Assignment 1
Due: Friday, October 4

There are four problems on this assignment. In addition to solving the problems at a technical level, think about how to interpret the facts you are asked to establish in problems 2, 3, and 4 at an intuitive level. (Problem 1 is not meant to have an interesting interpretation; it is just three independent mini-problems meant to get you used to some of the basic machinery we will use.)

1. (a) Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $A \in L(\mathcal{Y}, \mathcal{X})$ be an operator. Prove that there exists a complex Euclidean space $\mathcal{Z}$ along with vectors $u \in \mathcal{X} \otimes \mathcal{Z}$ and $v \in \mathcal{Z} \otimes \mathcal{Y}$ such that

$$A = (1_\mathcal{X} \otimes v^*)(u \otimes 1_\mathcal{Y}).$$

(b) Let $\mathcal{X}$ be a complex Euclidean space of dimension $n$, let $m$ be a positive integer, and let $\{A_1, \ldots, A_m\} \subset L(\mathcal{X})$ be an arbitrary collection of operators acting on $\mathcal{X}$. Prove that

$$\sum_{k=1}^{m} \text{vec}(A_k) \text{vec}(A_k)^* = \frac{1_\mathcal{X} \otimes 1_\mathcal{X}}{n}$$

if and only if

$$\sum_{k=1}^{m} A_k \otimes A_k = \frac{\text{vec}(1_\mathcal{X}) \text{vec}(1_\mathcal{X})^*}{n}.$$

(c) Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let $\Phi \in T(\mathcal{X}, \mathcal{Y})$ be a positive (but not necessarily completely positive) map, and let $\Delta \in C(\mathcal{Y})$ denote the completely dephasing channel with respect to the space $\mathcal{Y}$. Prove that $\Delta \Phi$ is completely positive.

2. Let $\Sigma$ be an alphabet, let $\mathcal{X}$ be a complex Euclidean space, and let $\phi : \text{Herm}(\mathcal{X}) \to \mathbb{R}^\Sigma$ be a linear function. Prove that $\phi(\rho) \in P(\Sigma)$ for every density operator $\rho \in D(\mathcal{X})$ if and only if there exists a measurement $\mu : \Sigma \to \text{Pos}(\mathcal{X})$ such that

$$(\phi(H))(a) = \langle \mu(a), H \rangle$$

for every $H \in \text{Herm}(\mathcal{X})$ and $a \in \Sigma$.

3. (a) Let $\{P_a : a \in \Sigma\} \subset \text{Pos}(\mathcal{X})$ be any collection of positive semidefinite operators, for $\mathcal{X}$ being a complex Euclidean space and $\Sigma$ being an alphabet, and define

$$Q = \sum_{a \in \Sigma} P_a.$$

Prove that there exists a measurement $\mu : \Sigma \to \text{Pos}(\mathcal{X})$ such that

$$P_a = \sqrt{Q} \mu(a) \sqrt{Q}$$

for all $a \in \Sigma$. 
(b) Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let $\Sigma$ be an alphabet, let $\sigma \in D(\mathcal{X})$ be a density operator, and let $\{P_a : a \in \Sigma\} \subset \text{Pos}(\mathcal{Y} \otimes \mathcal{X})$ be a collection of positive semidefinite operators such that

$$\sum_{a \in \Sigma} P_a = 1_{\mathcal{Y} \otimes \mathcal{X}}.$$ 

Prove that there exists a density operator $\rho \in D(\mathcal{X} \otimes \mathcal{X})$ and a measurement $\mu : \Sigma \to \text{Pos}(\mathcal{Y} \otimes \mathcal{X})$ for which the equation

$$\langle P_a, J(\Phi) \rangle = \langle \mu(a), (\Phi \otimes 1_{L(\mathcal{X})})(\rho) \rangle$$

is true for every choice of a channel $\Phi \in C(\mathcal{X}, \mathcal{Y})$.

4. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be complex Euclidean spaces and let $\Phi \in C(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ be a channel. Prove that

$$\text{Tr}_\mathcal{Z}(J(\Phi)) = \rho \otimes 1_{\mathcal{X}}$$

for some state $\rho \in D(\mathcal{Y})$ if and only if there exists a complex Euclidean space $\mathcal{W}$, a state $\sigma \in D(\mathcal{Y} \otimes \mathcal{W})$, and a channel $\Psi \in C(\mathcal{W} \otimes \mathcal{X}, \mathcal{Z})$ so that

$$\Phi(X) = (1_{L(\mathcal{Y})} \otimes \Psi)(\sigma \otimes X)$$

for all $X \in L(\mathcal{X})$.

Note that the existence of a complex Euclidean space $\mathcal{W}$, a state $\sigma \in D(\mathcal{Y} \otimes \mathcal{W})$, and a channel $\Psi \in C(\mathcal{W} \otimes \mathcal{X}, \mathcal{Z})$ such that

$$\Phi(X) = (1_{L(\mathcal{Y})} \otimes \Psi)(\sigma \otimes X)$$

for all $X \in L(\mathcal{X})$ means that $\Phi$ can be implemented as the following figure suggests: