Combinatorics of Hybrid Sets

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***Abstract*—Hybrid sets are generalizations of sets and multisets, in which the multiplicities of elements can take any integers. This construction was proposed by Whitney in 1933 in terms of characteristic functions. Hybrid sets have been used by combinatorists to give combinatorial interpretations for several generalizations of binomial coefficients and Stirling numbers and by computer scientists to design fast algorithms for symbolic domain decompositions. We present in this paper some combinatorial results on subsets and partitions of hybrid sets.**

I. Introduction

Set theory has been the logic foundation of mathematics since Cantor's work [1] in the 1870s. Boole in his book [2] initialized the algebraic study of set theory and this study was continued by Whitney in the 1930s [3], [4]. Multisets are generalization of sets in which elements can appear a positive finite number of times, which have many applications in mathematics, computer science, and molecular computing [5], [6]. The word *multiset* was first coined by de Bruijn in the 1970s and many other names, such as bags [7] and heaps are also used for this construction (see the note by Knuth in [8, p. 551]). Blizard in [9], [10] investigated this construction from the logical and historical point of view. More recently, Syropoulos wrote a nice survey [11] on the mathematics of multisets. Hybrid sets naturally generalizes multisets by allowing their elements to occur any *integer* number of times. This construction was first proposed by Whitney in 1933 via characteristic functions [4]. In 1992, Loeb and his collaborators presented a combinatorial interpretation for the generalized binomial coefficients and generalized Stirling numbers in [12], [13], [14].

Recently, hybrid sets have been used to simplify the treatment of expressions defined by cases in symbolic computation. They allow expressions to be constructed without regard to the relationship between regions, by allowing irrelevant inclusions to be later excluded. This reduces the complexity of the number of cases in symbolic expressions to be linear in the number of operations rather than exponential [15]. These ideas can be applied equally to the arithmetic of structured matrices with parts of symbolic dimension or piece-wise functions with subdomains with boundaries defined symbolically. In addition, the use of hybrid sets allows inclusion-exclusion to be handled as such, rather than introducing concepts such as negative area or oriented regions in integration [16].

This paper is motivated by the question: can we further extend the enumerative combinatorial results on sets and multisets to the hybrid setting? We first present a more direct characterization of all possible subsets of hybrid sets (Theorem 11), where subsets are defined by Loeb in a non-trivial way. In contrast, our idea is to define a special class of such subsets, called natural subsets (Definition 10), and we solve the enumeration problem on natural subsets (Proposition 12). In the last part of this paper, we define the notion of partitions and recursive partitions of hybrid sets (Definitions 13 and 15). The enumeration of recursive partitions is reduced to that of multisets, for which we sketch a generating-function based method by Devitt and Jackson [17] and then use it to derive an elegant formula of Bender in [18].

II. Hybrid sets

A unified way to define sets, multisets and hybrid sets is using characteristic functions over a universal set as did by Whiteney in [4]. Let $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ denote the set of all nonnegative integers. Let $\mathfrak{U}$ be the universal set (the universe of discourse). A classical *set* S is defined as a function $S : \mathfrak{U} \to \{0, 1\}$ and a *multiset* M is defined as a function $M : \mathfrak{U} \to \mathbb{N}$. The notion of sets and multisets is generalized as follows.

Definition 1: A *hybrid set* H is defined as a function $H : \mathfrak{U} \to \mathbb{Z}$. For any $u \in \mathfrak{U}$, we denote by $u \in H$ if $H(u) \neq 0$ and $u \notin H$ otherwise.

Example 2: Let $\mathfrak{U} := \{a, b, c\}$. Then $S : \mathfrak{U} \to \{0, 1\}$ with $S(a) = S(b) = 1$ and $S(c) = 0$ is a set, classically denoted by $\{a, b\}$; $M : \mathfrak{U} \to \mathbb{N}$ with $M(a) = 1, M(b) = 2$, and $M(c) = 3$ is a multiset, usually denoted by $\{a, b^2, c^3\}$; $H : \mathfrak{U} \to \mathbb{Z}$ with $H(a) = -1, H(b) = 2$, and $H(c) = 0$ is a hybrid set, which will be later denoted by $\{b^2 \mid a\}$.

The zero function from $\mathfrak{U}$ to $\mathbb{Z}$ is called the *empty hybrid set* over $\mathfrak{U}$, denoted by $\emptyset$. Let H be a hybrid set over $\mathfrak{U}$. For any $u \in \mathfrak{U}$, we call the number $H(u)$ the *multiplicity* of u in H and the set $S_H := \{u \in \mathfrak{U} \mid H(u) \neq 0\}$ the *supporting set* of H. Let $S_H^+ := \{u \in \mathfrak{U} \mid H(u) > 0\}$ and $S_H^- := \{u \in \mathfrak{U} \mid H(u) < 0\}$. We define two hybrid sets H^+ and H^- by

$$H^+(u) = \begin{cases} H(u), & u \in S_H^+; \\ 0, & u \notin S_H^+ \end{cases}$$

and

$$H^-(u) = \begin{cases} H(u), & u \in S_H^-; \\ 0, & u \notin S_H^- \end{cases}$$

for all $u \in \mathfrak{U}$ and we call them the *positive part* and *negative part* of H, respectively. If S_H is a finite set, we call the sums $\sum_{u\in\mathfrak{U}} H(u)$ and $\sum_{u\in\mathfrak{U}} |H(u)|$ the *cardinality* and *weight* of H, respectively. For a finitely supported hybrid set H, we will write it as

$$H = \{a_1^{m_1}, \ldots, a_s^{m_s} \mid b_1^{n_1}, \ldots, b_t^{n_t}\},$$

where the a_i's are elements with positive multiplicity and the b_j's are elements with negative multiplicity in H. A hybrid set of the form $\{a_1, \ldots, a_s \mid\}$ or $\{\mid b_1, \ldots, b_t\}$ is called a *new* set [12, Definition 2.2]. A hybrid set is said to be *proper* if its negative part is not empty.

Example 3: Let $\mathfrak{U} = \{a, b, c, d\}$. The hybrid sets $\{a, b, c \mid\}$ and $\{\mid b, c, d\}$ are new sets, $\{a, b \mid c\}$ is not a new set but a proper hybrid set. The hybrid set $\{a^2, b \mid\}$ is neither new nor proper.

As in the case of classical sets, we can introduce some basic operations on hybrid sets.

Definition 4 ([15]): Let H_1, H_2 be two hybrid sets over the universal set $\mathfrak{U}$. The sum $H_1 \oplus H_2$ of H_1 and H_2 is defined by

$$(H_1 \oplus H_2)(u) = H_1(u) + H_2(u) \text{ for all } u \in \mathfrak{U}.$$

Similarly, the product $H_1 \otimes H_2$ and the difference $H_1 \ominus H_2$ of H_1 and H_2 can also be defined pointwise.

Remark 5: The set $\mathbb{Z}^{\mathfrak{U}}$ of all hybrid sets over $\mathfrak{U}$ inherits the $\mathbb{Z}$-module structure from $\mathbb{Z}$ (This fact was proved in [15]). For any hybrid set H, we have $H = H^+ \oplus H^-$.

III. Subsets

In order to be useful in combinatorics, Loeb in [12] gave an intriguing definition of the inclusion relation among hybrid sets. To present this, we first recall a partial ordering in $\mathbb{Z}$.

Definition 6 ([12]): For integers $a, b \in \mathbb{Z}$, we say that $a \lessdot b$ if $a \leqslant b$ and either both a and b are nonnegative or both a and b are negative.

The relation $\lessdot$ defined above is a partial ordering in $\mathbb{Z}$ but not total, since two integers a, b with $a < 0$ and $b \geq 0$ are not comparable.

A. Loeb's subsets of hybrid sets

We now recall the notion of subsets of a hybrid set as follows.

Definition 7 ([12]): Let G and H be two hybrid sets over $\mathfrak{U}$. We say that G is a *subset* of H, denoted by $G \subseteq H$, if either $G(u) \lessdot H(u)$ for all $u \in \mathfrak{U}$, or $H(u) - G(u) \lessdot H(u)$ for all $u \in \mathfrak{U}$.

By [12, Proposition 5.1], the inclusion relation $\subseteq$ defined above is a well-defined partial ordering in $\mathbb{Z}^{\mathfrak{U}}$.

Example 8: Let $\mathfrak{U} = \{a, b, c, d\}$ and $H = \{a^2, b \mid c\}$. Then $\{a, b \mid c^2\}$ and $\{a, c \mid\}$ are subsets of H, but $G := \{b, c \mid a\}$ is not, since -1 and 2 are not comparable w.r.t. $\lessdot$ and $H(a) - G(a) \not\lessdot H(a)$.

Unlike the case of classical sets, a hybrid set may have infinitely many subsets if the negative part is not empty. Indeed, for all $i \in \mathbb{N}$, $\{a, b \mid c^i\}$ are subsets of H. It may also happen that $H = H_1 \oplus H_2$, but neither H_1 nor H_2 is subset of H. For example, let $H = \{a, b \mid c^3\}$, $H_1 = \{a, b \mid c\}$ and $H_2 = \{\mid c^2\}$. Then $H = H_1 \oplus H_2$, but H_1 and H_2 are not subsets of H.

Proposition 9: Let G, H be two hybrid sets over $\mathfrak{U}$ with $G \subseteq H$. Then $S_G \subseteq S_H$ and $H \ominus G$ is also a subset of H.

Proof. Suppose that there exists $u \in \mathfrak{U}$ such that $u \in S_G$ but $u \notin S_H$. Thus $G(u) \neq 0$ and $H(u) = 0$. This implies that either $G(u) > H(u)$ if $G(u) > 0$, or $G(u)$ and $H(u)$ are not comparable or $H(u) - G(u) > H(u)$ if $G(u) < 0$, which contradicts with the assumption that $G \subseteq H$. Therefore, we get $S_G \subseteq S_H$. Let $L = H \ominus G$. Since $G \subseteq H$ and $G(u) = H(u) - L(u)$ for all $u \in \mathfrak{U}$, we get either $L(u) \lessdot H(u)$ for all $u \in \mathfrak{U}$ or $H(u) - L(u) \lessdot H(u)$ for all $u \in \mathfrak{U}$. Thus $L \subseteq H$. ∎

The classical binomial coefficients $\binom{n}{k}$ counts the number of k-element subsets of a set of n elements. Several generalizations of binomial coefficients were proposed in [19], [12]. One such generalizations is called *Roman binomial coefficients* in [12], defined by the formula

$$\binom{n}{k}_R := \lim_{\epsilon\to 0} \frac{\Gamma(n+1+\epsilon)}{\Gamma(k+1+\epsilon)\Gamma(n-k+1+\epsilon)},$$

where $\Gamma(z)$ denotes the classical Gamma function. In terms of subsets of hybrid sets, Loeb gave a combinatorial interpretation of such binomial coefficients by showing that $\binom{n}{k}_R$ counts the number of k-element hybrid sets which are subsets of a new set H of cardinality n [12, Theorem 5.2].

B. Characterization and enumeration of Loeb's subsets

Definition 10: Let G and H be two hybrid sets over $\mathfrak{U}$. We call G a *natural subset* of H if $S_G \subseteq S_H$ and $0 \leq G(u) \leq H(u)$ for all $u \in S_H^+$ and $G(u) \geq 0$ if $u \in S_H^-$. The difference $H \ominus G$ is called the *complement* of G in H.

Note that all natural subsets of a hybrid set H are subsets of H and they are also multisets. We now can characterize all of the possible subsets of a hybrid set via its natural subsets, which is the main theorem in this section.

Theorem 11 (Hybrid subset characterization): Let G and H be two hybrid sets over $\mathfrak{U}$. Then G is a subset of H if and only if either G is a natural subset of H or G is the complement of some natural subset of H.

Proof. The sufficiency follows immediately from Definition 10 and Proposition 9. For the other direction assume that $G \subseteq H$. Suppose $G(u) \lessdot H(u)$ for all $u \in \mathfrak{U}$. We claim that $L = H \ominus G$ is a natural subset of H. For any $u \in S_H^+$, $H(u) > 0$. Since $G(u)$ and $H(u)$ are comparable, $G(u) \geq 0$. By the inequality $G(u) \leq H(u)$, we have

$$0 \leq L(u) = H(u) - G(u) \leq H(u).$$

For any $u \in S_H^-$, $H(u) < 0$. Since $G(u)$ and $H(u)$ are comparable, $G(u) < 0$. Then $G(u) \leq H(u)$ implies that

$$L(u) = H(u) - G(u) \geq 0.$$

So L is a natural subset of H. By the symmetry, we can show that G is a natural subset of H if $H(u) - G(u) \lessdot H(u)$ for all $u \in \mathfrak{U}$. This completes the proof. ∎

Let H be a finitely supported hybrid set of the form

$$H = \{a_1^{m_1}, \dots, a_s^{m_s} \mid b_1^{n_1}, \dots, b_t^{n_t}\}.$$

Let $\lambda_H(k)$ denote the number of all natural subsets of cardinality k of H. Let $[P]$ denote the Iverson bracket for a statement P, defined by $[P] = 1$ if P is true and $[P] = 0$ otherwise. We derive an explicit formula for $\lambda_H(k)$ in the following proposition.

Proposition 12:

$$\lambda_H(k) = k! \sum_{i_1+i_2+\cdots+i_{s+1}=k} \binom{t+i_{s+1}-1}{t-1} \prod_{j=1}^{s} [i_j \le m_j].$$

Proof. By definition, any natural subset of H consists of the a_i's with multiplicity nonnegative and at most m_i and the b_j's with any nonnegative multiplicity. By an old idea of Euler, we get the *crude generating function* (which was coined by MacMahon [20, Vol. II, p. 93]) for $\lambda_H(k)$ of the form

$$f(x) := \sum_{k\ge 0} \lambda_H(k) x^k = \prod_{i=1}^{s} (1 + x + \cdots + x^{m_i}) \cdot \frac{1}{(1-x)^t}.$$

By the multinomial theorem for iterated derivatives of a product of functions, we have for all kth differentiable functions $f_1(x), \dots, f_s(x)$,

$$(f_1 \cdot f_2 \cdots f_s)^{(k)} = \sum_{i_1+i_2+\cdots+i_s=k} \binom{k}{i_1, i_2, \dots, i_s} \prod_{j=1}^{s} f_j^{(i_j)},$$

where

$$\binom{k}{i_1, i_2, \dots, i_s} = \frac{k!}{i_1! i_2! \cdots i_s!}.$$

Then it is easy to verify that $p^{(k)}(0) = p_k k! [k \le d]$ for any polynomial $p = \sum_{i=0}^{d} p_i x^i$ and for any $i \in \mathbb{N}$, we have

$$((1-x)^{-t})^{(i)}|_{x=0} = t(t+1)\cdots(t+i-1) \triangleq t^{(i)}.$$

Since $\lambda_H(k) = f^{(k)}(0)/k!$, we then obtain the formula

$$\begin{aligned}
\lambda_H(k) &= \sum_{i_1+\cdots+i_{s+1}=k} \binom{k}{i_1, \dots, i_{s+1}} \prod_{j=1}^{s} (i_j! [i_j \le m_j]) t^{(i_{s+1})} \\
&= \sum_{i_1+\cdots+i_{s+1}=k} k! \frac{t^{(i_{s+1})}}{i_{s+1}!} \prod_{j=1}^{s} [i_j \le m_j] \\
&= k! \sum_{i_1+\cdots+i_{s+1}=k} \binom{t+i_{s+1}-1}{t-1} \prod_{j=1}^{s} [i_j \le m_j].
\end{aligned}$$

This completes the proof. ∎

By Theorem 11, an explicit formula for the number of all subsets of cardinality k of H can also be derived in a similar way.

IV. PARTITIONS

Partitions of sets and integers have been studied since Euler and now form an active research area which connects number theory and combinatorics. Comprehensive surveys on partitions have been given in [21], [22], [23]. The partitions of multisets was first investigated by Barón, Comtet, and de Bruijn in the 1960s, and later was studied more systematically by Bender and his collaborators in [18], [24]. A generating-function based approach for the enumeration of partitions of multisets was presented by Reilley in [25] and by Devitt and Jackson in [17]. For more recent works, one can also see [26], [23]. In this section, we generalize the partition problems to the case of hybrid sets.

A. Partitions and recursive partitions

Definition 13 (Partition): Let H be a hybrid set over $\mathfrak{U}$. A collection of nonempty hybrid sets over $\mathfrak{U}$, say $\{H_1, H_2, \dots, H_s\}$, is called a *partition* of H if for all i with $1 \le i \le s$, $H_i \subseteq H$ and

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_s.$$

The H_i's are called the *components or blocks* of H and we will also say s-partition to specify the number of parts.

Remark 14: (i) when H is a classical set, we obtain (for free) the disjointness property among the H_i's. That is $H_i \otimes H_j = \emptyset$ for all $1 \le i < j \le s$. (ii) A general hybrid set may have infinitely many partitions, since it may have infinitely many subsets and $H = G \oplus (H \ominus G)$ is a partition for any subset G of H.

Definition 15 (Recursive partition): A partition $H = H_1 \oplus \cdots \oplus H_s$ of H is said to be *recursive* if for any subset $\{i_1, \dots, i_t\}$ of $\{1, \dots, s\}$, the sum $H_{i_1} \oplus \cdots \oplus H_{i_t}$ also form a partition, i.e., all H_{i_j}'s are subsets of $H_{i_1} \oplus \cdots \oplus H_{i_t}$.

Example 16: Let $H = \{a^2, b \mid c\}$. Then the partition

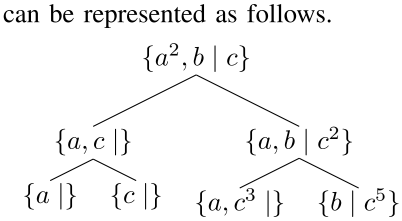
$$H = \{a \mid\} \oplus \{c \mid\} \oplus \{a, c^3 \mid\} \oplus \{b \mid c^5\}$$

is recursive but the partition

$$H = \{a \mid\} \oplus \{c^2 \mid\} \oplus \{a, b \mid c\} \oplus \{\mid c^2\},$$

since $\{a, b \mid c\}$ and $\{\mid c^2\}$ are not subsets of the hybrid set $\{a, b \mid c\} \oplus \{\mid c^2\}$.

For any recursive partition, we can construct a binary tree as follows. If $H = H_1 \oplus \cdots \oplus H_s$ is a recursive partition, then Theorem 11 implies that either H_i is natural or $H \ominus H_i$ is natural. We set H as root of the tree and split it into two branches with the unnatural subsets always left to the natural ones. It may happen that both parts are natural in which case any order is allowed. We can continue this splitting. For example, a tree associated with the recursive partition in Example 16 can be represented as follows.



B. Counting partitions and recursive partitions

In order to make the counting problems meaningful, we assume from now on that the hybrid sets in our discussion are all finitely supported and we also impose some constraints on the components of a partition.

Definition 17 (Deviation): Let G be a subset of the hybrid set H. For an element $u \in H$, we call the value

$$\delta_u(G, H) = \begin{cases} H(u) - G(u), & \text{if } H(u) \geq 0; \\ H(u) - G(u), & \text{if } H(u) < 0 \,\&\, G(u) < 0; \\ G(u), & \text{if } H(u) < 0 \,\&\, G(u) \geq 0. \end{cases}$$

the *deviation* of G at u from H. The value

$$\delta(G, H) := \max\{\delta_u(G, H) \mid u \in H\}$$

is called the *deviation* of G from H.

Remark 18: For a multiset H, the deviations of its subsets from H are finite and bounded by the largest multiplicity of elements in H. But the deviation can be arbitrary large if H has a non-trivial negative part.

We let $[z^k]f(z)$ denote the coefficient of the term z^k in the formal power series $f \in K[[z]]$.

Problem 19: For a given hybrid set H, compute the number of k-partitions of H with the deviations of all components bounded by d. We denote this number by $\mu_H(k, d)$.

It is still an open problem to find a closed form or find the generating funstions for $\mu_H(k, d)$, even for the the case of multisets. There are some special cases of this problem that have been solved. When $H = \{a_1, \ldots, a_n|\}$, the numbers $\mu_H(k, 1)$ are the Stirling numbers of the second kind, $S(n, k)$, and their generating function ((see [27, p. 74])) is

$$f(x, y) := \sum_{n \geq 0} \sum_{k \geq 0} S(n, k) x^k \frac{y^n}{n!} = \exp(x(\exp(y) - 1)).$$

When $H = \{a_1^m, \ldots, a_s^m \;|\}$, i.e., a multiset in which all elements are of the same multiplicity, four kinds of different partition problems have been studied in [18], [25], [17], [24].

We now show that the recursive partitions with components having bounded deviation can be reduced to partitions of multisets. Indeed, If $H = H_1 \oplus \cdots \oplus H_s$ is a recursive partition, then only one component of H is not natural. Suppose there are two components H_i and H_j are not natural. Then both H_i and H_j are subsets of $H_i \oplus H_j$. By Theorem 11, at least one of H_i and H_j is natural, which is an contradiction. Therefore, the counting of recursive partitions of H splits into two steps: 1) count the number of unnatural subsets of H; 2) for each such subset G, count the number of partitions of the multiset $H \ominus G$. Then the number of recursive partitions is obtained by the product rule for counting.

C. Multiset partitions: an example

We now illustrate an example for multiset partitions which applies the method in [17] to derive Bender's formula [18] on the numbers $v_H^*(k)$ of k-partitions of H with multiset components.

Let $[s] := \{1, 2, \ldots, s\}$ and we associate each element $a_i \in H$ with a recording variable z_i. Let W be the set of admissible components. For partitions with multiset components, we take

$$W = \left\{ \prod_{i=1}^{s} z_i^{m_i} \mid m_i \geq 0 \,\text{and}\; m_i\text{'s are not all zero} \right\},$$

and $\prod_{w \in W} w^t = \prod_{i=1}^{s} \frac{1}{1 - z_i^t} - 1$ for all $t \in \mathbb{N}$. Then the crude generating function for k-partitions with components in W is

$$g(x, z_1, \ldots, z_s) = \prod_{w \in W} \frac{1}{1 - xw}.$$

By the same argument as in [17], we have

$$f_{s,m}(x) = \sum_{k \geq 0} v_H^*(k) x^k = [z_1^m \cdots z_s^m](g(x, z_1, \ldots, z_s)).$$

We now start the computation of $f_{s,m}(x)$ and Bender's formula for the generating function

$$F_m(x, y) = \sum_{s \geq 0} f_{s,m}(x) \frac{y^s}{s!}.$$

Let $\mathbf{z} = (z_1, \ldots, z_s)$ and $\mathbf{u} = (u_1, \ldots, u_s)$. Then

$$\begin{aligned} f_{s,m}(x) &= [\mathbf{z}^m] \exp\left(\sum_{w \in W} \log \frac{1}{1 - xw} \right) \\ &= [\mathbf{z}^m] \exp\left(\sum_{w \in W} \left(\sum_{i=1}^{\infty} \frac{(xw)^i}{i} \right) \right) \\ &= [\mathbf{z}^m] \exp\left(\sum_{i=1}^{\infty} \frac{x^i}{i} \left(\sum_{w \in W} w^i \right) \right) \\ &= [\mathbf{z}^m] \exp\left(\sum_{i=1}^{\infty} \frac{x^i}{i} \left(\prod_{j=1}^{s} \frac{1}{1 - z_j^i} - 1 \right) \right) \end{aligned}$$

We now truncate the infinite sum at $i = m$ since only the previous m terms matter.

$$\begin{aligned} f_{s,m}(x) &= [\mathbf{z}^m] \exp\left(\sum_{i=1}^{m} \frac{x^i}{i} \left(\prod_{j=1}^{s} \frac{1}{1 - z_j^i} - 1 \right) \right) \\ &= \exp\left(-\sum_{i=1}^{m} \frac{x^i}{i} \right) [\mathbf{z}^m] \exp\left(\sum_{i=1}^{m} \frac{x^i}{i} \prod_{j=1}^{s} \frac{1}{1 - z_j^i} \right). \end{aligned}$$

By the Taylor expansion of the exponential function, we have

$$\begin{aligned} &[\mathbf{z}^m] \exp\left(\sum_{i=1}^{m} \frac{x^i}{i} \prod_{j=1}^{s} \frac{1}{1 - z_j^i} \right) \\ &= \sum_{u_1, \ldots, u_m \geq 0} \frac{x^{u_1 + \cdots + m u_m}}{u_1! \cdots u_m!} 2^{-u_1} \cdots m^{-u_m} [\mathbf{z}^m] L_m(\mathbf{u}), \end{aligned}$$

where

$$L_m(\mathbf{u}) = \left(\prod_{j=1}^{s} \frac{1}{1 - z_j} \right)^{u_1} \cdots \left(\prod_{j=1}^{s} \frac{1}{1 - z_j^m} \right)^{u_m}.$$

Moreover, we note that

$$\begin{aligned}&[z_1^m \cdots z_s^m] L_m(\mathbf{u})\\ &= \prod_{j=1}^{s} \left([z_j^m] \left(\frac{1}{1-z_j}\right)^{u_1} \cdots \left(\frac{1}{1-z_j^m}\right)^{u_m} \right)\\ &= (\chi_m(\mathbf{u}))^s,\end{aligned}$$

where

$$\chi_m(\mathbf{u}) := [z^m] \left(\left(\frac{1}{1-z}\right)^{u_1} \cdots \left(\frac{1}{1-z^m}\right)^{u_m} \right).$$

Then we have

$$\begin{aligned}&[\mathbf{z}^m] \exp\left(\sum_{i=1}^{m} \frac{x^i}{i} \prod_{j=1}^{s} \frac{1}{1-z_j^i} \right)\\ &= \sum_{u_1,\ldots,u_m \geq 0} \frac{x^{u_1+\cdots+mu_m}}{u_1! \cdots u_m!} 2^{-u_1} \cdots m^{-u_m} (\chi_m(\mathbf{u}))^s.\end{aligned}$$

This implies that

$$\begin{aligned}&F_m(x,y) = \sum_{s\geq 0} f_{s,m}(x) \frac{y^s}{s!}\\ &= \exp\left(-\sum_{i=1}^{m} \frac{x^i}{i} \right) \sum_{u_1,\ldots,u_m\geq 0} \frac{x^{u_1+\cdots+mu_m}}{u_1! \cdots u_m!} G(y,\mathbf{u}),\end{aligned}$$

where

$$G(y,\mathbf{u}) = 2^{-u_1} \cdots m^{-u_m} \exp(y\chi_m(\mathbf{u})).$$

In particular, the generating function for $m = 2$ is

$$F_2(x,y) = \exp\left(\frac{x^2(e^y - 1) - 2x}{2} \right) \sum_{u\geq 0} \frac{x^u}{u!} \exp\left(y \binom{u+1}{2} \right).$$

V. Conclusion

We have seen how some of the enumerative combinatorics of classical sets and multisets may be extended to the hybrid setting. These generalizations are interesting because we can get a unified treatment of some classical results shown in [12]. The on-going work in this direction would be finding combinatorial proofs of some generalizations of classical combinatorial identities in term of hybrid sets. We also anticipate that these techniques might be useful in the analysis of algorithms, by allowing flexibility in the order of operations.

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