Assignment 3
Due: Friday, June 30 at 4:00pm

Several of the languages described on this assignment involve encodings of certain objects. (For example, \( \langle M \rangle \) means the encoding of a DTM in questions 2 and 4.) In every such case, you should assume that a reasonable encoding scheme over some alphabet has been fixed. While the actual languages do of course depend on the encoding scheme that has been selected, the properties of these languages being or not being decidable or Turing recognizable do not.

1. [9 points] For this problem you will be asked to prove that three languages are decidable. You may do this by giving high-level descriptions of Turing machines that decide these languages. Please try to keep your Turing machines as simple as possible—making use of Turing machines described in the lecture notes as subroutines may help in this regard.

   (a) Prove that the following language is decidable:
   \[
   \{ \langle D, k \rangle : D \text{ is a DFA, } k \in \mathbb{N}, \text{ and } |L(D)| \geq k \}.
   \]

   (b) Prove that the following language is decidable:
   \[
   \left\{ \langle D \rangle : D \text{ is a DFA with alphabet } \{0, 1\}, \text{ and every string } w \text{ accepted by } D \text{ satisfies } |w|_0 \neq |w|_1 \right\}.
   \]

   (c) Prove that the following language is decidable:
   \[
   \{ \langle G \rangle : G \text{ is a CFG and } L(G) \text{ is finite} \}.
   \]

2. [4 points] Define a language
   \[
   A = \{ \langle M \rangle : M \text{ is a DTM that halts on at least one input string} \}.
   \]
   Prove that \( A \) is Turing-recognizable.

3. [5 points] Let \( \Sigma \) be an alphabet, and suppose that \( A, B \subseteq \Sigma^* \) are Turing-recognizable languages for which both \( A \cap B \) and \( A \cup B \) are decidable. Prove that \( A \) is decidable.

4. [6 points] Prove that the following language is not Turing recognizable:
   \[
   \{ \langle M \rangle : M \text{ is a DTM such that } |L(M)| = 1 \}.
   \]
   Hint: try to come up with a DTM that recognizes \( \text{DIAG} \) using a DTM that recognizes the language above as a subroutine.

5. [1 point] For each of the questions above, list the full name of each of your 360 classmates with whom you worked on that question. (If you didn’t work with anyone, that is fine: just indicate that you worked alone.)