CSE272: Assignment 4

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Abstract

While lights are often described by their origin and orientation in computer graphics, lights in the real world have particular states of vibration because lights are electromagnetic waves. Such state of vibration is called *polarization* state. Although human eyes cannot detect polarization state, sometimes lights behave differently according to their polarization states, which may be detected by human eyes. In this paper, we describe how to incorporate tracking of polarization states into conventional ray tracing. We propose *direct tracking* of polarization states, which uses two harmonic oscillations to describe polarization state. As applications of tracking of polarization states, we present how to render thin-film interference and birefringence. We show our direct tracking method is easy to be applied for thin-film interference compared to Stokes vector, which is one of the popular ways to describe polarization states in the field of optics. As for birefringence, we propose a method to calculate a refraction vector inside birefringent materials based on Huygens' principle. Our results suggest dealing with polarization is important for truly accurate image synthesis.

1 Introduction

It is well known that lights have duality of wave and particle. Optics based on particle properties of light is called *geometric optics*, and optics based on wave properties of light is called *wave optics*. Since human eyes cannot detect oscillations of light waves directly, geometric optics has been a popular model to describe light in computer graphics community. However, geometric optics fails to take into account someimportant phenomena caused by wave properties of light. Therefore, using only geometric optics is not enough to simulate appearances of the real world perfectly, so we need to take into account wave optics in some cases.

In this paper, we describe how to incorporate wave optics into conventional ray tracing, which is based on geometric optics. Especially, we present a way to incorporate *polarization* to ray tracing, which is fundamental phenomenon of wave optics. As applications of polarization, we present methods to render thin-film interference and birefringence. Thin-film interference is one of structure colors caused by diffraction. The rainbow-color of soap bubbles is famous examples of thin-film interference (Figure 6). Birefringence is a phenomenon that refractive index varies according to state of polarization and light direction.Doubling of refracted image by a calcite (Figure 13) is prominent example of birefringence. From the next section, we start by basic background theory of polarization, and then show how it can be applied to thin-film interference and birefringence.

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1.1 Polarization of Light

Based on geometric optics, we can describe propagation of light by using ray, which is defined by its origin and direction. However, since light is an electromagnetic wave [Donnelly et al. 2006], we should consider a transverse wave along ray direction as in Figure 1. This transverse wave oscillates perpendicular to the propagation

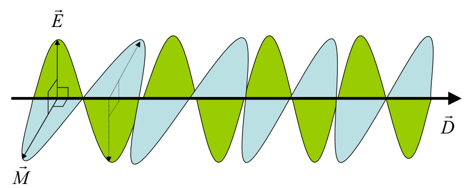

Figure 1: *Light as a transverse electromagnetic wave. $\vec{E}$ and $\vec{M}$ denote oscillation directions of electric wave and magnetic wave. Note that ray direction $\vec{R}$, $\vec{E}$ and $\vec{M}$ are orthogonal each other.*

direction $\vec{D}$. Note that electric wave $\vec{E}$ and magnetic wave $\vec{M}$ are orthogonally oscillating. We only deal with electric wave for the rest of the paper because magnetic wave is always perpendicular to electric wave.

Since light wave is transverse wave, it has direction of oscillation. If we choose direction of propagation $\vec{D}$ as a z axis and direction of oscillation can be expressed in 2D coordinates as in Figure 2. Therefore, we can describe oscillation of light wave in this coordinate system as

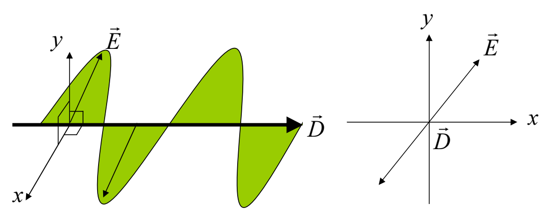

Figure 2: *Oscillation of electric wave in 2D coordinates. $\vec{E}$ is oscillating on a particular plane.*

$$\vec{E}(t) = A\vec{a}sin(\omega t + \phi) \quad (1)$$

Here $\vec{a}$ is an unit vector of direction of oscillation, ω is s frequency of light and ϕ is a phase shift of light wave. For simplicity, we look at the filed at $z = 0$. Note that the amplitude of light is $A = \|\vec{E}(t)\|$. The energy (or intensity) of light I is A^2, which is used in conventional ray tracing as intensity.

Since we can choose xy axes arbitrary as long as they are orthogonal to z axis and each other, we can rewrite Equation 1 in more general form by using combination of two harmonic oscillations.

$$\vec{E_{xy}}(t) = (A_x sin(\omega t + \phi_x), A_y sin(\omega t + \phi_y)) \quad (2)$$

The intensity of light in this case is ${A_x}^2 + {A_y}^2$. If $(\phi_x - \phi_y) = 0.5n\pi$ $(n = \cdots, -1, 0, 1 \cdots)$, these two harmonic oscillation are called to be *in phase*. This particular type of oscillation is called *linear po-*

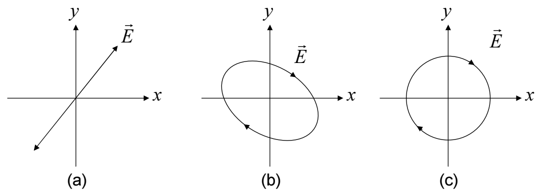

Figure 3: *Types of polarization. (a) linear polarization. (b) ellipsoidal polarization. (c) circular polarization.*

larization because light wave oscillates on a plane as in Figure 3 (a). If $(\phi_x - \phi_y) \neq 0.5n\pi$, a plane of oscillation will be rotating as light propagates. In this case, the tip of the wave swept an ellipsoid as in Figure 3 (b). This type of oscillation is called *elliptical polarization*. If the phase difference is exactly the quarter of the wavelength ($(\phi_x - \phi_y) = 0.25n\pi$) and amplitudes of waves are the same, $A_x = A_y$, the tip of the wave now swept a circle. This type of oscillation is called *circular polarization*. Note that neither elliptical nor circular polarized light rotate physically. Both cases are just superposition of two harmonic oscillations as in Equation 2, and the results of superposition just seem to be rotating wave. Therefore, the intensity of light is always calculated by ${A_x}^2 + {A_y}^2$ for all types of polarization.

If polarization state of light (while intensity is the same) is completely random in time, it is called a light is *unpolarized*. Note that while it is named "unpolarized", the exact description is "randomly polarized". If unpolarized light (=randomly polarized light) gets correlated by some phenomena such as reflection, light is now *polarized*. For polarized light, polarization states are not completely random but somewhat (or completely) correlated.

Since human eyes are not trained to capture polarization states of lights (e.g. we cannot discriminate between a circularly polarized light and a linearly polarized light if they have the same intensity), computer graphics community has been somewhat ignorant to polarization. However, polarization is actually important for visually accurate image synthesis because some materials behave differently for different polarization states (interest readers refer a survey by [Devlin et al. 2002]). One of the famous polarized lights is the sky light. Since the sky lights are linearly polarized by scattering, photographers often use a polarizer to enhance contrast of sky and landscape (Figure 4). In contrast to its importance in appearances of the

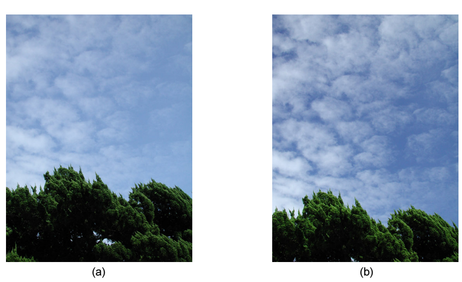

Figure 4: *Example of sky light polarization (©nikkei BPnet). (a) photograph taken without polarization filter. (b) photograph taken with polarization filter. Note that only the sky in (b) is darker than (a) because linearly polarized sky light is filtered out by polarization filter.*

real world, almost all commercial renderers do not take account of polarization and only a few graphics CADs, such as OptiCAD (see www.opticad.com), can handle polarization of lights. However, this little attention is somewhat misdirected because polarization affects resulting appearance in many cases, even if our eyes cannot detect polarization. Therefore, we need to incorporate polarization if we would like to render truly physically accurate images. As such examples, we describe thin-film interference and birefringence from the next chapter.

2 Interference with Polarization

2.1 Background

Thin-film interference is an interference caused by refracted/reflected lights from a film which is as thin as wavelength of incident light waves. Figure 5 shows how this interference occurs. Since reflected light on the bottom of the film goes through longer path than the reflected light on the top of the film, the interference affected by this difference of path (the details are in [Glassner 1999]). The most famous example of thin-film interference is

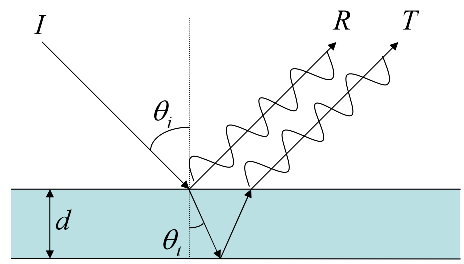

Figure 5: *Interference caused by a thin-film. Part of incident light (I) reflects off at the top of the film (R) or refracted into thin-film and reflects off at the bottom of the film (T). Thickness of the film (d) is small enough to make two rays (R and T) cause interference.*

rainbow colors of soap bubbles (Figure 6). Thickness of soap bubbles is usually several hundreds nm, which is close enough to the wavelength of visible light. Therefore, thin-film interference makes soap bubbles to be rainbow-colored. As described by

Figure 6: *Photograph of a soap bubble (©Photo 360 Limited). The rainbow color of soap bubble is due to thin-film interference.*

[Glassner 1999], a path difference due to a thin-film can be calculated as folllows.

$$w(\theta_t) = 2d\eta cos\theta_t \quad (3)$$

Here η is refractive index of a thin-film and θ_t is an angle between refracted light direction and the surface normal. Since light wave undergoes a half wavelength phase shift when it is reflected at the bottom of the film (iff the refractive index of a film is lager than the refractive index of incident media), the phase difference p between rays R and T due to thin-film is expressed as follows.

$$p(\theta_t, L) = 2\pi \frac{w(\theta_t)}{L} + \frac{\pi}{2} \quad (4)$$

Here L is wavelength of light. Therefore, given intensity of the incident light I, the interfered intensity F is calculated by

$$F(\theta_t, L) = 4Icos^2(p(\theta_t, L)) \quad (5)$$

Here we assume that there is no attenuation by reflection/refraction for simplicity. As a result, thin-film interference causes interference fringes over surface for particular wavelength. Since patterns of interference fringes depend on wavelength, it results in rainbow color as we can seen in soap bobbles.

Equation 5 well predicts the appearance of soap bubbles ([Glassner 1999]), but we should notice that light is actually polarized by reflection/refraction on thin-film according to *Fresnel equations*.Especially, we should take account the fact that two orthogonally and linearly polarized lights do not cause any interference. Previous formulation in Equation 5 ignores this effect, so resulting interference may be largely different if we take account of polarization as we describe from the next section.

2.2 Implementation

2.2.1 Direct Tracking

Before we consider interference with polarization, we need to decide how to keep track polarization state of light.One of the popular way to handle polarization state is Stokes vector, which has been used well in optics. Stokes vector describes a polarization state by 4-component vector as follows.

$$\vec{S} = (I, Q, U, V) \quad (6)$$

The first component of Stokes vector (I) denotes the intensity of a light, and the rest of components (Q, U, V) describe the polarization state. By using Stokes vector, we can express unpolarized light just by using 1 Stokes vector, $\vec{S_u} = (1, 0, 0, 0)$. If a light undergoes change of its polarization state by reflection/refraction, we can calculate the resulting Stokes vector $\vec{S_n}$ just by multiplying a matrix M.

$$\vec{S_n} = M\vec{S} \quad (7)$$

This matrix M is called *Muller matrix* and this calculation of polarization state is called *Muller calculus*. Stokes vector has been used well in optics because of its simplicity (such as [Bartel and Hielcher 2000]). However, as the fact that it can express unpolarized light just by one vector suggests, we should pay attention to that Stokes vector just describes *ensemble average* of several polarization states. We cannot use Stokes vector for thin-film interference because interference is interaction between particular polarization states. [Wilkie et al. 2001] and [Wolff and Kurlander 1990] have used *coherency matrix*, which is another way to express polarization state, but it also describes only the ensemble average.

For this reason, we propose *direct tracking* of polarization state in this paper. As Equation 2 suggests, all types of polarizations (linear, circular and ellipsoidal) can be expressed by combination of two harmonic oscillations. Direct tracking uses this fact to keep track polarization state. Therefore, polarization state is express by following four parameters.

$$\vec{S_d} = (A_x, \phi_x, A_y, \phi_y) \quad (8)$$

This vector is 4-component vector as the same as Stokes vector, but now we cannot use Muller calculus to change polarization state. We will describe how to calculate resulting polarization state from the next section when light is reflected/refracted. In contrast to Stokes vector, unpolarized light in direct tracking is expressed by multiple randomly polarized light (i.e. randomly changing ϕ_x and ϕ_y etc). Therefore, we calculate image with unpolarized light by taking ensemble average of results obtained from particular polarization state. This is similar to Monte-Carlo method, but notice that it actually imitates real behavior of unpolarized ligh (i.e. randomly polarized in time).

2.2.2 Reflection/Refraction by Direct Tracking

To perform direct tracking, we first need to define a coordinate system for two harmonic oscillations. We define such coordinate system as follows

$$\vec{R_x} = \frac{\vec{t} \times \vec{r}}{\|\vec{t} \times \vec{r}\|} \qquad \vec{R_y} = \frac{(\vec{t} \times \vec{r}) \times \vec{r}}{\|(\vec{t} \times \vec{r}) \times \vec{r}\|} \qquad \vec{R_z} = \vec{r} \quad (9)$$

Here $\vec{r}$ is the direction of ray and $\vec{t}$ is a vector which is defined by $\vec{t} = \vec{r}$ by replacing the component with the smallest absolute value by 1 (e.g. $\vec{r} = (-1, 0.5, 0.2)$ then $\vec{t} = (-1, 0.5, 1)$). Note that $\vec{R_z}$, $\vec{R_x}$ and $\vec{R_y}$ are orthogonal to each other. We call this coordinate system as *ray coordinate system*. The two harmonic oscillations, $A_x sin(\omega t + \phi_x)$ and $A_y sin(\omega t + \phi_y)$, are associated with this coordinate system. Therefore, $A_x sin(\omega t + \phi_x)$ is oscillating along $\vec{R_x}$ and $A_y sin(\omega t + \phi_y)$ is oscillating along $\vec{R_y}$.

If a light hits a dielectric surface, fraction of light is reflected off from the surface and the rest of light is refracted (Figure 5). The ratio of amplitudes of incident light and reflected/refracted light is described by *Fresnel equations*. Let the angle between incident ray and normal vector is θ_i, angle between refracted ray and normal vector is θ_t and ratio of refractive index (exitant media / incident media) is η. In this case, Fresnel coefficients obtained by Fresnel equations are

$$R_p = \frac{m - \eta}{m + \eta} \qquad R_s = \frac{1 - \eta m}{1 + \eta m}$$
$$T_p = \sqrt{\eta |m|} \frac{2}{m + \eta} \qquad T_s = \sqrt{\eta |m|} \frac{2}{1 + \eta m} \quad (10)$$
$$m = \frac{cos\theta_t}{cos\theta_i}$$

Note that $\sqrt{\eta |m|}$ is the term for change of "density of ray", which is required to obtain correct intensity. Here R_{ps} denote attenuation of amplitude for reflected ray and T_{ps} denote attenuation of amplitude for refracted ray. These Fresnel coefficients are defined on the coordinate system constructed by incident light direction and normal vector as:

$$\vec{F_p} = \frac{\vec{r} \times \vec{n}}{\|\vec{r} \times \vec{n}\|} \qquad \vec{F_s} = \frac{(\vec{r} \times \vec{n}) \times \vec{r}}{\|(\vec{r} \times \vec{n}) \times \vec{r}\|} \qquad \vec{F_r} = \vec{r} \quad (11)$$

where $\vec{n}$ is normal vector of the surface. We call this coordinate system as *p-s coordinate system*. If we consider a plane defined by incident light direction and normal vector p axis is parallel to

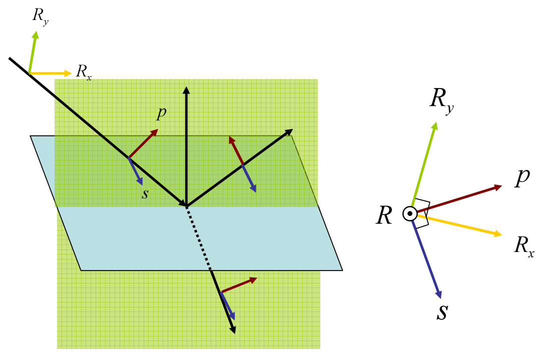

Figure 7: *Definition of p-s coordinate system. The green plane is a plane defined by incident ray and normal vector. Red and blue arrows denote p-s coordinate system. Note that ray coordinate system and p-s coordinate system share z axis as the direction of ray. The right shows relationship between ray coordinate system and p-s coordinate system by projecting to the plane perpendicular to the ray direction.*

this plane and s axis is perpendicular (s denotes *senkrecht*, "perpendicular" in German) to this plane (Figure 7). To multiply Fresnel coefficients to light by direct tracking, we need to transform values in ray coordinate system into p-s coordinate system. By considering ray coordinate system and p-s coordinate system share the same axis as ray direction, two harmonic oscillations in ray coordinate system are simply distributed to p-s coordinate system as follows. Therefore, given incident light $(A_x, \phi_x, A_y, \phi_y)$, resulting oscillations $\vec{E}_{ps}(t) = (E_p(t), E_s(t))$ are defined as

$$\begin{aligned} E_p(t) =& (\vec{F}_p \cdot \vec{R}_x) A_x sin(\omega t + \phi_x) + \\ & (\vec{F}_p \cdot \vec{R}_y) A_y sin(\omega t + \phi_y) \\ E_s(t) =& (\vec{F}_s \cdot \vec{R}_x) A_x sin(\omega t + \phi_x) + \\ & (\vec{F}_s \cdot \vec{R}_y) A_y sin(\omega t + \phi_y) \end{aligned} \tag{12}$$

The above equation can be rewritten as two harmonics oscillations $(A_p, \phi_p, A_s, \phi_s)$ in p-s coordinate system by using combination of harmonic oscillations as follows.

$$\begin{aligned} A_p &= \sqrt{\left(A_{px} cos\phi_x + A_{py} cos\phi_y\right)^2 + \left(A_{px} sin\phi_x + A_{py} sin\phi_y\right)^2} \\ A_s &= \sqrt{\left(A_{sx} cos\phi_x + A_{sy} cos\phi_y\right)^2 + \left(A_{sx} sin\phi_x + A_{sy} sin\phi_y\right)^2} \\ \phi_p &= tan^{-1} \frac{A_{px} sin\phi_x + A_{py} sin\phi_y}{A_{px} cos\phi_x + A_{py} cos\phi_y} \\ \phi_s &= tan^{-1} \frac{A_{sx} sin\phi_x + A_{sy} sin\phi_y}{A_{sx} cos\phi_x + A_{sy} cos\phi_y} \end{aligned} \tag{13}$$

Here we used notations $A_{px} = (\vec{F}_p \cdot \vec{R}_x) A_x$, $A_{py} = (\vec{F}_p \cdot \vec{R}_y) A_y$, $A_{sx} = (\vec{F}_s \cdot \vec{R}_x) A_x$, $A_{sy} = (\vec{F}_s \cdot \vec{R}_y) A_y$ for simplicity. Therefore, we finally obtain two harmonic oscillations on p-s coordinate system as follows.

$$\vec{E}_{ps}(t) = (A_p sin(\omega t + \phi_p), A_s sin(\omega t + \phi_s y)) \tag{14}$$

Note that the transformed light has the same intensity and difference of phase shift is the same as before transformation. After transforming into p-s coordinate system, Fresnel coefficients are multiplied to the amplitudes. Therefore the resulting reflected light $\vec{R}_{ps}(t)$ and refracted light $\vec{T}_{ps}(t)$ are expressed as follows.

$$\begin{aligned} \vec{R}_{ps}(t) &= (R_p A_p sin(\omega t + \phi_p), R_s A_s sin(\omega t + \phi_s)) \\ \vec{T}_{ps}(t) &= (T_p A_p sin(\omega t + \phi_p), T_s A_s sin(\omega t + \phi_s)) \end{aligned} \tag{15}$$

Notice that there is no change in phase shifts except for the total reflection because we assume there is no absorption within the thin-film. To complete this process, we again transform resulting light in p-s coordinate system into ray coordinate system. This transformation is almost the same as "ray to p-s" transformation using Equation 13. All we need to do is just swapping axis vectors for this transformation.

2.2.3 Thin-film Interference

The phase shift due to thin-film is exactly the same as formulation without considering polarization (Equation 4). However, if we consider polarization, we have to take into account polarization state of different light. If we denote $(R_x, \phi_{rx}, R_y, \phi_{ry})$ as reflected light at the top of the film and $(T_x, \phi_{tx}, T_y, \phi_{ty})$ as refracted light from the bottom of the film, the final oscillation of light $\vec{F}_{xy}(t) = (F_x(t), F_y(t))$ can be written as

$$\begin{aligned} \vec{F}_x(t) &= R_x sin(\omega t + \phi_{rx}) + T_x sin(\omega t + \phi_{tx}) \\ \vec{F}_y(t) &= R_y sin(\omega t + \phi_{ry}) + T_y sin(\omega t + \phi_{ty}) \end{aligned} \tag{16}$$

Here $p(\theta_t, L)$ is phase shift defined as Equation 4. Note that $\phi_{tx} = \phi_{rx} + p(\theta_t, L)$ and $\phi_{ty} = \phi_{ry} + p(\theta_t, L)$ because of phase shift due to the thin-film.

Finally, we combine two harmonic oscillations from two rays (R, T) into $(F_x, \phi_{fx}, F_y, \phi_{fy})$ similar to Equation 13.

$$\begin{aligned} F_x &= \sqrt{\left(R_x cos\phi_{rx} + T_x cos\phi_{tx}\right)^2 + \left(R_x sin\phi_{rx} + T_x sin\phi_{tx}\right)^2} \\ F_x &= \sqrt{\left(R_y cos\phi_{ry} + T_y cos\phi_{ty}\right)^2 + \left(R_y sin\phi_{ry} + T_x sin\phi_{ty}\right)^2} \\ \phi_{fx} &= tan^{-1} \frac{R_x sin\phi_{rx} + T_x sin\phi_{tx}}{R_x cos\phi_{rx} + T_x cos\phi_{tx}} \\ \phi_{fy} &= tan^{-1} \frac{R_y sin\phi_{ry} + T_y sin\phi_{ty}}{R_y cos\phi_{ry} + T_y cos\phi_{ty}} \end{aligned} \tag{17}$$

Therefore, intensity of interfered light is ${F_x}^2 + {F_y}^2$.

2.3 Results and Discussion

Figure 8 shows the rendered images for thin-film interference with/without polarization. For previous formulation (Equation 5), we fully incorporated Fresnel coefficients ($R = 0.5(R_p^2 + R_s^2)$ and $T = 0.5(T_p^2 + T_s^2)$) for intensity. For simplicity, we used RGB colors for all interference calculation. We used 32 samples to calculate results by unpolarized lights for direct tracking method. Since previous formulation (Equation 5) completely ignored polarization, the result is completely different even if we use unpolarized light as in Figure 8. Therefore, we can conclude previous formulation is not even approximation in unpolarized lighting condition. One of the reasons of this difference is that orthogonally linearly polarized lights do not interfere. The previous formulation (Equation 5) ignores this effect and estimate effect of interference too large. Figure 9 shows comparison with [Li and Peng 1996], which also incorporated polarization into Equation 5. Although result by [Li and Peng 1996] is well matched to our result in the case of single sphere (Figure 9: left), result by [Li and Peng 1996] is too bright in the case of interreflection (Figure 9: center and right). The reason is that [Li and Peng 1996] does not track polarization state but only calculates intensity with polarization. Therefore, reflected lights lack of information of polarization states, which is important for interreflection cases. Especially, it is well know that if light enters thin-film near

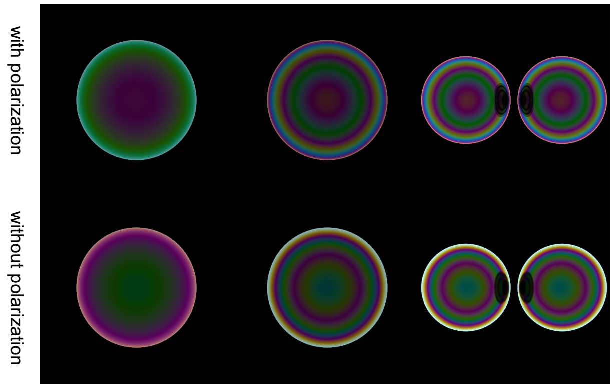

Figure 8: *Rendered images of thin-film interference. Left: Thickness of the film is 500nm. Center: Thickness of the film is 1500nm. Right: Thickness of the film is 1500nm. The intensity of light is multiplied by 1.5 to emphasize interreflection.*

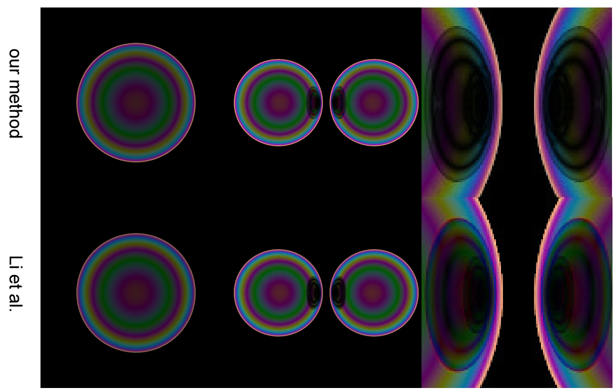

Figure 9: *Comparison with Li et al. Thickness of the film is 1500nm. Note that result by Li et al. for interreflection is too bright compared to our result*

the Brewster angle ($\theta_B = tan^{-1}(\eta)$) reflected lights become almost completely linearly polarized. Therefore, the effect of interference is limited compared to unpolarized light, because interference does not occur for orthogonally polarized component of light. As a conclusion, we found polarization plays important role for thin-film interference. Our direct tracking method is general method, so it can be applied to interreflection cases without any problem, as opposed to previous method like [Li and Peng 1996].

3 Birefringence

3.1 Background

In birefringent material, refractive index (more precisely speed of light) varies according to polarization state and propagation direction. Birefringence is defined by *optical axis*, which describes anisotropy of refractive index. If one optical axis is enough to define refractive index, it is called *uniaxial*. In the case of two optical axis, it is called *biaxial*. In this paper, we consider only uniaxial material. Either cases are called *anisotropic material* in contrast to *isotropic material*, which refractive index is just one value. For uniaxial material there are two refractive indices, denoted by η_o and η_e. Optical axis and these two refracted indices define *indicatrix* (i.e. ellipsoid defines refractive index) as in Figure 10. With

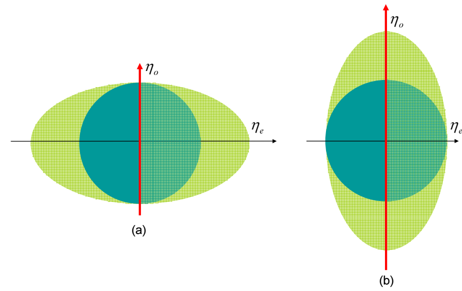

Figure 10: *Indicatrix defined by two refractive indices (η_o and η_e). The red axis is the optical axis. (a) If $\eta_o < eta_e$, it is called optically negative. (b) If $\eta_o > eta_e$, it is called optically positive.*

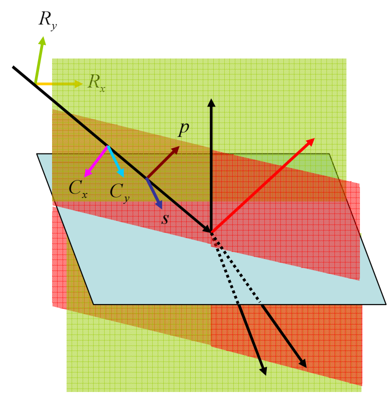

Figure 11: *Crystal coordinate system. The red arrow is optical axis and red plane is principal plane.*

optical axis $\vec{o}$ and ray direction $\vec{r}$, we can define *crystal coordinate system* as follows.

$$\vec{C}_x = \frac{\vec{r}\times\vec{o}}{\|\vec{r}\times\vec{o}\|} \qquad \vec{C}_y = \vec{o} \qquad \vec{C}_z = \frac{(\vec{r}\times\vec{o})\times\vec{o}}{\|(\vec{r}\times\vec{o})\times\vec{o}\|} \tag{18}$$

A plane defined by ray direction $\vec{r}$ and optical axis $\vec{o}$ is called *principal plane* (Figure 11).Note that $\vec{C}_x$ is perpendicular to the principal plane, and $\vec{C}_y$ is parallel to the principal plane.

If unpolarized light enters into an uniaxial material, incident light ray is split into two rays as in Figure 12. These two rays are orthogonally and linearly polarized (Figure 12). A light ray with perpendicularly polarized to the principal plane is called *ordinary ray*, and a light ray polarized parallel to the principal plane is called *extraordinary ray*.

As the name suggests, ordinary follows *Snell's law*:

$$\eta_i sin\theta_i = \eta_o sin\theta_o \tag{19}$$

where θ_i is angle between incident light direction and surface normal and θ_o is angle between refracted light direction and surface normal. Therefore, a ordinary light enters at right angle ($\theta_i = \frac{\pi}{2}$) to the uniaxial material does not refract. However, extraordinary ray *does not* follow Snell's law. It refracts even if incident ray enters at right angle. One of the example of such effect can be seen in calcite (Figure 13). As in Figure 13, refracted images through calcite is doubling. These images correspond to the images by ordinary ray and extraordinary ray. There are few research in computer graph-

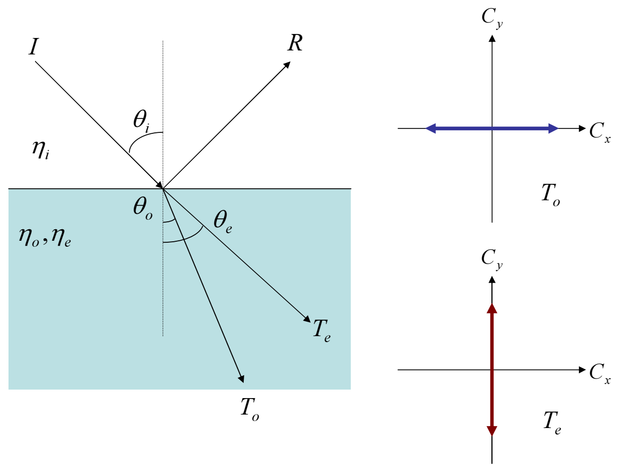

Figure 12: *Ordinary ray and extraordinary ray. Incident ray I split into ordinary ray T_o and extraordinary ray T_e. The direction of linear polarization of ordinary ray is perpendicular to the principal plane, while extraordinary ray vibrates parallel to the principal plane.*

Figure 13: *Birefringence of calcite (©Wikipedia). Note that even this photograph take at right angle to the surface of calcite, there is birefringence caused by ordinary ray and extraordinary ray.*

ics about birefringence. [Tannenbaum et al. 1994] extended coherent matrix to take account birefringence. [Guy and Soler 2004] mentioned birefringence in their gemstone rendering, however they only presented approximation appropriate for hardware rendering. To render birefringence phenomena, we need to deal with not only polarization states, but also calculation of propagation of rays inside anisotropic materials. Especially, since extraordinary ray does not follow Snell's law, we need another way to calculated refracted direction of extraordinary ray. In this paper, we propose an intuitive calculation of refracted vector based on Huygens' principle. For polarization state tracking, we used direct tracking method as in thin-film interference.

3.2 Implementation

3.2.1 Refraction by Huygens's principle

As we mentioned before, light ray splits into ordinary ray and extraordinary ray in birefringent materials ([Donnelly et al. 2006]). To understand the difference between ordinary ray and extraordinary ray, we need to think *wave normal* of light and *propagation direction of light* separately. The wave normal, which is perpendicular to wave front of light wave, of ordinary ray is the same as propagation direction of light. Since ordinary ray follows Snell's law, refracted vector can be calculated as

$$\vec{t_o} = \frac{\eta_i(\vec{r} - (\vec{r}\cdot\vec{n})\vec{n})}{\eta_o} - \vec{n}\sqrt{1 - \frac{\eta_i{}^2(1-(\vec{r}\cdot\vec{n})^2)}{\eta_o{}^2}} \qquad (20)$$

However, the wave normal of extraordinary ray is *different* from

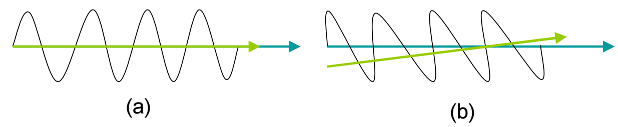

Figure 14: *Wave normal and ray direction for (a) ordinary ray and (b) extraordinary ray. Note that the wave normal and ray direction is different in the case of extraordinary ray.*

propagation direction of light, so direction of refracted ray follows Snell's law (Figure 14). More precisely, Snell's law is actually a law for wave normals. We cannot use this law to calculate refracted extraordinary ray direction because its wave normal is different from ray direction. Therefore, here we propose alternative way to calculate refraction vector based on *Huygens' principle*. Huygens' principle describes propagation of wave fronts by means of secondary wavelets. By Huygens' principle, we can find a propagated wave front in arbitrary shape by considering set of spherical secondary wavelets at the point set on the current wave front. The shape of wave front at the next moment is defined by enveloping surface of set of spherical secondary wavelets. We apply this principle to propagation of rays in anisotropic material.

For simplicity, assume light enters to uniaxial material at right angle as in Figure 15. Since refractive index of ordinary ray is independent of its direction, propagation of secondary wavelets at the point P_i will be spherical. Huygens' principle says wave front in this case is parallel to the material surface (Figure 15). For ordinary ray, propagation direction, which is defined by connecting P_i to the contact point of secondary wavelet and wave front P_o ([Gledhill 2000]), is the same as wave normal (normal vector of wave front) in this case. In contrast, since refractive index of extraordinary ray

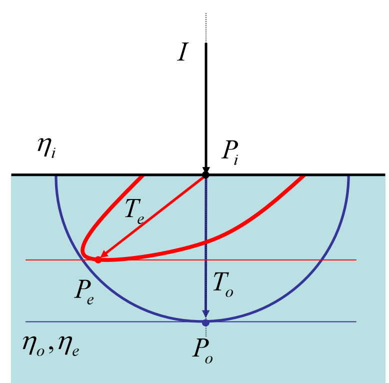

Figure 15: *Huygens' principle for uniaxial material. The blue line shows secondary wavelet for ordinary ray, and the red line shows secondary wavelet for extraordinary ray. Note that the wave front is parallel to the surface for both rays, but the direction of rays are different.*

is dependent on its direction in the case of extraordinary ray, propagation of secondary wavelet at the point will be ellipsoidal. Since refractive index is inversely proportional to speed of light in material, the shape of secondary wavelet is defined as follows ([Gledhill

2000])

$$x^2 + \frac{{\eta_o}^2}{{\eta_e}^2} y^2 + z^2 - 1 = 0 \tag{21}$$

Here we use crystal coordinate to define this ellipsoid.

As in Figure 15, Huygens' principle again says wave front for extraordinary ray is parallel to the material surface in this case. This is valid by considering Snell's law says wave normal is right angle to the surface for both ordinary and extraordinary rays. However, propagation of direction is different. As the same as in ordinary ray, propagation direction is by connecting P_i to the contact point of secondary wavelet and wave front P_e (Figure 15).

To calculate such direction, we can convert this problem as a maximization problem of the distance along wave normal for surface points on the ellipsoidal wavelet. Therefore, we can use Lagrange multiplier method to solve this problem and obtain:

$$\begin{aligned} \vec{T}_e &= \frac{\left(W_{ex}, \frac{{\eta_o}^2}{{\eta_e}^2} W_{ey}, W_{ez}\right)}{2\lambda} \\ \lambda &= \pm \frac{\sqrt{{W_{ex}}^2 + \frac{{\eta_o}^2}{{\eta_e}^2} {W_{ey}}^2 + {W_{ez}}^2}}{2} \\ \vec{W}_e &= \left(W_{ex}, W_{ey}, W_{ez}\right) \end{aligned} \tag{22}$$

where $\vec{T}_e$ is direction of propagation and $\vec{W}_e$ is wave normal calculated by Snell's law. Note that $\vec{T}_e$ is vector in crystal coordinate system. We need to transform $\vec{T}_e$ into the world coordinates after this calculation. Since refractive index varies according to direction for extraordinary ray, we need to take account for this variation to calculate $\vec{W}_e$. Given wave normal $\vec{W}_e$ and optical axis $\vec{o}$, refractive index for extraordinary ray is calculated by ([Hassler et al. 2002]):

$$\eta_c(\vec{W}_e) = \frac{\eta_o \eta_e}{\sqrt{(\vec{W}_e \cdot \vec{o})^2 {\eta_e}^2 + (1 - (\vec{W}_e \cdot \vec{o})^2) {\eta_o}^2}} \tag{23}$$

Notice that $\eta_c(\vec{W}_e)$ is different from η_e. While η_e is property of material, $\eta_c(\vec{W}_e)$ is function of angle between wave normal and optical axis. To calculate $\vec{W}_e$, we first calculate $\vec{W}_e$ based on Equation 20 with $\eta_c(\vec{W}_e) = \eta_o$. Next we calculate $\eta_c(\vec{W}_e)$ and again calculate $\vec{W}_e$ based on Equation 20 with $\eta_c(\vec{W}_e)$. We found just 5-10 numbers of iteration of this process is enough to find $\vec{W}_e$. While we have not applied this calculation to reflection inside uniaxial materials, we expect it can be easily extended to reflection.

3.2.2 Fresnel Reflection/Refraction

Fresnel coefficients shown in Equation 10 are for interface between isotropic materials. For birefringent materials, we need to consider generalized Fresnel equations for interface between anisotropic materials. In exact form, generalized Fresnel coefficients can be solved by the following matrix equation as follows ([McClain et al. 1993]).

$$\begin{aligned} M &= \begin{bmatrix} -\vec{v}_1 \cdot \vec{e}_r^o & -\vec{v}_1 \cdot \vec{e}_r^e & \vec{v}_1 \cdot \vec{e}_t^o & \vec{v}_1 \cdot \vec{e}_t^e \\ -\vec{v}_2 \cdot \vec{e}_r^o & -\vec{v}_2 \cdot \vec{e}_r^e & \vec{v}_1 \cdot \vec{e}_t^o & \vec{v}_1 \cdot \vec{e}_t^e \\ -n_r^o \vec{v}_1 \cdot \vec{h}_r^o & -n_r^e \vec{v}_1 \cdot \vec{h}_r^e & n_t^o \vec{v}_1 \cdot \vec{h}_t^o & n_t^e \vec{v}_1 \cdot \vec{h}_t^e \\ -n_r^o \vec{v}_2 \cdot \vec{h}_r^o & -n_r^e \vec{v}_2 \cdot \vec{h}_r^e & n_t^o \vec{v}_2 \cdot \vec{h}_t^o & n_t^e \vec{v}_2 \cdot \vec{h}_t^e \end{bmatrix} \\ M \begin{bmatrix} \alpha_r^o \\ \alpha_r^e \\ \alpha_t^o \\ \alpha_t^e \end{bmatrix} &= \begin{bmatrix} \vec{v}_1 \cdot \vec{e}_i \\ \vec{v}_2 \cdot \vec{e}_i \\ n_i \vec{v}_1 \cdot \vec{h}_i \\ n_i \vec{v}_2 \cdot \vec{h}_i \end{bmatrix} \end{aligned} \tag{24}$$

Here $e_{t,r,i}^{\vec{o},e}$ is vibration direction of electric wave and $h_{t,r,i}^{\vec{o},e}$ is vibration direction of electric wave. Since we only care about anisotropic/isotropic interface rather than anisotropic/anisotropic interface in this paper, solving generalized Fresnel equations is redundant. Therefore, we use approximation proposed by [Guy and Soler 2004], which uses Fresnel coefficients for isotropic material as follows.

$$\begin{aligned} \begin{bmatrix} \alpha_r^{oo} \\ \alpha_r^{eo} \\ \alpha_t^{oo} \\ \alpha_t^{eo} \end{bmatrix} &= \begin{bmatrix} R_s \\ 0 \\ T_s \\ 0 \end{bmatrix} \\ \begin{bmatrix} \alpha_r^{oe} \\ \alpha_r^{ee} \\ \alpha_t^{oe} \\ \alpha_t^{ee} \end{bmatrix} &= \begin{bmatrix} 0 \\ R_p \\ 0 \\ T_p \end{bmatrix} \end{aligned} \tag{25}$$

Here Fresnel coefficients R_s, R_p, T_s, T_p defined on p-s coordinate system where normal vector is replaced by optical axis. Each coefficient, $\alpha_r^{oo,eo}, \alpha_t^{oo,eo}$ denotes contribution of ordinary incident ray to exitant reflection/refraction, and $\alpha_r^{oe,ee}, \alpha_t^{oe,ee}$ denotes contribution of extraordinary incident ray to exitant reflection/refraction. Coefficient is null (zero) for cross cases like r^{oe}, t^{eo}. Therefore, ordinary ray only contributes to ordinary direction (perpendicular to optical axis) and extraordinary ray only contributes to extraordinary direction (parallel to optical axis). As [Guy and Soler 2004] have shown, this approximation works well for real materials where difference between η_o and η_e is not so large.

3.3 Results and Discussion

Figure 16 shows rendered images of birefringence material for different direction of optical axis. We used 64 samples to calculate results by unpolarized light. For all images, we only rendered refracted components to emphasize effects of birefringence. For the image on the left in Figure 16, even if light enters surface at right angle (center of the sphere),extraordinary exhibits refraction, and it results in doubling of refracted image.This effect cannot be achieved just by using Snell's law and well matched to real behavior of extraordinary ray (Figure 13). As we can see in Figure 16, even if refractive indices (η_o and η_e) and the geometry are the same, optical axis changes behaviour of extraordinary ray as we expected.

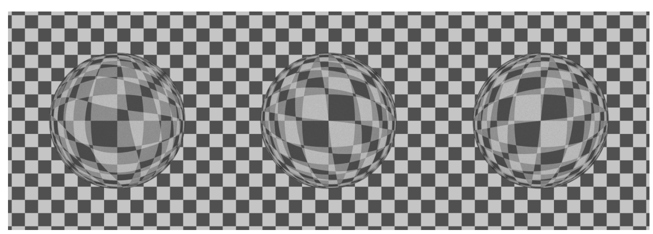

Figure 16: *Rendered images of birefringence material. Here we use $\eta_o = 1.658$ and $\eta_e = 1.486$, which correspond to calcite. These images use different optical axis as follows. Left: $\frac{(1,1,1)}{\sqrt{3}}$, Center: $\frac{(1,1,0)}{\sqrt{2}}$, Right: $\frac{(-1,1,0)}{\sqrt{2}}$*

Table 1 and Table 2 show numerical validations of our method. We compared results calculated by our method by results from [Beyerle and Stuart 1998]. As we can see in Table 2, refracted vectors (propagation vector and wave normal) are almost matched to the result from [Beyerle and Stuart 1998].

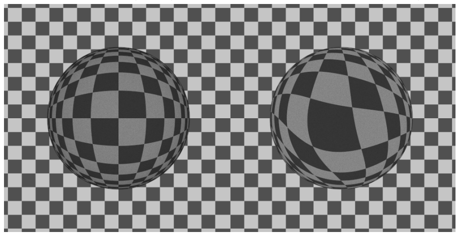

Figure 17: *Separation of ordinary ray and extraordinary ray by polarizer. Here we used polarizers defined by optical axis. Polarizer for ordinary ray is perpendicular to optical axis and polarizer for extraordinary ray is parallel to optical axis.*

Figure 17 shows result by applying polarizer between the camera and the object. Since ordinary ray and extraordinary ray are orthogonally and linearly polarized, the images in Figure 17 can separate images obtained by ordinary ray and extraordinary ray by polarizer.

	θ_i	t_x	t_y	t_z	s_x	s_y	s_z
Our Method	30	0.945087	0.326806	0.002785	0.946584	0.322457	0.000000
	45	0.888261	0.459334	0.002229	0.890116	0.455733	0.000000
	60	0.828028	0.560684	0.001739	0.829902	0.557909	0.000000
Beyerle et al.	30	0.945516	0.325546	0.004415	0.946288	0.323325	0.000000
	45	0.888783	0.458306	0.004536	0.889279	0.457365	0.000000
	60	0.828391	0.560131	0.004565	0.828365	0.560189	0.000000

Table 1: *Numerical validation of our method. Here* $\vec{t} = (t_x, t_y, t_z)$ *is refracted propagation direction and* $\vec{s} = (s_x, s_y, sz)$ *is refracted wave normal. We used surface normal* $(1,0,0)$*, optical axis* $(075, 0.5, 0.433)$ *and refractive indices* $\eta_o = 1.54426$*,* $\eta_e = 1.55335$ *as Beyerle et al.* θ_i *is angle between incident direction and surface normal and z component of incident direction is* 0.

	θ_i	$\vec{t}$	$\vec{s}$
inner product	30	0.999998	0.999999
	45	0.999997	0.999998
	60	0.999996	0.999996

Table 2: *Numerical errors of our method. The values are calculated by inner product of vectors in Table 1.*

4 Conclusion

In this paper, we presented how to incorporate polarization, where light behaves as wave, to conventional geometric optics based ray tracing. For applications of polarization, we presented methods to render thin-film interference and birefringence. To handle interference as well as polarization, we proposed direct tracking of polarization state, rather than using Stokes vector. As a result, we found appearance of interference is largely affected by incorporating polarization. For birefringence, we proposed a method to calculate refracted/reflected direction of extraordinary ray based on Huygens' principle. As a result, we show rendering of birefringent material by using ray tracing with direct tracking polarization. Future work includes thin-film interference with multiple layers, extension to biaxial material with absorption and integration of polarization into global illumination. , To the best of our knowledge, here is no research that fully incorporates polarization in global illumination system, so it may be interesting direction of future research.

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