Lecture 3: Superdense coding, quantum circuits, and partial measurements

January 24, 2006

Superdense Coding

Imagine a situation where two people (named Alice and Bob) are in different parts of the world. Alice has two bits: a and b. She would like to communicate these two bits to Bob by sending him just a single qubit. It turns out that there is no way they can can accomplish this task without additional resources. This is not obvious, but it is true—Alice cannot encode two classical bits into a single qubit in any way that would give Bob more than just one bit of information about the pair (a, b).

However, let us imaging that a long time ago, before Alice even knew what a and b are, that the two of them prepared two qubits A and B in the superposition

$$\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle .$$

Alice took the qubit A and Bob took the qubit B. We say that Alice and Bob *share an e-bit* or *share an EPR pair* in this situation. It is natural to view entanglement as a resource as we will see; and when Alice and Bob each have a qubit and the two of them are in the state above, it is natural to imagine that Alice and Bob share one unit (i.e., one entangled bit, or e-bit for short) of entanglement.

Given the additional resource of a shared e-bit of entanglement, Alice will be able to transmit both a and b to Bob by sending just one qubit. Here is how:

Superdense coding protocol

1. If $a = 1$, Alice applies the unitary transformation

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   to the qubit A. (If $a = 0$ she does not.)

2. If $b = 1$, Alice applies the unitary transformation

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   to the qubit A. (If $b = 0$ she does not.)

3. Alice sends the qubit A to Bob. (This is the only qubit that is sent during the protocol.)

4. Bob applies a controlled-NOT operation to the pair (A, B), where A is the control and B is the target. The corresponding unitary matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

5. Bob applies a Hadamard transform to A.
6. Bob measures both qubits A and B. The output will be (a, b) with certainty.

To see that the protocol works correctly, we simply compute the state of (A, B) after the steps involving transformations:

ab	state after step 1	state after step 2	state after step 4	state after step 5
00	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\left(\frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle\right)\|0\rangle$	$\|00\rangle$
01	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|10\rangle + \frac{1}{\sqrt{2}}\|01\rangle$	$\left(\frac{1}{\sqrt{2}}\|1\rangle + \frac{1}{\sqrt{2}}\|0\rangle\right)\|1\rangle$	$\|01\rangle$
10	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\left(\frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle\right)\|0\rangle$	$\|10\rangle$
11	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|10\rangle - \frac{1}{\sqrt{2}}\|01\rangle$	$\left(\frac{1}{\sqrt{2}}\|1\rangle - \frac{1}{\sqrt{2}}\|0\rangle\right)\|1\rangle$	$-\|11\rangle$

When Bob measures at the end of the protocol, it is clear that he sees ab as required.

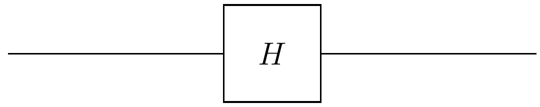
Quantum circuits

The previous discussion of superdense coding used a fairly inefficient way to describe the protocol; basically just plain English along with a few equations. We will need a more efficient way to describe sequences of quantum operations and measurements than this, particularly for when the complexity of the algorithms and protocols has increased. The most common way of doing this is to use *quantum circuit diagrams*, which is the way we will use. The basic idea is as follows:

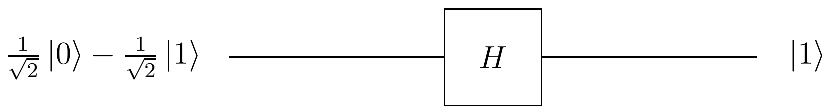
- Time goes from left to right.
- Horizontal lines represent qubits.
- Operations and measurements are represented by various symbols.

It is easiest to describe the model more specifically by using examples.

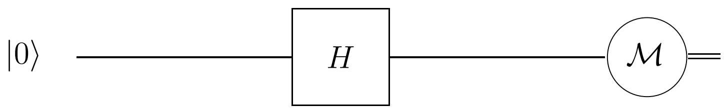
Example 1. The following diagram represents a Hadamard transform applied to a single qubit:



If the input is $|\psi\rangle$, the output is $H|\psi\rangle$. Sometimes when we want to explain what happens for a particular input, we label the inputs and outputs with superpositions, such as:

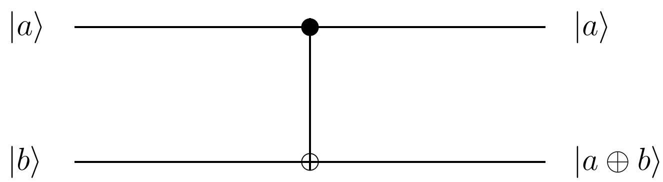


Example 2. Measurements are indicated by circles (or ovals) with the letter $\mathcal{M}$ inside. For example:

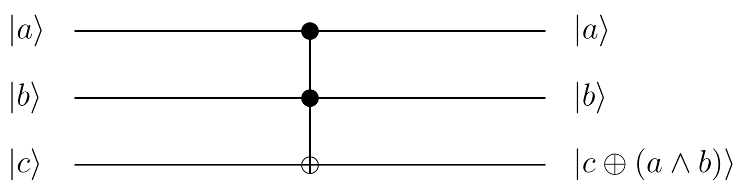


The result of the measurement is a classical value, and sometimes (as in the above diagram) we draw double lines to indicate classical bits. In this case the outcome is a uniformly distributed bit.

Example 3. Multiple-qubit gates are generally represented by rectangles, or have their own special representation. For instance, this is a *controlled-not* operation:

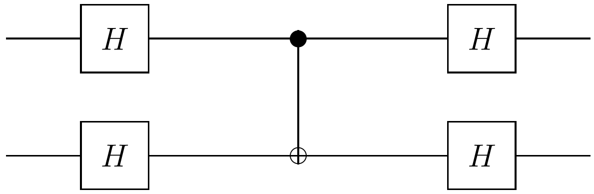


Here the action is indicated for classical inputs (meaning $a, b \in \{0, 1\}$). Along similar lines, here is a *controlled-controlled-not* operation, better known as a *Toffoli gate*:

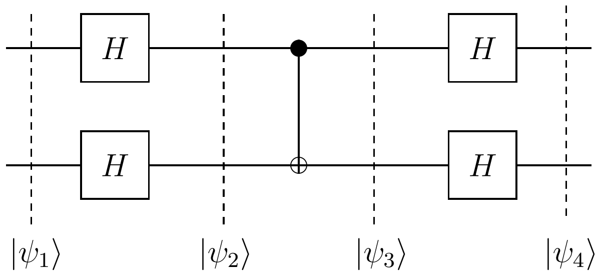


Here the action is described for each choice of $a, b, c \in \{0, 1\}$.

Example 4. Usually we have multiple operations, such as in this circuit:



It is not immediately obvious what this circuit does, so let us consider the superposition that would be obtained at various times for a selection of inputs:



Suppose first that $|\psi_1\rangle = |00\rangle$. Then

$$|\psi_2\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \;=\; \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle\,,$$

$$|\psi_3\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle + \frac{1}{2}|10\rangle \;=\; \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$|\psi_4\rangle = |00\rangle\,.$$

Next suppose that $|\psi_1\rangle = |01\rangle$. Then

$$|\psi_2\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \;=\; \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\,,$$

$$|\psi_3\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle - \frac{1}{2}|10\rangle \;=\; \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$|\psi_4\rangle = |11\rangle\,.$$

Next suppose that $|\psi_1\rangle = |10\rangle$. Then

$$|\psi_2\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \;=\; \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\,,$$

$$|\psi_3\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|11\rangle - \frac{1}{2}|10\rangle \;=\; \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$|\psi_4\rangle = |10\rangle\,.$$

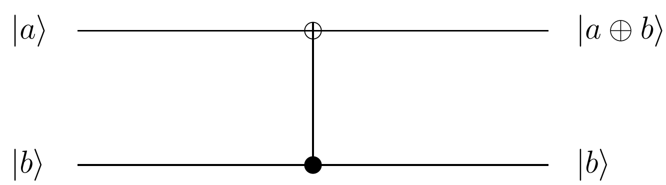
Finally, suppose that $|\psi_1\rangle = |11\rangle$. Then

$$|\psi_2\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle,$$

$$|\psi_3\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|11\rangle + \frac{1}{2}|10\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

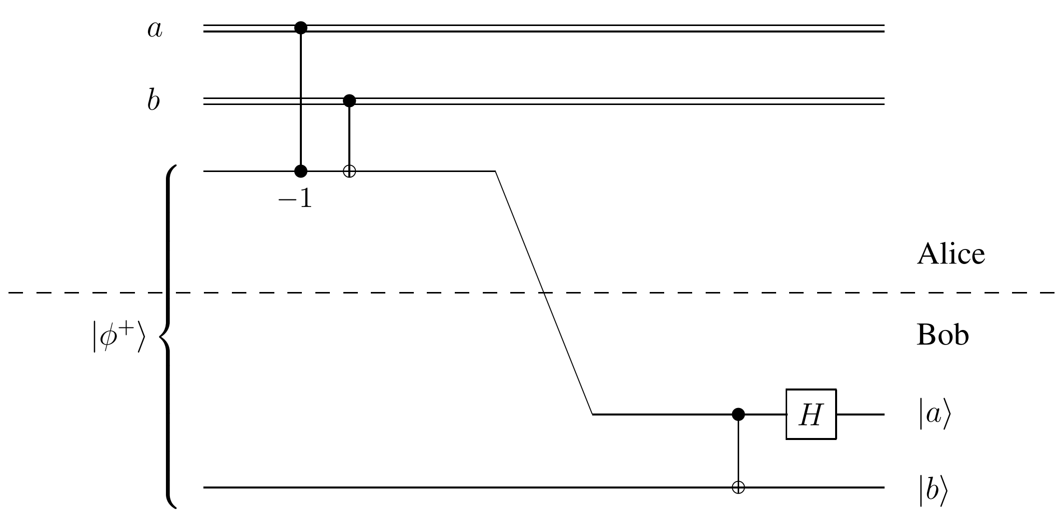
$$|\psi_4\rangle = |01\rangle.$$

It turns out that the circuit is equivalent to this gate:



This may seem counter to your intuition because the roles of control and target effectively switched in the controlled-not gate because of the Hadamard transforms. Don't let that bother you—instead take it as an example of how your intuition can be wrong.

Example 5. A quantum circuit diagram for superdense coding.



In this picture the first gate represents a controlled-σ_z gate, whose corresponding unitary matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It may seem strange that the first two gates are acting on one classical bit and one qubit. Generally we will only do this when the classical bit is a control bit for some operation. All that this means is that if the (classical) control bit is 1 then perform the corresponding operation on the qubit (σ_z or σ_x in the present case), otherwise do nothing.

Partial measurements

Suppose we have a system consisting of two or more qubits and we only measure one of them. For example, suppose the qubits are X and Y, and these qubits are in the state

$$|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle .$$

We know what the distribution of measurement outcomes would be if we measured both qubits:

$$\Pr[\text{outcome is } 00] = \frac{1}{4} \qquad \Pr[\text{outcome is } 01] = 0$$

$$\Pr[\text{outcome is } 10] = \frac{1}{4} \qquad \Pr[\text{outcome is } 11] = \frac{1}{2}.$$

The probability that a measurement of the first qubit, for instance, results in outcome 0 should therefore be $1/4+0 = 1/4$ and the probability of outcome 1 should be $1/4+1/2 = 3/4$. Intuitively, this should be the case regardless of whether or not the second qubit was actually measured. Indeed this is the case.

However, what is different between the case where the second qubit is measured and the case where it is not is the superposition of the system after the measurement (or measurements). If both qubits are measured, the superposition will be one of $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$, depending on the measurement outcome. If only the first qubit is measured, however, this may not be the case.

I'll explain how you determine the state of the system after the measurement of the first qubit for the example above, and the method for a general case should be clear. We begin by considering the possible outcome 0 for the measurement. We consider the state

$$|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

and cross off all of the terms in the sum that are inconsistent with measuring a 0 for the first qubit:

$$|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}\cancel{|10\rangle} + \frac{1}{\sqrt{2}}\cancel{|11\rangle}$$

which leaves

$$\frac{1}{2}|00\rangle .$$

The probability of measuring 0 is the norm-squared of this vector,

$$\Pr[\text{measurement outcome is } 0] = \left\| \frac{1}{2}|00\rangle \right\|^2 = \frac{1}{4},$$

and the state of the two qubits after the measurement conditioned on the measurement outcome being 0 is the vector itself *renormalized*:

$$\frac{\frac{1}{2}|00\rangle}{\left\| \frac{1}{2}|00\rangle \right\|} = |00\rangle .$$

The possible measurement outcome 1 is handled similarly. We cross off the terms inconsistent with measuring a 1:

$$\frac{1}{2}\xcancel{|00\rangle} - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

which leaves

$$-\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle .$$

The probability of measuring a 1 is then the norm-squared of this vector,

$$\Pr[\text{measurement outcome is 1}] = \left\| -\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle \right\|^2 = \left| -\frac{i}{2} \right|^2 + \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{3}{4},$$

and conditioned on measuring a 1 the state of the two qubits becomes

$$\frac{-\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle}{\left\| -\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle \right\|} = -\frac{i}{\sqrt{3}}|10\rangle + \sqrt{\frac{2}{3}}|11\rangle .$$

A completely equivalent way of performing this calculation is to begin by writing

$$|\psi\rangle = |0\rangle|\phi_0\rangle + |1\rangle|\phi_1\rangle$$

for some choice of $|\phi_0\rangle$ and $|\phi_1\rangle$. (There will always be a unique choice for each that works.) In this case we have

$$|\psi\rangle = |0\rangle\left(\frac{1}{2}|0\rangle\right) + |1\rangle\left(-\frac{i}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right),$$

so $|\phi_0\rangle = \frac{1}{2}|0\rangle$ and $|\phi_1\rangle = -\frac{i}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$. The probabilities for the two measurement outcomes are as follows:

$$\Pr[\text{outcome is 0}] = \| \, |\phi_0\rangle \, \|^2 = \left| \frac{1}{2} \right|^2 = \frac{1}{4}$$

$$\Pr[\text{outcome is 1}] = \| \, |\phi_1\rangle \, \|^2 = \left| -\frac{i}{2} \right|^2 + \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{3}{4}.$$

Conditioned on the measurement outcome, the state of the two qubits is as follows:

$$\text{measurement outcome is 0} \;\Rightarrow\; \text{state becomes } \frac{1}{\| \, |\phi_0\rangle \, \|}|0\rangle|\phi_0\rangle = |00\rangle$$

$$\text{measurement outcome is 1} \;\Rightarrow\; \text{state becomes } \frac{1}{\| \, |\phi_1\rangle \, \|}|1\rangle|\phi_1\rangle = \frac{-i}{\sqrt{3}}|10\rangle + \sqrt{\frac{2}{3}}|11\rangle .$$

The same method works for more than 2 qubits as well.

Example 6. Suppose 3 qubits are in the superposition

$$|\psi\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|100\rangle + \frac{1}{2}|101\rangle - \frac{1}{2}|111\rangle$$

and the **third** qubit is measured. What are the probabilities of the two possible measurement outcomes and what are the resulting superpositions of the three qubits for each case?

To determine the answer, we write

$$|\psi\rangle = \left(\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right)|0\rangle + \left(\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right)|1\rangle .$$

The probability that the measurement outcome is 0 is

$$\left\|\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right\|^2 = \frac{1}{2}$$

and in this case the resulting superposition is

$$\sqrt{2}\left(\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right)|0\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|100\rangle .$$

The probability that the measurement outcome is 1 is

$$\left\|\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right\|^2 = \frac{1}{2}$$

and in this case the resulting superposition is

$$\sqrt{2}\left(\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right)|1\rangle = \frac{1}{\sqrt{2}}|101\rangle - \frac{1}{\sqrt{2}}|111\rangle .$$