Assignment 1 solutions

1. This problem is not intended to reveal anything profound—it is just meant to give you some practice in working with vectors, operators, and such.

(a) Let \( \mathcal{X} \) and \( \mathcal{Y} \) be complex Euclidean spaces and let \( A \in L(\mathcal{Y}, \mathcal{X}) \) be any nonzero operator. Prove that there exists a complex Euclidean space \( \mathcal{Z} \) along with vectors \( u \in \mathcal{X} \otimes \mathcal{Z} \) and \( v \in \mathcal{Z} \otimes \mathcal{Y} \) such that

\[
A = (1_\mathcal{X} \otimes v^*) (u \otimes 1_\mathcal{Y}).
\]

What is the minimum possible dimension of \( \mathcal{Z} \) that is required to write a given \( A \) in this way? (Unless stated otherwise, your answers should always be supported by a proof or argument of some form—so in this case you should not only give an expression for the minimum dimension of \( \mathcal{Z} \), but also a proof showing that your expression is indeed the minimum possible dimension.)

(b) Let \( \mathcal{X} \) and \( \mathcal{Y} \) be complex Euclidean spaces and let \( \Phi \in CP(\mathcal{X}, \mathcal{Y}) \) be a completely positive map. Prove that there exists an operator \( B \in L(\mathcal{X} \otimes \mathcal{Z}, \mathcal{Y}) \), for some choice of a complex Euclidean space \( \mathcal{Z} \), such that

\[
\Phi(X) = B(X \otimes 1_Z)B^*
\]

for all \( X \in L(\mathcal{X}) \). Identify a condition on the operator \( B \) that is equivalent to \( \Phi \) preserving trace.

Solution. (a) Consider first a singular-value decomposition of \( A \):

\[
A = \sum_{k=1}^{r} s_k x_k y_k^*,
\]

where \( r = \text{rank}(A) \). Let \( \mathcal{Z} = \mathbb{C}^r \) and define vectors \( u \in \mathcal{X} \otimes \mathcal{Z} \) and \( v \in \mathcal{Z} \otimes \mathcal{Y} \) as follows:

\[
u = \sum_{k=1}^{r} s_k e_k \otimes y_k.
\]

It holds that

\[
(1_\mathcal{X} \otimes v^*) (u \otimes 1_\mathcal{Y}) = \sum_{j=1}^{r} \sum_{k=1}^{r} \sqrt{s_j} \sqrt{s_k} (1_\mathcal{X} \otimes e_j^* \otimes y_j^*)(x_k \otimes e_k \otimes 1_\mathcal{Y}) = \sum_{k=1}^{r} s_k x_k y_k^* = A,
\]

as required.

The minimum dimension of \( \mathcal{Z} \) that is required is \( \text{rank}(A) \). The vectors described above show that it is possible to write \( A \) in this form when \( \mathcal{Z} \) has dimension \( r = \text{rank}(A) \). To see that it is not possible to write \( A \) in this way for \( \mathcal{Z} \) having smaller dimension than \( \text{rank}(A) \), consider an arbitrary choice of a complex Euclidean space \( \mathcal{Z} = \mathbb{C}^\Sigma \). For any two vectors \( u \in \mathcal{X} \otimes \mathcal{Z} \) and \( v \in \mathcal{Z} \otimes \mathcal{Y} \), one may write

\[
u = \sum_{a \in \Sigma} e_a \otimes y_a
\]

and

\[
u = \sum_{a \in \Sigma} e_a \otimes y_a
\]
for some choice of (not necessarily orthogonal) vectors \(\{x_a : a \in \Sigma\} \subset \mathcal{X}\) and \(\{y_a : a \in \Sigma\} \subset \mathcal{Y}\). It holds that

\[
(1_{\mathcal{X}} \otimes \nu^*)(u \otimes 1_{\mathcal{Y}}) = \sum_{a,b \in \Sigma} (1_{\mathcal{X}} \otimes e_a^* \otimes y_a^*)(x_b \otimes e_b \otimes 1_{\mathcal{Y}}) = \sum_{a \in \Sigma} x_a y_a^*,
\]

and therefore, assuming \(A = (1_{\mathcal{X}} \otimes \nu^*)(u \otimes 1_{\mathcal{Y}})\), one has

\[
\text{rank}(A) = \text{rank}((1_{\mathcal{X}} \otimes \nu^*)(u \otimes 1_{\mathcal{Y}})) \leq |\Sigma| = \dim(\mathcal{Z}).
\]

(b) Because \(\Phi\) is a completely positive map, we know that there must exist an alphabet \(\Sigma\) and a collection of operators \(\{A_a : a \in \Sigma\} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})\) satisfying

\[
\sum_{a \in \Sigma} A_a X A_a^* = \Phi(X)
\]

for all \(X \in \mathcal{L}(\mathcal{X})\). Let \(\mathcal{Z} = \mathbb{C}^\Sigma\) and define \(B \in \mathcal{L}(\mathcal{X} \otimes \mathcal{Z}, \mathcal{Y})\) as

\[
B = \sum_{a \in \Sigma} A_a \otimes e_a^*.
\]

It holds that

\[
B(X \otimes 1_{\mathcal{Z}})B^* = \sum_{a,b \in \Sigma} (A_a \otimes e_a^*)(X \otimes 1_{\mathcal{Z}})(A_b^* \otimes e_b) = \sum_{a \in \Sigma} A_a X A_a^* = \Phi(X)
\]

for every \(X \in \mathcal{L}(\mathcal{X})\).

Under the assumption that \(\Phi(X) = B(X \otimes 1_{\mathcal{Z}})B^*\) for all \(X \in \mathcal{L}(\mathcal{X})\), a condition on \(B\) that is equivalent to \(\Phi\) preserving trace is

\[
\text{Tr}_\mathcal{Z}(B^*B) = 1_{\mathcal{X}}.
\]

One can verify that this is so by first observing that

\[
\text{Tr}(B(X \otimes 1_{\mathcal{Z}})B^*) = \text{Tr}(B^*B(X \otimes 1_{\mathcal{Z}})) = \text{Tr}(\text{Tr}_\mathcal{Z}(B^*B)X) = (\text{Tr}_\mathcal{Z}(B^*B), X),
\]

for every \(X \in \mathcal{L}(\mathcal{X})\). Therefore, the condition that \(\Phi\) preserves trace is equivalent to

\[
\langle 1_{\mathcal{X}}, X \rangle = \text{Tr}(X) = \text{Tr}(\Phi(X)) = (\text{Tr}_\mathcal{Z}(B^*B), X)
\]

for all \(X \in \mathcal{L}(\mathcal{X})\), which is equivalent to \(\text{Tr}_\mathcal{Z}(B^*B) = 1_{\mathcal{X}}\).

2. Let \(\Sigma\) be an alphabet, let \(\mathcal{X}\) be a complex Euclidean space, and let \(\phi : \text{Herm}(\mathcal{X}) \to \mathbb{R}^\Sigma\) be a linear function. Prove that these two statements are equivalent:

Statement 1. It holds that \(\phi(\rho) \in \mathcal{P}(\Sigma)\) for every density operator \(\rho \in \mathcal{D}(\mathcal{X})\).

Statement 2. There exists a measurement \(\mu : \Sigma \to \text{Pos}(\mathcal{X})\) such that

\[
(\phi(H))(a) = \langle \mu(a), H \rangle
\]

for every \(H \in \text{Herm}(\mathcal{X})\) and \(a \in \Sigma\).

A correct solution to this problem implies that the definition of how measurements work is simply a mathematical way of representing what measurements obviously need to be: linear functions that map quantum states to probability distributions of measurement outcomes.
Solution. Assume first that statement 1 holds.

For every linear function of the form $\psi : \text{Herm}(\mathcal{X}) \to \mathbb{R}$, there must exist a unique Hermitian operator $K \in \text{Herm}(\mathcal{X})$ such that

$$\psi(H) = \langle K, H \rangle$$

for all $H \in \text{Herm}(\mathcal{X})$. The existence of such an operator $K$ is established by taking

$$K = \sum_{a,b \in \Gamma} \phi(H_{a,b}) H_{a,b}$$

assuming $\mathcal{X} = \mathbb{C}^\Gamma$ and taking $\{H_{a,b} : a, b \in \Gamma\}$ to be any orthonormal basis for $\text{Herm}(\mathcal{X})$ (such as the basis described in equation (1.103) in the book). Uniqueness is straightforward: if $K_0$ and $K_1$ both satisfy the required property, then

$$\langle K_0, K_0 - K_1 \rangle = \phi(K_0 - K_1) = \langle K_1, K_0 - K_1 \rangle,$$

so

$$\|K_0 - K_1\|_2^2 = \langle K_0 - K_1, K_0 - K_1 \rangle = 0,$$

and therefore $K_0 = K_1$.

For a given linear function $\phi : \text{Herm}(\mathcal{X}) \to \mathbb{R}^\Sigma$, there must therefore exist uniquely determined Hermitian operators $\{K_a : a \in \Sigma\} \subset \text{Herm}(\mathcal{X})$ such that

$$\phi(H)(a) = \langle K_a, H \rangle$$

for all $H \in \text{Herm}(\mathcal{X})$ and all $a \in \Sigma$. For each $a \in \Sigma$ this implies that $\langle K_a, \rho \rangle \geq 0$ for every $\rho \in \text{D}(\mathcal{X})$, and therefore $K_a \in \text{Pos}(\mathcal{X})$. Moreover

$$\left\langle \sum_{a \in \Sigma} K_a, \rho \right\rangle = 1$$

for every $\rho \in \text{D}(\mathcal{X})$, which implies

$$\sum_{a \in \Sigma} K_a = 1_{\mathcal{X}}.$$  

Defining $\mu(a) = K_a$ for each $a \in \Sigma$ establishes that statement 2 holds.

The fact that statement 2 implies statement 1 is routine: for every $a \in \Sigma$ and every measurement operator $\mu(a)$, one has $\langle \mu(a), \rho \rangle \geq 0$, and moreover

$$\sum_{a \in \Sigma} \langle \mu(a), \rho \rangle = \langle 1_{\mathcal{X}}, \rho \rangle = 1,$$

implying that $\phi(\rho) \in \mathcal{P}(\Sigma)$ for every $\rho \in \text{D}(\mathcal{X})$.

3. Interesting structural properties of channels are sometimes reflected in a simple way by their Choi representations. This problem is concerned with one example along these lines.

Let $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ be complex Euclidean spaces, let $\Phi \in \mathcal{C}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ be a channel, and consider the following two statements.

\textit{Statement 1.} There exists a density operator $\rho \in \text{D}(\mathcal{Y})$ such that

$$\text{Tr}_\mathcal{Z}(J(\Phi)) = \rho \otimes 1_{\mathcal{X}}.$$  


There exists a complex Euclidean space $W$, a density operator $\sigma \in D(Y \otimes W)$, and a channel $\Psi \in C(W \otimes \mathcal{X}, Z)$ so that

$$\Phi(X) = (1_{L(Y)} \otimes \Psi)(\sigma \otimes X)$$

for all $X \in L(\mathcal{X})$.

It may be helpful to think about a channel $\Phi$ satisfying statement 2 as being one that can be implemented as the following figure suggests:

```
  \sigma
  \downarrow
  W
  \downarrow
  \mathcal{X}
  \Psi
  \downarrow
  Z
```

Prove that statements 1 and 2 are equivalent.

**Solution.** Assume first that statement 1 holds. Choose $W$ to be any complex Euclidean space with $\dim(W) \geq \text{rank}(\rho)$, and choose $u \in Y \otimes W$ to be any purification of $\rho$. (All we require of $W$ is that it is large enough to admit a purification of $\rho$.)

Now consider the vector

$$u \otimes \text{vec}(1_X) \in Y \otimes W \otimes \mathcal{X} \otimes \mathcal{X}.$$ 

If we trace out the middle two tensor factors, we obtain

$$\text{Tr}_{W \otimes X}(uu^* \otimes \text{vec}(1_X) \text{vec}(1_X)^*) = \rho \otimes 1_X = \text{Tr}_Z(J(\Phi)).$$

By Proposition 2.29, there must therefore exist a channel $\Psi \in C(W \otimes \mathcal{X}, Z)$ such that

$$(1_{L(Y)} \otimes \Psi \otimes 1_{L(\mathcal{X})})(uu^* \otimes \text{vec}(1_X) \text{vec}(1_X)^*) = J(\Phi).$$

Let us now check that the channel $\Psi$ satisfies the requirements of statement 2, assuming we take $\sigma = uu^*$. In the interest of clarity, let us define a new channel $\Xi \in C(\mathcal{X}, Y \otimes Z)$ as

$$\Xi(X) = (1_{L(Y)} \otimes \Psi)(uu^* \otimes X)$$

for every $X \in L(\mathcal{X})$, so that our goal is to prove that $\Xi = \Phi$. In fact, this task is essentially done already—for if we compute the Choi representation of $\Xi$ we obtain

$$J(\Xi) = (\Xi \otimes 1_{L(\mathcal{X})})(\text{vec}(1_X) \text{vec}(1_X)^*)$$

$$= (1_{L(Y)} \otimes \Psi \otimes 1_{L(\mathcal{X})})(uu^* \otimes \text{vec}(1_X) \text{vec}(1_X)^*) = J(\Phi),$$

and because Choi representations uniquely determine maps we find that $\Xi = \Phi$. It has been proved that statement 1 implies statement 2.

The fact that statement 2 implies statement 1 is fairly straightforward. Let us assume that $\mathcal{X} = \mathbb{C}^\Sigma$, so that

$$J(\Phi) = \sum_{a,b \in \Sigma} (1_{L(Y)} \otimes \Psi)(\sigma \otimes E_{a,b}) \otimes E_{a,b}.$$
Tracing out $Z$ yields
\[
\text{Tr}_Z(f(\Phi)) = \sum_{a,b \in \Sigma} \text{Tr}_{W \otimes X}(\sigma \otimes E_{a,b} \otimes E_{a,b}) = \sum_{a \in \Sigma} \text{Tr}_W(\sigma) \otimes E_{a,a} = \text{Tr}_W(\sigma) \otimes 1_X.
\]

Statement 1 therefore holds for $\rho = \text{Tr}_W(\sigma)$.

4. Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let $\Sigma$ be an alphabet, and let $\eta : \Sigma \to \text{Pos}(\mathcal{X})$ be an ensemble of states. Suppose further that $u \in \mathcal{X} \otimes \mathcal{Y}$ is a vector such that
\[
\text{Tr}_\mathcal{Y}(uu^*) = \sum_{a \in \Sigma} \eta(a).
\]

Prove that there exists a measurement $\mu : \Sigma \to \text{Pos}(\mathcal{Y})$ for which it holds that
\[
\eta(a) = \text{Tr}_\mathcal{Y}(1_X \otimes \mu(a) uu^*)
\]
for all $a \in \Sigma$.

One interpretation of this problem is as follows. Suppose Alice holds $X$ and Bob holds $Y$, and that the state of $(X, Y)$ is pure. If Bob performs a measurement on $Y$ and sends the outcome to Alice, the state of $X$ (together with Bob’s measurement outcome) will be described by some ensemble $\eta$. The fact you are asked to prove implies that if Bob selects his measurement appropriately, he can cause the state of $X$ to be described by any ensemble he chooses, so long as the original state purified the average state of that ensemble.

**Solution.** First, define $Z = C^\Sigma$, and let
\[
P = \sum_{a \in \Sigma} \eta(a) \otimes E_{a,a}.
\]

It holds that $P \in \text{Pos}(\mathcal{X} \otimes Z)$ and
\[
\text{Tr}_\mathcal{Y}(uu^*) = \sum_{a \in \Sigma} \eta(a) = \text{Tr}_Z(P).
\]

From Proposition 2.29 in the book, there must exist a channel $\Phi \in \mathcal{C}(\mathcal{Y}, Z)$ such that
\[
(1_L(\mathcal{X}) \otimes \Phi)(uu^*) = P.
\]

Now, we will use the channel $\Phi$ to define the measurement $\mu$ that the problem statement requires. Intuitively speaking, $\mu$ will correspond to the measurement on $\mathcal{Y}$ that is obtained by first applying $\Phi$ and then measuring in the standard basis of $Z$. More succinctly, we define
\[
\mu(a) = \Phi^*(E_{a,a})
\]
for each $a \in \Sigma$. It holds that
\[
\text{Tr}_\mathcal{Y}(1_X \otimes \mu(a) uu^*) = \text{Tr}_\mathcal{Y}(1_X \otimes \Phi^*(E_{a,a}) uu^*) = \text{Tr}_Z(1_X \otimes E_{a,a})(1_L(\mathcal{X}) \otimes \Phi)(uu^*) = \text{Tr}_Z(1_X \otimes E_{a,a} P) = \eta(a)
\]
for each $a \in \Sigma$, as required.

**Remark.** The fact that this problem establishes is sometimes known as the Hughston–Josza–Wootters theorem.