Combined Lagrangian-Eulerian Approach for Accurate Advection

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1 Introduction

In this poster, we propose a new algorithm to accurately calculate advection equations. Even the latest fluid simulations [Fattal and Lischinski 2004] have been suffering from the numerical errors in advection equations. These numerical errors cause mass dissipation and motion damping of fluid, as a result, the detail of fluid animation is filtered out. Unlike some previous methods [Enright et al. 2002], the proposed method can deal with not only fields with sharp boundaries but also blurry fields (*i.e.*no boundaries) very accurately. The proposed method mainly consists of the advection phase and the non-advection phase. In the advection phase, the method calculates the current field from the initial field by using the combined method of lagrangian method and eulerian method. In the non-advection phase, influences of non-advection terms are added back to the initial field by using the mapping functions of advection equations. The method can calculate highly detailed fluid animations using a relatively coarse grid.

2 Methods

In the advection phase, we need to calculate the advection, which can be expressed as the identical equation along the trajectory of particle $\vec{p}(\vec{x},t)$ as:

$$a(\vec{x},t_{n+1}) = a(\vec{p}(\vec{x},-\Delta t),t_n) \tag{1}$$

Semi-lagrangian methods, which are popular methods to calculate the advection for fluid animations, use equation(1) directly. In these methods, we need to interpolate $a(\vec{x},t_n)$ at positions of $\vec{p}(\vec{x},-\Delta t)$ when calculating equation(1), because $a(\vec{x},t_n)$ is usually given as a discretized form for numerical calculations. These interpolation errors are the major factor of the numerical errors of advections. In the proposed method, we do not use equation(1) directly. Instead, $\vec{G}(\vec{x},t_n)$ is introduced such as:

$$\begin{aligned} &a(\vec{x},t_n) = a\left(\vec{G}(\vec{x},t_n),0\right) \\ &\left(\vec{G}(\vec{x},t_n) \equiv \vec{p}(\vec{x},-t_n),\vec{G}(\vec{x},0) \equiv \vec{x}\right) \end{aligned} \tag{2}$$

By using equation(2), the current field $a(\vec{x},t_n)$ is always reconstructed from the initial field $a(\vec{x},0)$, so the numerical errors do not accumulate in time. The update process of $\vec{G}(\vec{x},t_n)$ is performed by tracking mass-less particles that have positions of $\vec{G}(\vec{x},t_n)$. In the non-advection phase, we need to add the influences of non-advection terms to the retained initial field. Simply adding non-advection terms to the current field causes no effect to the fields at the next time step, because our method always reconstructs the current field by the initial field. Since using $\vec{G}(\vec{x},t_n)$ to this non-advection phase forms a forward mapping, which tends to make holes in the destination of mappings, the inverse of $\vec{G}(\vec{x},t_n)$ as $\vec{F}(\vec{x},t_n)$ is introduced as:

$$\begin{aligned} &\hat{a}(\vec{x},0) = a(\vec{x},0) + d\left(\vec{F}(\vec{x},t_n),t_n\right) \\ &\left(\vec{F}(\vec{x},t_n) \equiv \vec{p}(\vec{x},t_n),\vec{F}(\vec{x},0) \equiv \vec{x}\right) \end{aligned} \tag{3}$$

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where $d(\vec{x},t_n)$ is the current non-advection terms, such as pressure terms and diffusion terms. By using equation(3), non-advection terms are added back in time by using $\vec{F}(\vec{x},t_n)$ as backward mapping. From the property of advection equations, $\vec{F}(\vec{x},t_n)$ can be easily obtained by backwardly tracing the mass-less particles, similar to the calculation of $\vec{G}(\vec{x},t_n)$. We need to reset these mapping functions when the distortions of mapping functions became large. This process, called *remapping*, copies the current field into the initial field and initializes the mapping functions. The concept of the remapping is similar to Particle reseeding [Enright et al. 2002].

3 Results

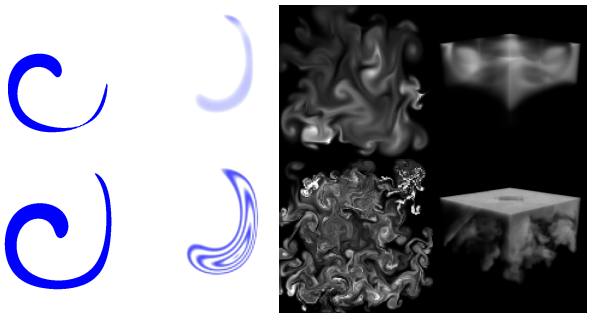

Figure 1: Comparisons of semi-lagrangian method and our method (at the same resolution: 100^2, 100^2, 256^2, 64^3)

Figure 1 shows comparisons of various calculations between semi-lagrangian method (top row) and our method (bottom row). Both of the methods are 1st-order in space and 2nd-order in time. The left two columns show the results of 2D deformation tests [Enright et al. 2002] for the sharp field and blurry field. Note that the proposed method calculates the advection of both blurry fields and sharp fields very accurately. The next column shows the results of 2D fluid simulation, and the rightmost column shows the results of 3D smoke simulation. Notice that the semi-lagrangian method largely blurs both density and velocity fields. By using our method, higher-frequency components, such as small vortexes, are very well depicted. On the contrary, semi-lagrangian methods always produce blurry results. Since our method can calculate advections of blurry fields like density of smoke, the results illustrate the strength of our method.

References

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