CS 245
Logic and Computation

Lecture 7

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Propositional Logic

1. syntax (well-formed formulas)
   - Syntax and well-formed formulas
2. semantics
   - Truth tables, Boolean valuations
   - Logical implication ($\models$)
   - Logical equivalence ($\Longleftrightarrow$)
3. proof procedures ($\vdash$)
   - Transformational proof ($\underset{TP}{\Longleftrightarrow}$)
   - Natural deduction ($\vdash_{\text{ND}}$)
   - Semantic tableaux ($\vdash_{\text{ST}}$)

Today’s Agenda

- Semantic Tableaux ($\vdash_{ST}$)
  - General form of tableaux
  - Using a tableau to show a set of formulas is inconsistent
  - Using a tableau to show an argument is valid
  - Tableaux expansion rules
  - Heuristic for tableaux expansion
  - Soundness and completeness

Notes

For semantic tableaux, we are using Ch. 2 from Kelly (see the course pack).

Kelly uses different symbols for implication and equivalence:

Kelly	Nissanke
$\longrightarrow$	$\Longrightarrow$
$\longleftrightarrow$	$\Longleftrightarrow$

Also Kelly uses “mutually consistent” where we have used “consistent” when talking about a set of formulas.

Semantic Tableaux

A semantic tableau is a tree representing all the ways the conjunction of the formulas at the root can be true.

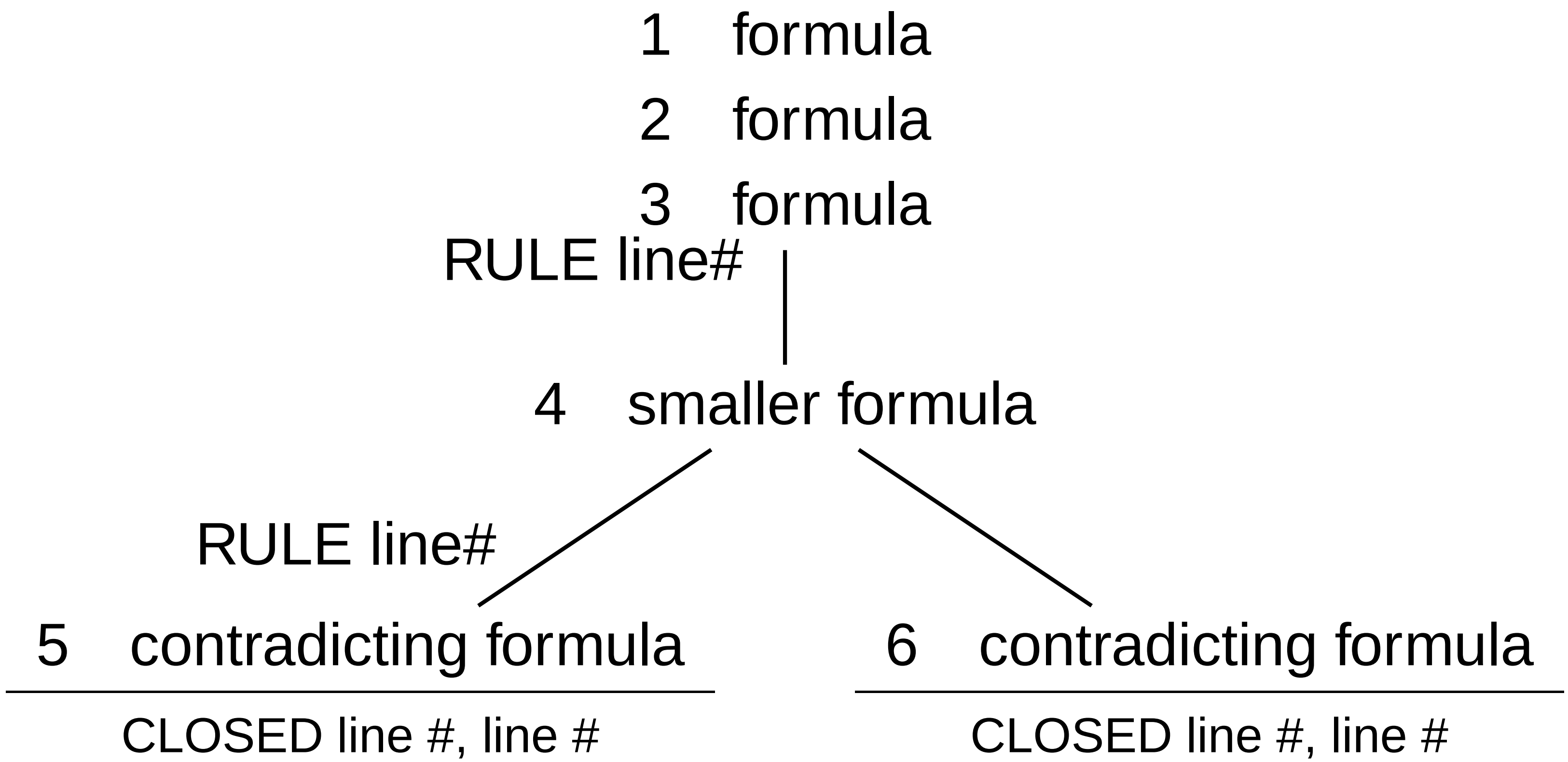
We expand the formulas based on the structure of the compound formulas. This expansion forms a tree.

If all branches in the tableau lead to a contradiction, then there is no way the conjunction of the formulas at the root can be true.

A path of the tree represents the conjunction of the formulas along the path.

Semantic tableaux was invented by E.W. Beth and J. Hintikka (1965).

General Form of Tableaux



The rules are based on the logical connectives in the formulas.

Semantic Tableaux

A branch is closed is a and $\neg a$ both appear on the path from the root of the tree to the leaf of the branch (i.e., there is a contradiction on the branch). a need not be a prime proposition.

If all branches of the tree are closed, then the tableau is closed and we can conclude the conjunction of the formulas at the root are not satisfiable, therefore the set of formulas is inconsistent.

We will number all the formulas in the tableau, and use these along with rules to justify the expansion of the tableau. (The order that formulas get assigned numbers doesn't matter, as long as each formula has a unique number.)

Showing Inconsistency

If all branches of a tableau are closed, the set of formulas at the root are inconsistent.

We can use semantic tableaux to show a set of formulas is inconsistent.

Showing Validity

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To show an argument is valid, we put the premises and the negation of the conclusion at the root of a tableau.

If we can close all the branches of the tableau, then this set of formulas is inconsistent. This means the argument is valid and we can write:

$$\boxed{p_1, p_2, p_3, \ldots \vdash_{\text{ST}} q}$$

Showing Validity

1 premise

2 premise

3 negation of conclusion

RULE line#

4 smaller formula

RULE line#

5 contradicting formula

CLOSED line #, line #

6 contradicting formula

CLOSED line #, line #

Showing Validity

In semantic tableaux, we are proving $p_1, p_2, p_3 \models q$ by showing $p_1, p_2, p_3, \neg q$ is an inconsistent set of formulas.

Semantic tableaux is based on the idea of proof by contradiction. It is a refutation-based system.

Semantic tableaux is a form of backward proof because we start from the conclusion and decompose it and the premises into to smaller and smaller parts until we reach a contradiction.

Tableaux Expansion Rules

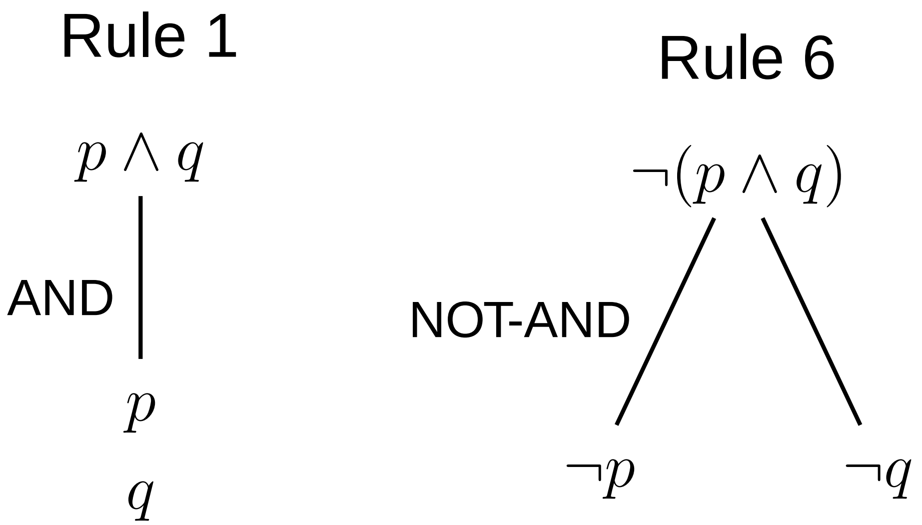
There are rules for:

- each of the binary logical connectives
- the negation of a formula with each binary logical connective
- double negation

The rule numbers are provided to show you the correspondence with Kelly's text book. We will use names rather than numbers for the rules.

There is a summary sheet available on the course web page with the semantic tableaux expansion rules.

Rules for Conjunction



This rule can be applied to a formula with more than two conjuncts in a single step.

Example

Show $p \wedge q, r \vdash_{\text{ST}} q \wedge r$

1 $p \wedge q$

2 r

3 $\neg(q \wedge r)$

AND 1

4 p

5 q

NOT-AND 3

6 $\neg q$

CLOSED 5, 6

7 $\neg r$

CLOSED 2, 7

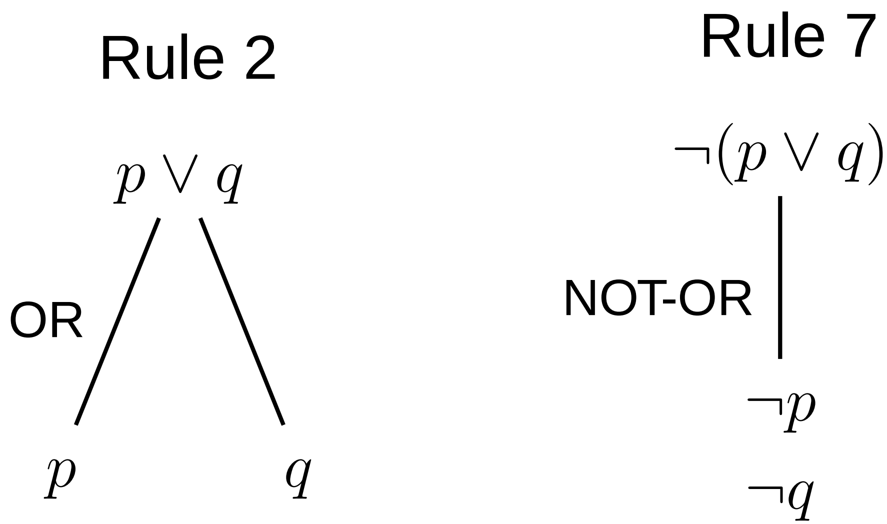
Notes

A semantic tableaux rule only applies to one formula (i.e., one line of the tree).

Closing a tableau requires two formulas that contradict each other (i.e., two lines of the proof).

A branch means there are two ways to make the formula true. A branch captures disjunction.

Rules for Disjunction



This rule can be applied to a formula with more than two disjuncts in a single step.

Heuristic

Apply the non-branching rules first.

Usually this will result in shorter proofs.

Rule for Negation

Rule 5

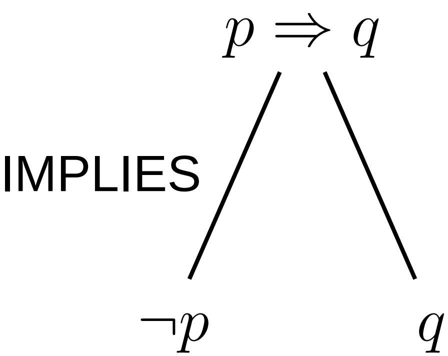
$\neg\neg p$

NOT-NOT

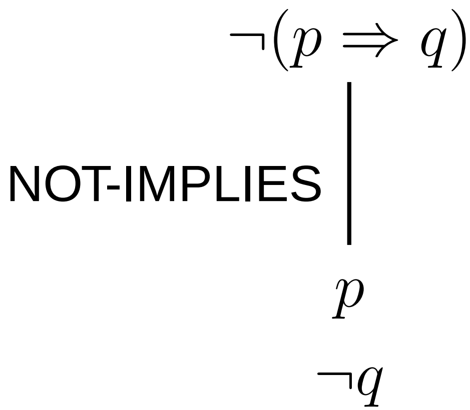
p

Rules for Implication

Rule 3



Rule 8



Rules for Equivalence

Rule 4

$p \Leftrightarrow q$

IFF

$p \wedge q$ | $\neg p \wedge \neg q$

Rule 9

$\neg(p \Leftrightarrow q)$

NOT-IFF

$p \wedge \neg q$ | $\neg p \wedge q$

Soundness and Completeness of $\vdash_{\text{ST}}$

Semantic tableaux for propositional logic is sound and complete.

Soundness: if $p_1, p_2, \ldots, p_n \vdash_{\text{ST}} q$ then $p_1, p_2, \ldots, p_n \models q$

Semantic tableaux only proves tautologies.

Completeness: if $p_1, p_2, \ldots, p_n \models q$ then $p_1, p_2, \ldots, p_n \vdash_{\text{ST}} q$

Semantic tableaux can be used to prove all tautologies.

Showing Inconsistency

1. Sales of houses fall off if interest rates rise. $r \Rightarrow s$
2. Auctioneers are not happy if sales of houses fall off. $s \Rightarrow \neg h$
3. Interest rates are rising. r
4. Auctioneers are happy. h

where

s = sales of houses fall off
r = interest rates rise
h = auctioneers are happy

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Exercises and Next Lecture

Suggested exercises: Kelly 2.1, 2.2, Misc Exercises 1 and 2

Try redoing any examples done using natural deduction in semantic tableaux.

Topic: Example using Propositional Logic

- QUESTIONS?