Assignment 1
Due: Thursday, October 5 at 4:00pm

1. This problem is not intended to reveal anything profound—it is just meant to give you some practice in working with vectors, operators, and such.

(a) Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $A \in \mathbb{L}(\mathcal{Y}, \mathcal{X})$ be any nonzero operator. Prove that there exists a complex Euclidean space $\mathcal{Z}$ along with vectors $u \in \mathcal{X} \otimes \mathcal{Z}$ and $v \in \mathcal{Z} \otimes \mathcal{Y}$ such that
$$A = (1_\mathcal{X} \otimes v^*) (u \otimes 1_\mathcal{Y}).$$

What is the minimum possible dimension of $\mathcal{Z}$ that is required to write a given $A$ in this way? (Unless stated otherwise, your answers should always be supported by a proof or argument of some form—so in this case you should not only give an expression for the minimum dimension of $\mathcal{Z}$, but also a proof showing that your expression is indeed the minimum possible dimension.)

(b) Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $\Phi \in \mathbb{CP}(\mathcal{X}, \mathcal{Y})$ be a completely positive map. Prove that there exists an operator $B \in \mathbb{L}(\mathcal{X} \otimes \mathcal{Z}, \mathcal{Y})$, for some choice of a complex Euclidean space $\mathcal{Z}$, such that
$$\Phi(X) = B(X \otimes 1_\mathcal{Z}) B^*$$
for all $X \in \mathbb{L}(\mathcal{X})$. Identify a condition on the operator $B$ that is equivalent to $\Phi$ preserving trace.

2. Let $\Sigma$ be an alphabet, let $\mathcal{X}$ be a complex Euclidean space, and let $\phi : \text{Herm}(\mathcal{X}) \to \mathbb{R}^\Sigma$ be a linear function. Prove that these two statements are equivalent:

Statement 1. It holds that $\phi(\rho) \in \mathcal{P}(\Sigma)$ for every density operator $\rho \in \mathcal{D}(\mathcal{X})$.

Statement 2. There exists a measurement $\mu : \Sigma \to \text{Pos}(\mathcal{X})$ such that
$$(\phi(H))(a) = \langle \mu(a), H \rangle$$
for every $H \in \text{Herm}(\mathcal{X})$ and $a \in \Sigma$.

A correct solution to this problem implies that the definition of how measurements work is simply a mathematical way of representing what measurements obviously need to be: linear functions that map quantum states to probability distributions of measurement outcomes.

3. Interesting structural properties of channels are sometimes reflected in a simple way by their Choi representations. This problem is concerned with one example along these lines.

Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be complex Euclidean spaces, let $\Phi \in \mathbb{C}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ be a channel, and consider the following two statements.

Statement 1. There exists a density operator $\rho \in \mathcal{D}(\mathcal{Y})$ such that
$$\text{Tr}_\mathcal{Z}(J(\Phi)) = \rho \otimes 1_\mathcal{X}.$$
Statement 2. There exists a complex Euclidean space $W$, a density operator $\sigma \in D(\mathcal{Y} \otimes W)$, and a channel $\Psi \in \mathcal{C}(W \otimes \mathcal{X}, \mathcal{Z})$ so that

$$\Phi(X) = (\mathbb{1}_{L(\mathcal{Y})} \otimes \Psi)(\sigma \otimes X)$$

for all $X \in L(\mathcal{X})$.

It may be helpful to think about a channel $\Phi$ satisfying statement 2 as being one that can be implemented as the following figure suggests:

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  σ
   \arrow{Y

 W
   \arrow{X

 Ψ
   \arrow{Z
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Prove that statements 1 and 2 are equivalent.

4. Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let $\Sigma$ be an alphabet, and let $\eta : \Sigma \to \text{Pos}(\mathcal{X})$ be an ensemble of states. Suppose further that $u \in \mathcal{X} \otimes \mathcal{Y}$ is a vector such that

$$\text{Tr}_\mathcal{Y}(uu^*) = \sum_{a \in \Sigma} \eta(a).$$

Prove that there exists a measurement $\mu : \Sigma \to \text{Pos}(\mathcal{Y})$ for which it holds that

$$\eta(a) = \text{Tr}_\mathcal{Y}((\mathbb{1}_\mathcal{X} \otimes \mu(a))uu^*)$$

for all $a \in \Sigma$.

One interpretation of this problem is as follows. Suppose Alice holds a register $X$ and Bob holds $Y$, and that the state of the pair $(X, Y)$ is pure. If Bob performs a measurement on $Y$ and sends the outcome to Alice, the state of $X$ (together with Bob’s measurement outcome) will be described by some ensemble $\eta$. The fact you are asked to prove implies that if Bob selects his measurement appropriately, he can cause the state of $X$ to be described by any ensemble he chooses, so long as the original state purifies the average state of that ensemble.