Assignment 3
Due: Monday, November 11

1. Assume \( X, Y, \) and \( Z \) are registers in some arbitrary given state \( \rho \in D(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}) \). Prove that each of the following inequalities is true:

   (a) \( H(X|Z) - H(X,Y|Z) \leq H(Y|Z) + 2H(Z) \).
   
   (b) \( I(X,Y : Z) \leq I(Y : X,Z) + 2H(X) \).
   
   (c) \( I(X : Z) - I(Y : Z) \leq I(X,Y : Z) \).

2. Let \( X, Y, \) and \( Z \) be registers, let \( \Sigma \) be an alphabet, let \( \eta : \Sigma \rightarrow \text{Pos}(\mathcal{X}) \) be an ensemble of states of \( X \), and let \( \{\sigma_a : a \in \Sigma\} \subseteq D(\mathcal{Y} \otimes \mathcal{Z}) \) be an arbitrary collection of states of \( (Y,Z) \). Prove that, with respect to the state

\[
\rho = \sum_{a \in \Sigma} \eta(a) \otimes \sigma_a \in D(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}),
\]

it is the case that

\[
I(X, Y : Z) \leq I(Y : X,Z) + \chi(\eta).
\]

Notice that, because \( \chi(\eta) \leq H(X) \), a correct solution to this problem demonstrates that the inequality in part (b) of problem 1 (which is, in fact, tight for some states) can be improved for states of the form described above. In words, these are simply states for which \( X \) and \( (Y,Z) \) are not entangled.

3. Let \( \mathcal{X} \) be a complex Euclidean space of dimension \( n \) and let \( \Phi \in \mathcal{C}(\mathcal{X}) \) be a unital channel. Following our usual convention for singular-value decompositions, let \( s_1(X) \geq \cdots \geq s_n(X) \) denote the singular values of a given operator \( X \in \mathcal{L}(\mathcal{X}) \), ordered from largest to smallest, and taking \( s_k(X) = 0 \) when \( k > \text{rank}(X) \).

Prove that, for every operator \( X \in \mathcal{L}(\mathcal{X}) \), it holds that

\[
s_1(X) + \cdots + s_m(X) \geq s_1(\Phi(X)) + \cdots + s_m(\Phi(X))
\]

for every \( m \in \{1, \ldots, n\} \).

4. For every positive integer \( n \geq 2 \), define a unital channel \( \Phi_n \in \mathcal{C}(\mathcal{C}^n) \) as

\[
\Phi_n(X) = \frac{\text{Tr}(X)1_n - X^T}{n-1}
\]

for every \( X \in \mathcal{L}(\mathcal{C}^n) \), where \( 1_n \) denotes the identity operator on \( \mathcal{C}^n \). Prove that \( \Phi_n \) is not mixed-unitary when \( n \) is odd.

Hint: This is proved in the book in Example 4.3 for the case that \( n = 3 \), but this proof will not extend to larger odd values of \( n \). Instead, for any fixed choice of \( n \geq 2 \), think about an arbitrary Kraus representation

\[
\Phi_n(X) = \sum_{a \in \Sigma} A_a X A_a^*
\]

of \( \Phi_n \). Try to identify a property that every Kraus operator \( A_a \) must have, and then prove that no nonzero scalar multiple of a unitary operator can have this property when \( n \) is odd.