CS 137

Programming Principles

Chapter 7

Floating Numbers, Math Library, Polynomials, and Root Finding

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Floating Point Numbers

Let us explore how to work with decimal numbers. You can write decimal numbers using decimal points such as `3.125`, or in scientific notation such as `-2.61202 × 10^30`, where `-2.61202` is the precision and `30` is the range. In C we will use two data types:

Type	Size	Precision	Range (approx.)
float	4 bytes	~7 significant digits	$\pm 3.4 \times 10^{38}$
double	8 bytes	~15 significant digits	$\pm 1.7 \times 10^{308}$

> You will almost always use the type double.

printf and Decimal Numbers

We can display floating-point numbers using `printf` in many different ways. The general format is `%± m.pX` where:

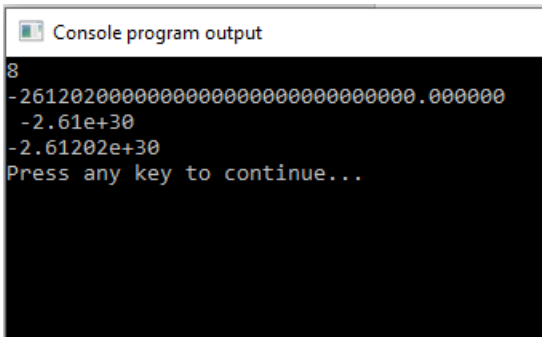
- `±` is the right or left justification (positive = right, negative = left).
- `m` is the minimum field width — how many spaces to reserve for the number.
- `p` is the precision (meaning depends on X).
- `X` is a letter specifying the type:

- `%d` — decimal integer. Precision = minimum number of digits to display (default 1).
- `%e` — float in exponential form. Precision = digits after the decimal point (default 6).
- `%f` — float in fixed decimal format. Precision same as %e.
- `%g` — float in either %e or %f form depending on the number's size. Precision = maximum number of significant digits. Most versatile; useful when the magnitude of the number is unknown.

Example

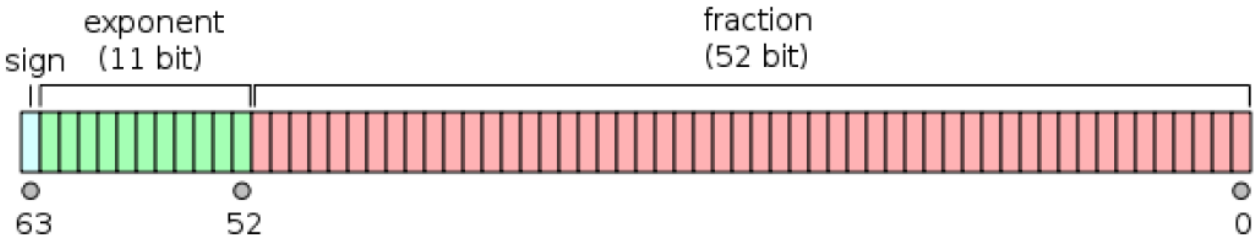
```
#include <stdio.h>

int main(void)
{
    double x = -2.61202e30;
    printf("%zu\n",    sizeof(double));
    printf("%f\n",     x);
    printf(" %.2e\n", x);
    printf("%g\n",     x);
    return 0;
}
```



IEEE 754 Floating Point Standard

IEEE — Institute of Electrical and Electronics Engineers.



(Picture courtesy of Wikipedia)

A floating-point number is stored as:

```
(-1)^(sign)  ×  fraction  ×  2^(exponent)
```

The sign bit, exponent, and mantissa (fraction) are stored in separate bit fields. A single-precision (`float`) uses 32 bits; a double-precision (`double`) uses 64 bits.

> 📌 The formula above is a simplification that is good enough for our purposes. The full details can get complex — see the Wikipedia article on IEEE 754 for more information.

Converting Decimal to Binary

To convert the fractional part to binary:

- Multiply the fractional part by 2 and record the bit that appears before the decimal point.
- Repeat with the new fractional part until you reach 1.0 or have enough bits.

Example 1 — 0.25

```
0.25 × 2 = 0.50  →  take 0,  carry 0.50
0.50 × 2 = 1.00  →  take 1,  stop

Result: 0.25 = (0.01)₂
```

Example 2 — 0.33

```
0.33 × 2 = 0.66  →  0
0.66 × 2 = 1.32  →  1   carry 0.32
0.32 × 2 = 0.64  →  0
0.64 × 2 = 1.28  →  1   carry 0.28
0.28 × 2 = 0.56  →  0
0.56 × 2 = 1.12  →  1   carry 0.12
0.12 × 2 = 0.24  →  0
0.24 × 2 = 0.48  →  0
0.48 × 2 = 0.96  →  0
0.96 × 2 = 1.92  →  1   carry 0.92
0.92 × 2 = 1.84  →  1   carry 0.84
0.84 × 2 = 1.68  →  1   carry 0.68
0.68 × 2 = 1.36  →  1   carry 0.36   ← note: 0.36 will repeat
0.36 × 2 = 0.72  →  0
0.72 × 2 = 1.44  →  1   carry 0.44
0.44 × 2 = 0.88  →  0
0.88 × 2 = 1.76  →  1   carry 0.76
0.76 × 2 = 1.52  →  1   carry 0.52
          ... 0.32 repeats — the pattern is periodic

0.33 ≈ (0.01010100011100101 1)₂   (23 bits stored by the hardware)
```

📌 Some fractional numbers never terminate in binary (just as 1/3 never terminates in decimal). The computer allocates 23 bits for the fractional part of a float, so the conversion stops after 23 iterations.

Errors

Floating-point numbers can only store rational numbers; they cannot store arbitrary real numbers exactly. Rational numbers can approximate real numbers as accurately as we need, but approximation errors exist.

Let r be the real number being approximated, and p be the approximate value. We use two error measures:

```
Absolute error  =  |p - r|
Relative error  =  |p - r| / |r|    (when r ≠ 0)
```

⚠ Relative error can be significant when r is small, even if the absolute error appears small.

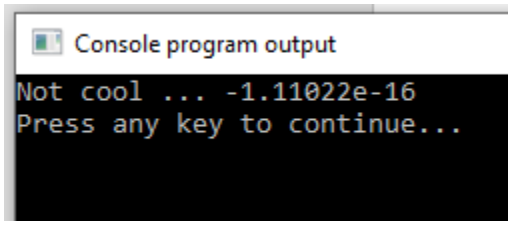
Be wary of:

- Subtracting nearly equal numbers.
- Dividing by very small numbers.
- Multiplying by very large numbers.
- Testing for equality with ==.

Example 1 — Equality comparison pitfall

```c
#include <stdio.h>

int main(void)
{
    double a = 7.0 / 12.0;
    double b = 1.0 / 3.0;
    double c = 1.0 / 4.0;
    if (b + c == a)
        printf("Everything is Awesome !");
    else
        printf("Not cool ... %g\n", b + c - a);
}
```



⚠ Comparing x == y is often risky with floating-point numbers. Instead of if (x == y), use a tolerance: if (x - y < 0.0001 || y - x < 0.0001) or equivalently: if (fabs(x - y) < 0.0001) The value ε = 0.0001 is called the tolerance.

Example 2 — Integer division pitfall

```c
#include <stdio.h>

int main(void)
{
    double a = 1 / 3;    // integer division! result is 0, not 0.333...
    printf("%g\n", a);
    return 0;
}
```

In C, most operators are overloaded. When C sees `1/3`, it reads this as integer division and returns 0. To fix this, make at least one operand a `double`:

```c
#include <stdio.h>

int main(void)
{
    double a = 1.0 / 3;        // double literal — works correctly
    double b = (double)1 / 3;  // explicit typecast — also works
    printf("a = %g\n", a);
    printf("b = %g\n", b);
    return 0;
}
```

Math Library

Reference: http://www.tutorialspoint.com/c_standard_library/math_h.htm

```c
#include <math.h>
```

Useful functions include:

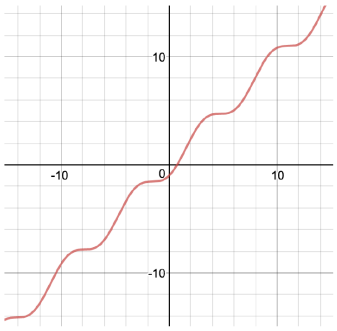
- `double sin(double x)` — and similarly `cos`, `tan`, `asin`, `acos`, `atan`, etc.
- `double exp(double x)` — and similarly `log` (natural log), `log10`, `log2`, `sqrt`, `ceil`, `floor`, etc.
- `double fabs(double x)` — absolute value for floating-point (integer `abs` is in `<stdlib.h>`).
- `double pow(double x, double y)` — computes x^y.
- Constants (caution — not in the basic standard): `M_PI`, `M_PI_2`, `M_PI_4`, `M_E`, `M_LN2`, `M_SQRT2`.
- Other special values: `INFINITY`, `NAN` (e.g., √−1), `MAXFLOAT`.

⚠ IMPORTANT: To compile a C program that includes <math.h> on Linux, you must add -lm at the end of the compile command: gcc -std=c11 program.c -lm

Root Finding

Given a function f(x), how can we find a root — that is, a value of x for which f(x) = 0?

Example: f(x) = x − cos(x).



Claim: There is a root in the interval [−10, 10].

Idea:

- Notice that f(−10) < 0 < f(10), so a root must lie in [−10, 10].
- Look at the midpoint (0) and evaluate f(0).
- If f(0) > 0, search in [−10, 0]. Otherwise, search in [0, 10].
- Repeat until a root is found.

Method 1: Bisection Method

Based on the theorem: if f(x) is continuous on [a, b] and f(a) × f(b) < 0, then there exists c in (a, b) such that f(c) = 0.

> 📌 f(a) × f(b) < 0 means f(a) and f(b) have opposite signs.

The algorithm ends when |f(c)| is less than a defined tolerance ε, or a maximum number of iterations is reached.

Two stopping conditions:

- Stop when |f(m)| < ε for some fixed ε > 0, where m is the interval midpoint. (Not great — the actual root may still be far away.)
- Stop when |m_(n−1) − m_(n)| < ε, where m_(n) is the nth midpoint. (Much better.)

Algorithm

```
Given a and b with f(a) and f(b) of opposite sign, set m = (a + b) / 2.

If f(m) and f(b) have the same sign, set b = m.
Otherwise, set a = m.

Loop until |f(m)| < ε, |m_(n-1) - m_(n)| < ε, or max iterations reached.
```

Interface File — bisection.h

```
#ifndef BISECTION_H
#define BISECTION_H

double f(double x);

/*
 * Pre:  epsilon > 0 is a tolerance, iterations > 0,
 *       f(x) has only one root in [a,b], f(a)*f(b) < 0
 * Post: Returns an approximate root of f(x) using the bisection method.
 *       Stops when either the number of iterations is exceeded
 *       or |f(m)| < epsilon.
 */
double bisect(double a, double b, double epsilon, int iterations);

#endif
```

Implementation File — bisection.c

```
#include <assert.h>
#include <math.h>
#include "bisection.h"

double f(double x) {
    return x - cos(x);
}

double bisect(double a, double b, double epsilon, int iterations)
{
    double m = a;
    double fb = f(b);  // cached to avoid recomputing f(b) each iteration
    assert(epsilon > 0.0 && f(a) * f(b) < 0);

    for (int i = 0; i < iterations; i++) {
        m = (a + b) / 2.0;
        if (fabs(b - a) < epsilon) {
            return m;
        }
        // Alternatively: if (fabs(f(m)) < epsilon) return m;
        if (f(m) * fb > 0) {
            b = m;
            fb = f(b);
        } else {
            a = m;
        }
    }
    return m;
}
```

Main Program

```
#include <stdio.h>
#include "bisection.h"

int main(void) {
    printf("%g\n", bisect(-10, 10, 0.0001, 50));
    return 0;
}
```

Tracing:

```
a=-10    b=10
i=  0         m=         0       a=         0       b=        10
i=  1         m=         5       a=         0       b=         5
i=  2         m=       2.5       a=         0       b=       2.5
i=  3         m=      1.25       a=         0       b=      1.25
i=  4         m=     0.625       a=     0.625       b=      1.25
i=  5         m=    0.9375       a=     0.625       b=    0.9375
i=  6         m=   0.78125       a=     0.625       b=   0.78125
i=  7         m=  0.703125       a=  0.703125       b=   0.78125
i=  8         m=  0.742188       a=  0.703125       b=  0.742188
i=  9         m=  0.722656       a=  0.722656       b=  0.742188
i= 10         m=  0.732422       a=  0.732422       b=  0.742188
i= 11         m=  0.737305       a=  0.737305       b=  0.742188
i= 12         m=  0.739746       a=  0.737305       b=  0.739746
i= 13         m=  0.738525       a=  0.738525       b=  0.739746
i= 14         m=  0.739136       a=  0.738525       b=  0.739136
i= 15         m=  0.738831       a=  0.738831       b=  0.739136
i= 16         m=  0.738983       a=  0.738983       b=  0.739136
i= 17         m=  0.739059       a=  0.739059       b=  0.739136
i= 18         m=  0.739098
0.739098
```

Method 2: Fixed Point Iteration

Goal: Given a function g(x), find a value x_0 such that $g(x_0) = x_0$. Such x_0 is called a **fixed point**.

In our example, finding a root of f(x) = x − cos(x) is equivalent to finding a fixed point of g(x) = cos(x).

> 📌 Not all functions have fixed points, but many root-finding problems can be reformulated as fixed-point problems.

Algorithm

```
Start with some initial point x₀.
Compute x₁ = g(x₀).
If |x₁ - x₀| < ε, stop and return x₁.
Otherwise, set x₀ = x₁ and repeat.
```

Interface File — fixed.h

```
#ifndef FIXED_H
#define FIXED_H

/* Pre:  none
 * Post: Returns the value of cos(x) */
double g(double x);

/*
 * Pre:  epsilon > 0 is a tolerance, iterations > 0,
 *       x0 is sufficiently close to a stable fixed point
 * Post: Returns an approximate fixed point of g(x) using cobwebbing.
 *       Stops when either the number of iterations is exceeded
 *       or |g(xi) - xi| < epsilon, where xi is x0 after i iterations.
 */
double fixed(double x0, double epsilon, int iterations);

#endif
```

Implementation File — fixed.c

```
#include <assert.h>
#include <math.h>
#include "fixed.h"

double g(double x)
{
    return cos(x);
}

double fixed(double x0, double epsilon, int iterations)
{
    double x1;
```

```
    assert(epsilon > 0.0);
    for (int i = 0; i < iterations; i++)
    {
        x1 = g(x0);
        if (fabs(x1 - x0) < epsilon)
            return x1;
        x0 = x1;
    }
    return x0;
}
```

Main Program

```
#include <stdio.h>
#include "fixed.h"

int main(void)
{
    printf("%g\n", fixed(0, 0.0001, 50));
    return 0;
}
```

Tracing:

```
i=  0         x1=         1         x0=         1
i=  1         x1=  0.540302         x0=  0.540302
i=  2         x1=  0.857553         x0=  0.857553
i=  3         x1=   0.65429         x0=   0.65429
i=  4         x1=   0.79348         x0=   0.79348
i=  5         x1=  0.701369         x0=  0.701369
i=  6         x1=   0.76396         x0=   0.76396
i=  7         x1=  0.722102         x0=  0.722102
i=  8         x1=  0.750418         x0=  0.750418
i=  9         x1=  0.731404         x0=  0.731404
i= 10         x1=  0.744237         x0=  0.744237
i= 11         x1=  0.735605         x0=  0.735605
i= 12         x1=  0.741425         x0=  0.741425
i= 13         x1=  0.737507         x0=  0.737507
i= 14         x1=  0.740147         x0=  0.740147
i= 15         x1=  0.738369         x0=  0.738369
i= 16         x1=  0.739567         x0=  0.739567
i= 17         x1=   0.73876         x0=   0.73876
i= 18         x1=  0.739304         x0=  0.739304
i= 19         x1=  0.738938         x0=  0.738938
i= 20         x1=  0.739184         x0=  0.739184
i= 21         x1=  0.739018         x0=  0.739018
i= 22         x1=   0.73913         x0=   0.73913
i= 23         x1=  0.739055
0.739055
```

Function Pointers

In the previous examples, the target function was hard-coded. Ideally, the function itself would be passed as a parameter. C supports this via **function pointers**.

Syntax

```
double (*f)(double)   // pointer to a function taking a double and returning a
double
```

📌 The parentheses around (*f) are essential. Without them, double *f(double) would declare a function that returns a pointer to double — a very different thing.

Interface File — bisection2.h

```
#ifndef BISECTION2_H
#define BISECTION2_H
#include <math.h>

/*
 * Pre:  epsilon > 0, iterations > 0,
 *       f(x) has only one root in [a,b], f(a)*f(b) < 0
 * Post: Returns an approximate root of f using the bisection method.
 *       Stops when iterations exceeded or |b - a| < epsilon.
 */
double bisect2(double a, double b, double epsilon, int iterations,
               double (*f)(double));

#endif
```

Implementation File — bisection2.c

```
#include <assert.h>
#include "bisection2.h"
// No need to include math.h here; it was included in the interface file.

double bisect2(double a, double b, double epsilon, int iterations, double
(*f)(double))
{
    double m = a;
    double fb = f(b);
    assert(epsilon > 0.0 && f(a) * f(b) < 0);
    for (int i = 0; i < iterations; i++)
    {
        m = (a + b) / 2.0;
        if (fabs(b - a) < epsilon)
            return m;
        // Alternatively: if (fabs(f(m)) < epsilon) return m;
        if (f(m) * fb > 0)
        {
            b = m;
```

```
            fb = f(b);
        }
        else
        {
            a = m;
        }
    }
    return m;
}
```

Main Program

```
#include <stdio.h>
#include "bisection2.h"

double f1(double x)
{
    return x - cos(x);
}

double f2(double x)
{
    return x * x * x - x + 1;
}

int main(void)
{
    // bisect2 called twice, each time with a different function
    printf("%g\n", bisect2(-10, 10, 0.0001, 50, f1));
    printf("%g\n", bisect2(-10, 10, 0.0001, 50, f2));
    return 0;
}
```

Polynomials

A polynomial is an expression with at least one indeterminate and coefficients lying in some set. In general:

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

For example: $3x^3 + 4x^2 + 9x + 2$. We will primarily use integers or doubles for the coefficients.

> ? Think about different ways we can represent polynomials in memory. Consider the pros and cons of each approach before reading further.

Representation

We represent a polynomial as an array of n+1 coefficients where n is the degree. Coefficients are stored in ascending order of power:

```
// 3x³ + 4x² + 9x + 2
double p[] = {2.0, 9.0, 4.0, 3.0};
//             a₀   a₁   a₂   a₃
```

The function signature for polynomial evaluation:

```
double eval(double p[], int n, double x);
```

Traditional Method

Evaluate $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ directly:

- Compute $x, x^2, x^3, \ldots, x^n$ — requires $n-1$ multiplications.
- Multiply each power by its coefficient $a_1, a_2, \ldots, a_n$ — another n multiplications.
- Add all terms including a_0 — n additions.
- Total: $2n-1$ multiplications and n additions.

> ? Multiplication is computationally more expensive than addition. Is there a way to reduce the number of multiplications?

Horner's Method

Idea: factor the polynomial to remove redundant multiplications.

```
a₀ + a₁x + a₂x² + … + aₙxⁿ
  = a₀ + x(a₁ + x(a₂ + … + x(aₙ₋₁ + aₙx)…))

Example: 2 + 9x + 4x² + 3x³  =  2 + x(9 + x(4 + 3x))
```

Total operations: **n multiplications + n additions** — compared to $2n-1$ multiplications and n additions for the traditional method.

Implementation

```
#include <stdio.h>
#include <assert.h>

double horner(double p[], int n, double x) {
    assert(n > 0);
    // n is the number of elements in the array; polynomial has degree n-1
    double y = p[n - 1];
    for (int i = n - 2; i >= 0; i--)
        y = y * x + p[i];
    return y;
}

int main(void) {
    double p[] = { 2, 9, 4, 3 };
    int len = sizeof(p) / sizeof(p[0]);
    printf("f(0)  = %g\n", horner(p, len,  0));
    printf("f(1)  = %g\n", horner(p, len,  1));
    printf("f(2)  = %g\n", horner(p, len,  2));
    printf("f(-1) = %g\n", horner(p, len, -1));
    return 0;
}
```

Extra Practice Problems

1.

Write a function named find_sqrt that uses Newton's Iteration to compute √n.

Parameters: double n, double tolerance
Returns: double (the square root)

Start with $x_0 = 1$. Each subsequent guess is:
x_(k+1) = (x_k + n / x_k) / 2

Stop and return x when $|x^2 - n| \leq$ tolerance.

Note: Try to solve this before looking at the provided solution.

```
int main(void)
{
    assert(find_sqrt(100.500000, 0.01) * find_sqrt(100.500000, 0.01)
           - 100.500000 <= 0.01);
    assert(find_sqrt(1123.525, 0.00123) * find_sqrt(1123.525, 0.00123)
           - 1123.525 <= 0.00123);
    assert(find_sqrt(10000.250000, 0.00000000001) *
           find_sqrt(10000.250000, 0.00000000001)
           - 10000.250000 <= 0.00000000001);
}
```

2.

What is the output of the following program? Trace manually before running.

```
#include <stdio.h>
int main(void)
{
    int i = 1, g = 2, *pi, *ptr = &i;
    float f, *pf;
    char c = 0, *pc;
    pi  = &g;
    pc  = &c;
    pf  = &f;
    *pi += 43;
    *pf  = (float)*pi + 0.54;
    ptr -= (int)*pf;
    *pc += '0';
    printf("%c\n", c);
    printf("%d\n", c);
    *pc += 4;
    printf("%d,%d\n", i, g);
    printf("%.2f\n", f);
    printf("%c\n", c);
    printf("%d\n", c);
    return 0;
}
```

3. Cosine via Taylor series:

> Write a program to compute the cosine of x using the Taylor series.
> The user supplies x and a positive integer n. Include all terms up through the term involving x^n.
>
> $\cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$

Solution to problem 1:

```c
#include <assert.h>
#include <stdbool.h>

// within_tolerance(a, b, tolerance) returns true if the difference between
//   a and b is <= tolerance, otherwise false.
// requires: tolerance >= 0
bool within_tolerance(double a, double b, double tolerance)
{
    if (a < b)
    {
        return (b - a <= tolerance);
    }
    else
    {
        return (a - b <= tolerance);
    }
}

// find_sqrt(n, tolerance) uses Newton's Iteration algorithm to find the
//   sqrt of n within the given tolerance
// requires: n > 0, tolerance > 0
double find_sqrt(double n, double tolerance)
{
    double g = 1;

    while (!within_tolerance(n, g * g, tolerance))
    {
        g = (g + n / g) / 2;
    }
    return g;
}

int main(void)
{

    assert(find_sqrt(100.500000, 0.01) * find_sqrt(100.500000, 0.01)
        - 100.500000 <= 0.01);
    assert(find_sqrt(1123.525, 0.00123) * find_sqrt(1123.525, 0.00123)
        - 1123.525 <= 0.00123);
    assert(find_sqrt(10000.250000, 0.00000000001) * find_sqrt(10000.250000, 0.00000000001)
        - 10000.250000 <= 0.00000000001);
}
```