Because of the limited capacity of the 4381, we cannot promise that all requests to use our system will be accepted. We will try to accommodate as many as possible.

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Algebra Snapshot: Linear Ordinary Differential Operators

Here we report on our work on operators in Scratchpad II. The package we describe provides linear ordinary differential operators. We decided to restrict our attention to this limited, well-understood domain for two reasons. First, this domain was a tool to be used in some work we were doing on ordinary differential equations. Second, we felt that if we could create the differential operator domain using only user-level facilities, then it could be used as an example for other operator domains.

As far as we are aware, operator algebra has not been part of the initial design of any computer algebra system. After-the-fact addition of classes of operators can be awkward, requiring either special functions for operator manipulation ("opPlus", "opTimes", etc.), which is ugly, or requiring modification of the base system, which is impossible for most users.

Scratchpad II addresses this issue by providing generic operations and by making application itself an operation. For example, the domain Mapping(A, B) of functions from A to B exports the application operation

".": (Mapping(A,B), A) -> B

which takes a mapping from A to B and an element of A to produce an element of B.

The notion of generic application is fundamental to the system and is used, not only for function invocation but also for subscripting arrays, list element extraction, record field selection and indexing tables. Two syntaxes are provided for application, one associating to the left and another to the right:

f a b means f applied to (a applied to b)
f.a.b means (f applied to a) applied to b

To create a ring R of operators on a set S in Scratchpad II, one adds the operation

"."; (R, S) -> S

to the constructor of R.
**LODO**

**LODO**(A,M) is the domain of linear ordinary differential operators over an A-module M, where A is a differential ring. This includes the cases of operators which are polynomials in D acting on scalars or vectors depending on a single variable. The coefficients of the operator polynomials can be integers, rational functions, matrices or elements of other domains.

**Differential operators with constant coefficients**

We begin by making some type assignments and declaring Dk to be a linear differential operator. Please refer to “Some Scratchpad II Constructor Name Abbreviations” on page 8 for expansions of the type abbreviations.

\[ \text{PZ := UPRD}(\text{P}[\text{x}], \text{I}); \]
\[ \text{PQ := UPRD}(\text{P}[\text{x}], \text{RN}); \]
\[ \text{DX := LODO}(\text{RN}, \text{PQ}); \]

PZ is the domain of univariate polynomials in x over the integers with some additional properties that make it into a differential ring. Similarly, PQ is a domain of univariate polynomials over the rationals. (Within a few months one should be able to use P[x] I and P[x] RN directly, without resorting to the UPRD constructor.) Operators are created as polynomials in D().

\[ \text{DX := D}(1) \]
\[ (5) \quad "x" \]
\[ a := \text{DX} + 1 \]
\[ (6) \quad D + 1 \]
\[ b := a + 1/2*Dk**2 - 1/2 \]
\[ (7) \quad \frac{1}{2}D + D + \frac{1}{2} \]
\[ c := (1/3)*b*(a + b)**2 \]
\[ (8) \quad \frac{1}{3}D + (\frac{5}{6})D + (\frac{5}{12})D + \frac{19}{32} \]
\[ + \frac{7}{36} \]
\[ (\frac{7}{12})D + (\frac{7}{12})D + \frac{7}{12} \]

To apply the operator a to the value p the usual function call syntax is used.

\[ p := 4*x**2 + 2/3; \]
\[ a \quad p \]
\[ (10) \quad 4x + 8x + \frac{2}{3} \]

Operator multiplication is defined by the identity \((a \times b)p = a(b(p))\).

\[ (a*b) \quad p = a \quad b \quad p \]
\[ (11) \quad 2x + 12x + \frac{37}{3} = \frac{2}{3}x + 12x + \frac{37}{3} \]

Operator expressions may be applied directly.

\[ (a**2 - 3/4*b + c) \quad (p + 1) \]
\[ (12) \quad \frac{2}{3}x + \frac{4}{3}x + \frac{5}{6}x \]
\[ 3 \quad 36 \]

When the operator coefficients are rational, it is possible to factor.

\[ \text{factor c} \]
\[ (13) \quad \frac{1}{72} \quad (\frac{1}{2}D + 3)^2 \quad (D + 1) \]

**Differential operators with rational function coefficients**

We clear the values of most of the variables, but retain those of PZ and PQ.

\[ \text{clear value DX a b p e f} \]
\[ (dx, a, b) := \text{LODO}(\text{QF PZ, QF PZ}) \]

\[ \text{DX := D}(1) \]
\[ (16) \quad "x" \]
\[ b := 3*x**2*Dk**2 + 2*DX + 1/x \]
\[ (17) \quad \frac{2}{3}x + 2D + \frac{1}{x} \]
\[ a := b*(5*x*Dk + 7) \]
\[ (18) \quad 3x + 2D + 19 + 10D + 2D + 29D + \frac{7}{x} \]
\[ p := \text{QF PZ} := x**2 + 1/x**2 \]
\[ (19) \quad \frac{4}{x} + 1 \]

Since the operator coefficients depend on x the operator multiplication is not commutative.
(a*b - b*a) / 4
- 75x + 540x - 75
----------
4
x

When the coefficients come from a field it is possible to define left and right division of the operators. The results of \texttt{ldiv} and \texttt{rdiv} are quotient/remainder pairs.

\texttt{ldiv}(a,b)
\begin{align*}
(21) & \frac{5x+7.0}{a} \\
& \text{"\%" means the result of the previous expression} \\
& a - (b \times \%)\text{.quotient} + \%\text{.remainder} \\
& (22) 0
\end{align*}

\texttt{rdiv}(a,b)
\begin{align*}
(23) & \frac{5x+7.10D}{a} + \frac{5}{x} \\
& a - (\%\text{.quotient} \times b + \%\text{.remainder} \\
& (24) 0
\end{align*}

The divisions allow the computation of left and right greatest common divisors via remainder sequences, and consequently the computation of left and right least common multiples.

\begin{align*}
\texttt{e := lgcd}(a,b) \\
(25) & \frac{2 2}{x} \\
& 3x D + 2 D + \frac{1}{x} \\
\end{align*}

A GCD doesn't necessarily divide \(a\) and \(b\) on both sides—here the left GCD does not divide \(a\) on the right.

\begin{align*}
\texttt{ldrem}(a, a) & \quad \text{the remainder from left division} \\
(26) & 0 \\
\texttt{rdrem}(a, a) & \quad \text{the remainder from right division} \\
(27) & 10D + \frac{5}{x} \\
\end{align*}

Likewise, an LCM is not necessarily divisible from both sides.

\begin{align*}
\texttt{f := rlcma}(a,b) \\
(28) & \frac{5 5}{a} \\
& \frac{604x + 80x}{20x D + (\%\text{.quotient}) D} \\
& 3 \\
& + \\
& \frac{5832x + 1656x^2 + 80x}{9} \\
& (\%\text{.remainder}) D \\
& 3
\end{align*}

\texttt{rdrem}(f, b)
\begin{align*}
(29) & 0 \\
\texttt{ldrem}(f, b)
\end{align*}

\begin{align*}
\frac{2}{x} \\
\frac{3672x + 2040x + 352}{a} \\
\frac{9}{x} \\
\frac{172}{9x} + \frac{-28}{9x^2} \\
\frac{9x}{2} \\
\frac{-1176x + 160}{9x} + \frac{312x - 80}{2} \\
\frac{9x}{9x}
\end{align*}

\textbf{Differential operators with elementary function coefficients}

\textbf{Problem:} find the first few coefficients of \(e^x x^{-i}\) in \(L_3 \phi\), where

\[ L_3 = D^3 + \frac{G}{x^2} D + \frac{H}{x^3} - 1 \]

and

\[ \phi = \sum_{i=0}^{\infty} s_i e^x x^{-i} \]

We will compute the first five coefficients. We start by defining the domain \texttt{EDF} to be the elementary functions viewed as a differential ring in the variable \(x\).

\texttt{clear} value n Dx L3 phi rho num xnum
\begin{align*}
n & := 5 \\
(32) & 5 \\
\texttt{EDF} & := \texttt{ODR}(\texttt{SE}, \texttt{EF} P L, x); \\
\end{align*}

Next we assign the differential operator \(L\) and the first terms of the function \(\phi\).

\begin{align*}
\texttt{(Dx, L3)} & := \texttt{LODO}(\texttt{EDF}, \texttt{EDF}) \\
\texttt{Dx} & := \texttt{D(1)} \\
(35) & "D" \\
\texttt{L3} & := \texttt{Dx**3} + \frac{G}{x^2} \texttt{Dx} + \frac{H}{x^3} - 1 \\
(36) & \frac{3}{x} + \frac{G}{x} + \frac{H - x^3}{x^3} \\
\end{align*}
\[ \phi := \frac{\exp(x)}{x} \text{ for } i \in 0..n \]
\[
\begin{pmatrix}
2 & 3 & 4 & 5 \\
5 & 4 & 3 & 2 & 1 & 0
\end{pmatrix}
\]
\[ \rho := \text{EF P I := L3 } \phi \]
\[
\begin{pmatrix}
2 & 3 & 4 & 5 \\
5 & 4 & 3 & 2 & 1 & 0
\end{pmatrix}
\]
\[ \text{clear value D} x \ a \ b \ t \ p0 \ p1 \ p2 \ p q \ m \]
\[ \text{Mat := SMDR(3, P2);} \]
\[ \text{Vect := DPMK(3, P2, Mat, P2);} \]

The matrix \( m \) will be used as a coefficient and the vectors \( p \) and \( q \) will be operated upon.

\[ t := x^2; \]
\[ m := \text{zero(3, 3)} \times \text{Mat}; \]
\[ m(0, 0) := t; \]
\[ m(0, 1) := 1; \]
\[ m(1, 0) := 1; \]
\[ m(1, 1) := t^2; \]
\[ m(2, 2) := 4 \times t; \]
\[ m \]
\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 4 & 0 \\
0 & 0 & 4 x
\end{pmatrix}
\]
\[ p0 := 3 \times x^2 + 1; \]
\[ p1 := 2 \times x; \]
\[ p2 := 7 \times x^3 + 2 \times x; \]
\[ p := \text{Vect := [p0, p1, p2] \times V(P2);} \]
\[ \text{q := Vect := m } \times \text{ p;} \]
\[ q \]
\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 4 & 0 \\
0 & 0 & 4 x
\end{pmatrix}
\]

\[ \text{Dx := D();} \]
\[ \text{Dx := D()}; \]
\[ a := Dx + m \]
\[ a \]
\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 4 & 0 \\
0 & 0 & 4 x
\end{pmatrix}
\]
\[ b := a \times Dx + 1 \]
\[ b \]
\[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 4 & 0 \\
0 & 0 & 4 x
\end{pmatrix}
\]

**Differential operators with matrix coefficients acting on vectors**

Define the differential ring of 3 by 3 matrices of polynomials in \( x \) over the integers and the three dimensional vector space of these polynomials.
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Recent Demonstrations of Scratchpad II

August, 1985  IJCAI ’85, Los Angeles, California.
August, 1985  AMS Summer Conference in Computational Number Theory, Humboldt State University, Arcata, California.
September, 1985  CIRM Conference on Computers in Algebraic Geometry, Trento, Italy.
October, 1985  NSF Workshop on Comp. in Algebra and Number Theory, University of California, Berkeley.
November, 1985  Computer Science Department Chairmen’s Conference, Yorktown Heights, New York.
December, 1985  Future Directions in Computing, Grenoble, France.

For your information ...

The Scratchpad II Computer Algebra System currently runs only under the IBM VM/SP operating system on mainframe computers. A three megabyte segment is shared among all users on a given computer, and each user requires at least a four megabyte virtual machine.

At last count, the Scratchpad II library consisted of 53 category constructors, 102 domain constructors and 59 package constructors.