Assignment 4
Due: Monday, July 24 at 4:00pm

1. [5 points] Define a language

\[ \text{ALL} = \{ \langle M \rangle : M \text{ is a DTM that accepts every string over its input alphabet} \} \]

Prove that \( \text{ALL} \) is not Turing recognizable.

(If you are having trouble proving that \( \text{ALL} \) is not Turing recognizable, then try proving \( \text{ALL} \) is not decidable—a correct solution demonstrating this is worth 3 points.)

2. [5 points] Define two languages \( A, B \subseteq \Sigma^* \) as follows:

\[ A = \{ \langle M \rangle : M \text{ is a DTM that rejects } \langle M \rangle \}, \]
\[ B = \{ \langle M \rangle : M \text{ is a DTM that accepts } \langle M \rangle \}. \]

Prove that there does not exist a decidable language \( C \subseteq \Sigma^* \) such that \( A \subseteq C \) and \( B \subseteq \overline{C} \).

3. [4 points] Define

\[ A_{\text{DTM}} = \{ \langle M, w \rangle : M \text{ is a DTM and } w \in L(M) \} \]

as usual, and let \( \Sigma \) be the alphabet over which this language is defined (i.e., \( \Sigma \) is the alphabet that is used for the encoding scheme mentioned in the definition of \( A_{\text{DTM}} \)). Let \( B \subseteq \Sigma^* \) be any Turing-recognizable language over the same alphabet \( \Sigma \). Prove that \( B \leq_m A_{\text{DTM}} \).

4. [6 points] Let \( \Sigma = \{0, 1\} \), let \( A \subseteq \Sigma^* \) be a language contained in \( \text{DTIME}(4^n) \), and define

\[ B = \{ xx : x \in A \}. \]

(a) Give a high-level argument supporting the claim that \( B \in \text{DTIME}(2^n) \).
(b) Prove that \( A \leq_p B \).
(c) Prove that \( \text{DTIME}(2^n) \neq \text{NP} \).

You may use the following fact (which is stated in the Lecture 19 notes without proof) without proving it: if \( C, D \subseteq \{0, 1\}^* \) are languages, \( D \in \text{NP} \), and \( C \leq_m D \), then \( C \in \text{NP} \).

5. [4 points] Let \( \Sigma = \{0, 1\} \) and let \( A, B \subseteq \Sigma^* \) be languages. Prove that if \( A \) is NP-hard, \( B \) is in \( \text{P} \), \( A \cap B = \emptyset \), and \( A \cup B \neq \Sigma^* \), then \( A \cup B \) is NP-hard.

6. [1 point] For each of the questions above, list the full name of each of your 360 classmates with whom you worked on that question. (If you didn’t work with anyone, that is fine: just indicate that you worked alone.)