad (105)$$

which again implies

$$\|\tilde{\xi}_k - \sigma_k\|_2 \leq \sqrt{\eta}. \quad (106)$$

The computation of each $P_k = \text{proj}(\Xi_k(\sigma_{-k}) - \alpha_k \mathbb{1}_{\mathcal{V}_k} \mid \mathcal{K}_k)$ may be performed in almost exactly the same manner, through almost exactly the same semidefinite program. We note in particular that the optimal value is no larger than $\|\Xi_k(\sigma_{-k}) - \alpha_k \mathbb{1}_{\mathcal{V}_k}\|_2^2$, as the projection of $\Xi_k(\sigma_{-k}) - \alpha_k \mathbb{1}_{\mathcal{V}_k}$ onto \mathcal{K}_k is no further away from this operator than the zero operator, which is contained in \mathcal{K}_k , and so once again the feasible region may be bounded. Thus we may compute, again in polynomial time, $R_k \in \mathcal{K}_k$ satisfying $\|P_k - R_k\|_2 \leq \sqrt{\eta}$.

All of the other computations required to approximate G can be performed exactly. The lemma follows by choosing η to be sufficiently small while polynomial in δ and the input length to the problem. \square

It therefore follows from Corollary 4 that, on input H_1, \dots, H_m and $\delta > 0$, the problem of computing a δ -approximate fixed point of G is contained in PPAD. It remains to prove that from such an approximate fixed point of G , we obtain an approximate Nash equilibrium for a game described by H_1, \dots, H_m .

At this point we face a minor inconvenience: an approximate fixed point (ρ_1, \dots, ρ_m) of G provided by the PPAD computation whose existence is implied by Corollary 4 might not be contained in $\mathcal{C}_1 \times \dots \times \mathcal{C}_m$, although by necessity it will be close. Because we require an approximate Nash equilibrium to consist of strategies and not “near strategies,” we must project these density operators onto the sets $\mathcal{C}_1, \dots, \mathcal{C}_m$. Specifically, suppose that (ρ_1, \dots, ρ_m) is an $(\eta/2)$ -approximate fixed point of G , for

$$\eta = \frac{\varepsilon^2}{(3nM)^4}. \quad (107)$$

Writing $\sigma_k = \text{proj}(\rho_k \mid \mathcal{C}_k)$ for each $k \in \{1, \dots, m\}$ as before, we find by the definition of G that

$$G(\rho_1, \dots, \rho_m) = G(\sigma_1, \dots, \sigma_m) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_m, \quad (108)$$

and combining this observation with the fact that projections are 1-Lipschitz, it follows that $(\sigma_1, \dots, \sigma_m)$ is also an $(\eta/2)$ -approximate fixed point of G . Although the density operators $(\sigma_1, \dots, \sigma_m)$ cannot be computed exactly from (ρ_1, \dots, ρ_m) , the analysis used in the proof of the previous lemma implies that, in polynomial time, one may compute from (ρ_1, \dots, ρ_m) an m -tuple $(\tilde{\xi}_1, \dots, \tilde{\xi}_m) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_m$ (with this containment guaranteed by the ellipsoid method) satisfying

$$\|(\sigma_1, \dots, \sigma_m) - (\tilde{\xi}_1, \dots, \tilde{\xi}_m)\|_2 < \frac{\eta}{4K}. \quad (109)$$

It follows that

$$\begin{aligned} & \|G(\xi_1, \dots, \xi_m) - (\xi_1, \dots, \xi_m)\|_2 \\ & \leq \|G(\xi_1, \dots, \xi_m) - G(\sigma_1, \dots, \sigma_m)\|_2 + \|G(\sigma_1, \dots, \sigma_m) - (\sigma_1, \dots, \sigma_m)\|_2 \\ & \quad + \|(\sigma_1, \dots, \sigma_m) - (\xi_1, \dots, \xi_m)\|_2 \leq \eta. \end{aligned} \quad (110)$$

Thus, (ξ_1, \dots, ξ_m) is an η -approximate fixed point of G .

One more lemma is needed, which will imply that by scaling the density operators (ξ_1, \dots, ξ_m) , an ε -approximate Nash equilibrium is obtained.

Lemma 6. *Let $\mathcal{C} \subseteq D(\mathbb{C}^n)$ be a nonempty, convex, and compact set of density operators, let $\mathcal{K} = \text{cone}(\mathcal{C})$ be the cone generated by \mathcal{C} , and let $A \in \text{Herm}(\mathbb{C}^n)$ be a Hermitian operator. For a given density operator $\sigma \in \mathcal{C}$, define*

$$P = \text{proj}(A - \langle A, \sigma \rangle \mathbb{1} \mid \mathcal{K}), \quad (111)$$

and assume that

$$\left\| \frac{\sigma + P}{1 + \text{Tr}(P)} - \sigma \right\|_2 \leq \eta \quad (112)$$

for $\eta > 0$. It is the case that

$$\langle A, \sigma \rangle \geq \sup_{\xi \in \mathcal{C}} \langle A, \xi \rangle - \delta \quad (113)$$

for

$$\delta = (1 + 3n \|A\|)^2 \sqrt{\eta}. \quad (114)$$

Proof. The operator P is defined to be the closest element of the cone \mathcal{K} to the operator $A - \langle A, \sigma \rangle \mathbb{1}$ with respect to the Frobenius norm, which is to say that

$$\|(A - \langle A, \sigma \rangle \mathbb{1}) - P\|_2 \leq \|(A - \langle A, \sigma \rangle \mathbb{1}) - \lambda \xi\|_2 \quad (115)$$

for every choice of $\lambda \geq 0$ and $\xi \in \mathcal{C}$. We may first consider the case that $\lambda = 0$, from which the bound

$$\|P\|_2 \leq 2 \|A - \langle A, \sigma \rangle \mathbb{1}\|_2 \leq 4\sqrt{n} \|A\|, \quad (116)$$

is obtained, implying that $\text{Tr}(P) \leq 4n \|A\|$. It follows that

$$\|P - \text{Tr}(P)\sigma\|_2 = (1 + \text{Tr}(P)) \left\| \frac{\sigma + P}{1 + \text{Tr}(P)} - \sigma \right\|_2 \leq (1 + 4n \|A\|) \eta. \quad (117)$$

Next, by squaring both sides of the inequality (115) and simplifying, one obtains

$$\lambda \langle A - \langle A, \sigma \rangle \mathbb{1}, \xi \rangle \leq \langle A - \langle A, \sigma \rangle \mathbb{1}, P \rangle + \frac{\lambda^2}{2} \|\xi\|_2^2 - \frac{1}{2} \|P\|_2^2. \quad (118)$$

Disregarding the negative final term and observing the inequality $\|\xi\|_2 \leq 1$ and the equality $\langle A - \langle A, \sigma \rangle \mathbb{1}, \sigma \rangle = 0$, we find that

$$\lambda \langle A - \langle A, \sigma \rangle \mathbb{1}, \xi \rangle \leq \langle A - \langle A, \sigma \rangle \mathbb{1}, P - \text{Tr}(P)\sigma \rangle + \frac{\lambda^2}{2}, \quad (119)$$

again for every $\lambda \geq 0$ and $\xi \in \mathcal{C}$. Setting $\lambda = \sqrt{\eta}$ and applying the Cauchy–Schwarz inequality yields

$$\begin{aligned}
\langle A, \xi \rangle - \langle A, \sigma \rangle &= \langle A - \langle A, \sigma \rangle \mathbf{1}, \xi \rangle \\
&\leq \frac{1}{\sqrt{\eta}} \|A - \langle A, \sigma \rangle \mathbf{1}\|_2 \|P - \text{Tr}(P)\sigma\|_2 + \frac{\sqrt{\eta}}{2} \\
&\leq \left(2\sqrt{n}\|A\|(1 + 4n\|A\|) + \frac{1}{2}\right) \sqrt{\eta} \\
&\leq (1 + 3n\|A\|)^2 \sqrt{\eta}.
\end{aligned} \tag{120}$$

As this bound holds for every $\xi \in \mathcal{C}$, the lemma is proved. \square

We conclude from this lemma that

$$\langle \Xi_k(\xi_{-k}), \xi_k \rangle \geq \sup_{\tau \in \mathcal{C}_k} \langle \Xi_k(\xi_{-k}), \tau \rangle - (1 + 3n_k \|\Xi_k(\xi_{-k})\|)^2 \sqrt{\eta}. \tag{121}$$

Define $Q_k = d_k \xi_k$ for each $k \in \{1, \dots, m\}$ so that

$$(Q_1, \dots, Q_m) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_m. \tag{122}$$

By (121) it follows that

$$\begin{aligned}
\langle \Xi_k(Q_{-k}), Q_k \rangle &\geq \sup_{R \in \mathcal{S}_k} \langle \Xi_k(Q_{-k}), R \rangle - d_1 \dots d_m (1 + 3n_k \|\Xi_k(\sigma_{-k})\|)^2 \sqrt{\eta} \\
&\geq \sup_{R \in \mathcal{S}_k} \langle \Xi_k(Q_{-k}), R \rangle - \varepsilon,
\end{aligned} \tag{123}$$

and therefore (Q_1, \dots, Q_m) is an ε -approximate Nash equilibrium of the interactive quantum game having associated payoff operators H_1, \dots, H_m . As (Q_1, \dots, Q_m) has been obtained from ε together with the approximate fixed point (ρ_1, \dots, ρ_m) of G by a polynomial-time computation, Theorem 2 is proved.

5. Discussion of directions for further research

We conclude the paper with a collection of open problems and suggestions of topics that we hope might inspire further work on quantum game theory and its connections to theoretical computer science.

1. Is there a quantum extension or variant of the Lemke–Howson algorithm [LH64] for computing or approximating a Nash equilibrium in a non-interactive two-player quantum game?
2. It is interesting to consider quantum players having different restrictions placed on their strategies. For example, we might insist that players process quantum information using limited resources, or restrict player’s actions so that they represent adversarial models of noise. What can be said about quantum games in contexts such as these?

3. We have limited our focus to a non-cooperative setting, in which players must play independently, representing an inability for the players to form collusions. The consideration of collusions, and more generally the study of *cooperative quantum game theory*, is an interesting research direction.

For instance, let us imagine that there exists a shared quantum state that allows players to implement a strategy in a quantum game that is good by some measure. Non-local games, for instance, may naturally be viewed as non-interactive games in a purely cooperative setting where shared quantum states can lead to improved strategies. In the general, not completely cooperative setting, such a shared state could be provided by a trusted non-participant in the game, as in a correlated (or entangled) equilibrium—but an alternative is a setting in which such a state must arise from an unmediated interaction between colluding players, in which case players could deviate from any prescribed protocol that produces this state.

4. Closely related to the notion of an unmediated interaction, one may consider games in which there is no referee. Coin-flipping may be cast as an example, and its evidently complicated structure suggests nothing less in a setting in which the goal is, perhaps, to produce a quantum state of interest.
5. Generally speaking, can quantum game theory provide a foundation through which one may discover quantum protocols having either theoretical or practical utility?

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