Assignment 1

Due: Thursday, October 5 at 4:00pm

1. This problem is not intended to reveal anything profound—it is just meant to give you some practice in working with vectors, operators, and such.

   (a) Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $A \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$ be any nonzero operator. Prove that there exists a complex Euclidean space $\mathcal{Z}$ along with vectors $u \in \mathcal{X} \otimes \mathcal{Z}$ and $v \in \mathcal{Z} \otimes \mathcal{Y}$ such that
   $$A = \big(\mathbb{1}_{\mathcal{X}} \otimes v^*\big)\big(u \otimes \mathbb{1}_{\mathcal{Y}}\big).$$
   What is the minimum possible dimension of $\mathcal{Z}$ that is required to write a given A in this way? (Unless stated otherwise, your answers should always be supported by a proof or argument of some form—so in this case you should not only give an expression for the minimum dimension of $\mathcal{Z}$, but also a proof showing that your expression is indeed the minimum possible dimension.)

   (b) Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces and let $\Phi \in \mathrm{CP}(\mathcal{X}, \mathcal{Y})$ be a completely positive map. Prove that there exists an operator $B \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Z}, \mathcal{Y})$, for some choice of a complex Euclidean space $\mathcal{Z}$, such that
   $$\Phi(X) = B(X \otimes \mathbb{1}_{\mathcal{Z}})B^*$$
   for all $X \in \mathrm{L}(\mathcal{X})$. Identify a condition on the operator B that is equivalent to Φ preserving trace.

2. Let Σ be an alphabet, let $\mathcal{X}$ be a complex Euclidean space, and let $\phi : \mathrm{Herm}(\mathcal{X}) \to \mathbb{R}^{\Sigma}$ be a linear function. Prove that these two statements are equivalent:

   Statement 1. It holds that $\phi(\rho) \in \mathcal{P}(\Sigma)$ for every density operator $\rho \in \mathrm{D}(\mathcal{X})$.

   Statement 2. There exists a measurement $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ such that
   $$(\phi(H))(a) = \langle \mu(a), H \rangle$$
   for every $H \in \mathrm{Herm}(\mathcal{X})$ and $a \in \Sigma$.

   A correct solution to this problem implies that the definition of how measurements work is simply a mathematical way of representing what measurements obviously need to be: linear functions that map quantum states to probability distributions of measurement outcomes.

3. Interesting structural properties of channels are sometimes reflected in a simple way by their Choi representations. This problem is concerned with one example along these lines.

   Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be complex Euclidean spaces, let $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$ be a channel, and consider the following two statements.

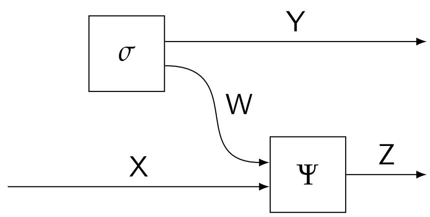
   Statement 1. There exists a density operator $\rho \in \mathrm{D}(\mathcal{Y})$ such that
   $$\mathrm{Tr}_{\mathcal{Z}}(J(\Phi)) = \rho \otimes \mathbb{1}_{\mathcal{X}}.$$

Statement 2. There exists a complex Euclidean space $\mathcal{W}$, a density operator $\sigma \in \mathrm{D}(\mathcal{Y} \otimes \mathcal{W})$, and a channel $\Psi \in \mathrm{C}(\mathcal{W} \otimes \mathcal{X}, \mathcal{Z})$ so that

$$\Phi(X) = \big(\mathbb{1}_{\mathrm{L}(\mathcal{Y})} \otimes \Psi\big)(\sigma \otimes X)$$

for all $X \in \mathrm{L}(\mathcal{X})$.

It may be helpful to think about a channel Φ satisfying statement 2 as being one that can be implemented as the following figure suggests:



Prove that statements 1 and 2 are equivalent.

4. Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Euclidean spaces, let Σ be an alphabet, and let $\eta : \Sigma \rightarrow \mathrm{Pos}(\mathcal{X})$ be an ensemble of states. Suppose further that $u \in \mathcal{X} \otimes \mathcal{Y}$ is a vector such that

$$\mathrm{Tr}_{\mathcal{Y}}(uu^*) = \sum_{a \in \Sigma} \eta(a).$$

Prove that there exists a measurement $\mu : \Sigma \rightarrow \mathrm{Pos}(\mathcal{Y})$ for which it holds that

$$\eta(a) = \mathrm{Tr}_{\mathcal{Y}}\big((\mathbb{1}_{\mathcal{X}} \otimes \mu(a))uu^*\big)$$

for all $a \in \Sigma$.

One interpretation of this problem is as follows. Suppose Alice holds a register X and Bob holds Y, and that the state of the pair (X, Y) is pure. If Bob performs a measurement on Y and sends the outcome to Alice, the state of X (together with Bob's measurement outcome) will be described by some ensemble η. The fact you are asked to prove implies that if Bob selects his measurement appropriately, he can cause the state of X to be described by *any ensemble he chooses*, so long as the original state purifies the average state of that ensemble.