Geometric algebra: a computational framework for geometrical applications (part I: algebra)

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Abstract

Geometric algebra is a consistent computational framework in which to define geometric primitives and their relationships. This algebraic approach contains all geometric operators and permits specification of constructions in a coordinate-free manner. Thus, the ideas of geometric algebra are important for developers of CAD systems. This paper gives an introduction to the elements of geometric algebra, which contains primitives of any dimensionality (rather than just vectors), and an introduction to three of the products of geometric algebra, the geometric product, the inner product, and the outer product. These products are illustrated by using them to solve simple geometric problems.

Keywords: geometric algebra, Clifford algebra, subspaces, blades, geometric product, inner product, outer product

1 Introduction

In the usual way of defining geometrical objects in fields like computer graphics, robotics and computer vision, one uses vectors to characterize the constructions. To do this effectively, the basic concept of a vector as an element of a linear space is extended by an inner product and a cross product, and some additional constructions such as homogeneous coordinates to encode compactly the intersection of, for instance, offset planes in space. Many of these techniques work rather well in 3-dimensional space, although some problems have been pointed out: the difference between vectors and points [3], and the characterization of planes by normal vectors (which may require extra computation after linear transformations, since the normal vector of a transformed plane is not the transform of the normal vector). These problems are then traditionally fixed by the introduction of data structures and combination rules; object-oriented programming can be used to implement this patch tidily [6].

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Yet there are deeper issues in geometric programming that are still accepted as 'the way things are'. For instance, when you need to intersect linear subspaces, the intersection algorithms are split out in treatment of the various cases: lines and planes, planes and planes, lines and lines, et cetera, need to be treated in separate pieces of code. After all, the outcomes themselves can be points, lines or planes, and those are essentially different in their further processing.

Yet this need not be so. If we could see subspaces as basic elements of computation, and do direct algebra with them, then algorithms and their implementation would not need to split their cases on dimensionality. For instance, $A \wedge B$ could be 'the subspace spanned by the spaces A and B', the expression $A \rfloor B$ could be 'the part of B perpendicular to A'; and then we would always have the computation rule $(A \wedge B) \rfloor C = A \rfloor (B \rfloor C)$ since computing the part of C perpendicular to the span of A and B can be computed in two steps, perpendicularity to B followed by perpendicularity to A. Subspaces therefore have computational rules of their own that can be used immediately, independent of how many vectors were used to span them (i.e. independent of their dimensionality). In this view, the split in cases for the intersection operator could be avoided, since intersection of subspaces always leads to subspaces. We should consider using this structure, since it would enormously simplify the specification of geometric programs.

This and a subsequent paper intend to convince you that subspaces form an algebra with well-defined products that have direct geometric significance. This algebra can then be used as a language for geometry, and we claim that it is a better choice than a language always reducing everything to vectors (which are just 1-dimensional subspaces). Along the way, we will see that this framework allows us to divide by vectors (in fact, we can divide by any subspace), and we will see several familiar computer graphics constructs (quaternions, normals, Plücker coordinates) that fold in naturally to the framework and need no longer be considered as clever but extraneous tricks. This algebra is called *geometric algebra*. Mathematically, it is like *Clifford algebra*, but carefully wielded to have a clear geometrical interpretation, which excludes some constructions and suggests others. In most literature, the two terms are used interchangeably.

In this paper, we primarily introduce subspaces (the basic element of computation in geometric algebra) and the products of geometric algebra. Our intent is to introduce these ideas, and we will not always give proofs of what we present. The proofs we do give are intended to illustrate use of the geometric algebra; the missing proofs can be found in the references. In a subsequent paper, we will give some examples of how these products can be used in elementary but important ways, and look at more advanced topics such as differentiation, linear algebra, and homogeneous representation spaces.

Since subspaces are the main 'objects' of geometric algebra we introduce them first, which we do by combining vectors that span the subspace in Section 2. We then introduce the geometric product, and look at products derived from the geometric product in Section 3. Some of the derived products, like the inner and outer products, are so basic that it is natural to treat them in this section also, even though the geometric product is all we really need to do geometric algebra. Other products (such meet, join, and rotation through 'sandwiching') are better introduced in the context of their geometrical meaning, and we develop them in a subsequent paper. This approach reduces the amount of new notation, but it may make it seem as if geometric algebra needs to invent a new technique for every new kind of geometrical operation one wants to embed. This is *not* the case: all you need is the geometric product and its (anti-)commutation properties.

2 Subspaces as elements of computation

As in the classical approach, we start with a real vector space V^m that we use to denote 1-dimensional directed magnitudes. Typical usage would be to employ a vector to denote a translation in such a space, to establish the location of a point of interest. (Points are not vectors, but their locations relative to a fixed point are [3].) We now want to extend this capability of indicating directed magnitudes to higher-dimensional directions such as facets of objects, or tangent planes. We will start with the simplest subspaces: the 'proper' subspaces of a linear vector space, which are lines, planes, etcetera through the origin, and develop their algebra of spanning and perpendicularity measures. In our follow-up paper, we show how to use the same algebra to treat "offset" subspaces, and even spheres.

2.1 Constructing subspaces

So we start with a real m-dimensional linear space V^m, of which the elements are called *vectors*. Many approaches to geometry make explicit use of coordinates. While coordinates are needed for input and output, and while they are also needed to perform low level operations on objects, most of the formulas and computations in geometric algebra can work directly on subspaces without resorting to coordinates. Thus, we will always view vectors geometrically: a vector denotes a '1-dimensional direction element', with a certain 'attitude' or 'stance' in space, and a 'magnitude', a measure of length in that direction. These properties are well characterized by calling a vector a 'directed line element', as long as we mentally associate an orientation and magnitude with it: $\mathbf{v}$ is not the same as $-\mathbf{v}$ or $2\mathbf{v}$. These properties are independent of any coordinate system, and in this and in our follow-up paper, we will not refer to coordinates, except for times when we feel a coordinate example clarifies an explanation.

Algebraic properties of these geometrical vectors are: they can be added and weighted with real coefficients, in the usual way to produce new vectors; and they can be multiplied using an *inner product*, to produce a scalar $\mathbf{a} \cdot \mathbf{b}$ (in both of these papers, we use a metric vector space with well-defined inner product).

In geometric algebra, higher-dimensional oriented subspaces are also basic elements of computation. They are called *blades*, and we use the term k-*blade* for a k-dimensional homogeneous subspace. So a vector is a 1-blade.

A common way of constructing a blade is from vectors, using a product that constructs the span of vectors. This product is called the *outer product* (sometimes the *wedge product*) and denoted by $\wedge$. It is codified by its algebraic properties, which have been chosen to make sure we indeed get m-dimensional space elements with an

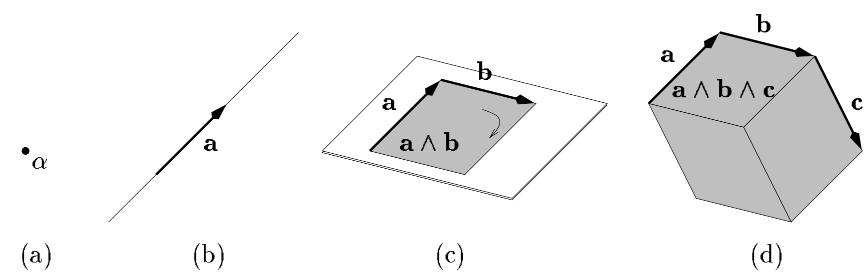

Figure 1: *Spanning proper subspaces using the outer product.*

appropriate magnitude (area element for $m = 2$, volume elements for $m = 3$; see Figure 1). As you have seen in linear algebra, such magnitudes are determinants of matrices representing the basis of vectors spanning them. But such a definition would be too specifically dependent on that matrix representation. Mathematically, a determinant is viewed as an anti-symmetric linear scalar-valued function of its vector arguments. That gives the clue to the rather abstract definition of the outer product in geometric algebra:

> The *outer product* of vectors $\mathbf{a}_1, \cdots, \mathbf{a}_k$ is anti-symmetric, associative and linear in its arguments. It is denoted $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k$, and called a k-blade.

The only thing that is different from a determinant is that the outer product is *not* forced to be scalar-valued; and this gives it the capability of representing the 'attitude' of a k-dimensional subspace element as well as its magnitude.

2.2 2-blades in 3-dimensional space

Let us see how this works in the geometric algebra of a 3-dimensional space V^3. For convenience, let us choose a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in this space, relative to which we denote any vector. Now let us compute $\mathbf{a} \wedge \mathbf{b}$ for $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. By linearity, we can write this as the sum of six terms of the form $a_1b_2\mathbf{e}_1 \wedge \mathbf{e}_2$ or $a_1b_1\mathbf{e}_1 \wedge \mathbf{e}_1$. By anti-symmetry, the outer product of any vector with itself must be zero, so the term with $a_1b_1\mathbf{e}_1 \wedge \mathbf{e}_1$ and other similar terms disappear. Also by anti-symmetry, $\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_1 \wedge \mathbf{e}_2$, so some terms can be grouped. You may verify that the final result is

$$\begin{aligned}
&\mathbf{a} \wedge \mathbf{b} = \\
&\quad = \quad (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \wedge (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\
&\quad = \quad (a_1b_2 - a_2b_1)\,\mathbf{e}_1 \wedge \mathbf{e}_2 + (a_2b_3 - a_3b_2)\,\mathbf{e}_2 \wedge \mathbf{e}_3 + (a_3b_1 - a_1b_3)\,\mathbf{e}_3 \wedge \mathbf{e}_1 \qquad (1)
\end{aligned}$$

We cannot simplify this further. Apparently, the axioms of the outer product permit us to decompose any 2-blade in 3-dimensional space onto a basis of three elements. This '2-blade basis' (also called 'bivector basis') $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1\}$ consists of 2-blades spanned by the basis vectors. Linearity of the outer product implies that the set of 2-blades forms a linear space on this basis. We will interpret this as the space of all *plane elements* (or *area elements*).

Let us show that $\mathbf{a} \wedge \mathbf{b}$ indeed has the correct magnitude for an area element. That is particularly clear if we choose a specific orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, chosen such that $\mathbf{a}$ lies in the $\mathbf{e}_1$-direction, and $\mathbf{b}$ lies in the $(\mathbf{e}_1, \mathbf{e}_2)$-plane (we can always do this). Then $\mathbf{a} = a\mathbf{e}_1$, $\mathbf{b} = b \cos\phi\, \mathbf{e}_1 + b \sin\phi\, \mathbf{e}_2$ (with ϕ the angle from $\mathbf{a}$ to $\mathbf{b}$), so that

$$\mathbf{a} \wedge \mathbf{b} = (a\, b\, \sin\phi)\, \mathbf{e}_1 \wedge \mathbf{e}_2 \tag{2}$$

This single result contains both the correct magnitude of the area $a\, b\, \sin\phi$ spanned by $\mathbf{a}$ and $\mathbf{b}$, and the plane in which it resides – for we recognize $\mathbf{e}_1 \wedge \mathbf{e}_2$ as 'the unit directed area element of the $(\mathbf{e}_1, \mathbf{e}_2)$-plane'. Since we can always adapt our coordinates to vectors in this way, this result is universally valid: $\mathbf{a} \wedge \mathbf{b}$ is an area element of the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ (see Figure 1c). Denoting the unit area element in the $(\mathbf{a}, \mathbf{b})$-plane by $\mathbf{I}$, the coordinate-free formulation of the above is

$$\mathbf{a} \wedge \mathbf{b} = (a\, b\, \sin\phi)\, \mathbf{I} \tag{3}$$

The result extends to blades of higher grades: each is proportional to the unit hypervolume element in its subspace, by a factor that is the hypervolume.

2.3 Volumes as 3-blades

We can also form the outer product of *three* vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Considering each of those decomposed onto their three components on some basis in our 3-dimensional space (as above), we obtain terms of three different types, depending on how many common components occur: terms like $a_1 b_1 c_1\, \mathbf{e}_1 \wedge \mathbf{e}_1 \wedge \mathbf{e}_1$, like $a_1 b_1 c_2\, \mathbf{e}_1 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$, and like $a_1 b_2 c_3\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. Because of associativity and anti-symmetry, only the last type survives, in all its permutations. The final result is

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = (a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_1 c_3 - a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1)\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3.$$

The scalar factor is the determinant of the matrix with columns $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, which is proportional to the signed volume spanned by them (as is well known from linear algebra). The term $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ is the denotation of which volume is used as unit: that spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The order of the vectors gives its orientation, so this is a 'signed volume'. In 3-dimensional space, there is not really any other choice for the construction of volumes than (possibly negative) multiples of this volume (see Figure 1d). But in higher dimensional spaces, the attitude of the volume element needs to be indicated just as much as we needed to denote the attitude of planes in 3-space.

2.4 The pseudoscalar as hypervolume

Forming the outer product of *four* vectors $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d}$ in 3-dimensional space will always produce zero (since they must be linearly dependent). The highest order blade that is non-zero in an m-dimensional space is an m-blade. Such a blade, representing an m-dimensional volume element, is called a *pseudoscalar* for that space, for historical reasons; unfortunately a rather abstract term for the elementary geometric concept of 'oriented hypervolume element'.

2.5 Scalars as subspaces

To make scalars fully admissible elements of the algebra we have so far, we can define the outer product of two scalars, and a scalar and a vector, by identifying it with the familiar scalar product in the vector space we started with:

$$\alpha \wedge \beta = \alpha\,\beta \quad \text{and} \quad \alpha \wedge \mathbf{v} = \alpha\,\mathbf{v}$$

Since the scalars are constructed by the outer product of 'no vectors at all', we can interpret them geometrically as the representation of '0-dimensional subspace elements'. These are like points with masses. So scalars are geometrical entities as well, if we are willing to stretch the meaning of 'subspace' a little. We will denote scalars mostly by Greek lower case letters.

2.6 The linear space of subspaces

Collating what we have so far, we have constructed a geometrically significant algebra containing only two operations: the addition $+$ and the outer multiplication $\wedge$ (subsuming the usual scalar multiplication). Starting from scalars and a 3-dimensional vector space we have generated a 3-dimensional space of 2-blades, and a 1-dimensional space of 3-blades (since all volumes are proportional to each other). In total, therefore, we have a set of elements that naturally group by their dimensionality. Choosing some basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write what we have as spanned by the set

$$\left\{ \underbrace{1}_{\textit{scalars}}\ ,\ \underbrace{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3}_{\textit{vector space}}\ ,\underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_2 \wedge \mathbf{e}_3,\ \mathbf{e}_3 \wedge \mathbf{e}_1}_{\textit{bivector space}},\ \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\textit{trivector space}} \right\} \tag{4}$$

Every k-blade formed by $\wedge$ can be decomposed on the k-vector basis using $+$. The 'dimensionality' k is often called the *grade* or *step* of the k-blade or k-vector, reserving the term *dimension* for that of the vector space that generated them. A k-blade represents a k-dimensional oriented subspace element.

If we allow the scalar-weighted addition of arbitrary elements in this set of basis blades, we get an 8-dimensional linear space from the original 3-dimensional vector space. This space, with $+$ and $\wedge$ as operations, is called the *Grassmann algebra* of 3-space. In an m-dimensional space, there are $\binom{m}{k}$ basis elements of grade k, for a total basis of 2^m elements for the Grassmann algebra. The same basis is used for the geometric algebra of the space, although we will construct the objects in it in a different manner.

3 The Products of Geometric Algebra

In this section, we describe the geometric product, the most important product of geometric algebra. The fact that the geometric product can be applied to k-blades and has an inverse considerably extends algebraic techniques for solving geometrical problems. We can use the geometric product to derive other meaningful products. The most

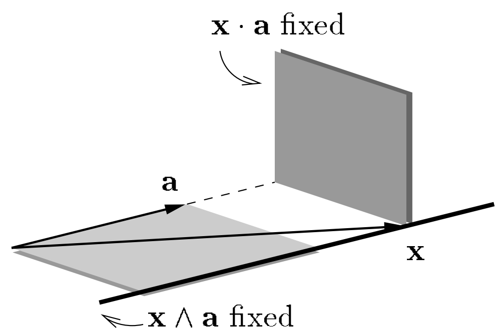

Figure 2: *Invertibility of the geometric product.*

elementary are the inner and outer products, also discussed in this section; the useful but less elementary products giving reflections, rotations and intersection are treated later, and in more detail in our follow-up paper.

3.1 The Geometric Product

For vectors in our metric vector space V^m, the geometric product is defined in terms of the inner and outer product as

$$\mathbf{a}\,\mathbf{b} \equiv \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{5}$$

So the geometric product of two vectors is an element of mixed grade: it has a scalar (0-blade) part $\mathbf{a} \cdot \mathbf{b}$ and a 2-blade part $\mathbf{a} \wedge \mathbf{b}$. It is therefore *not* a blade; rather, it is an operator on blades (as we will soon show). Changing the order of $\mathbf{a}$ and $\mathbf{b}$ gives

$$\mathbf{b}\,\mathbf{a} \equiv \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b}$$

The geometric product of two vectors is therefore neither fully symmetric (unlike the inner product), nor fully anti-symmetric (unlike the outer product). However, the geometric product is invertible.

A simple drawing may convince you that the geometric product is indeed invertible, whereas the inner and outer product separately are not. In Figure 2, we have a given vector $\mathbf{a}$. We have indicated the set of vectors $\mathbf{x}$ with the same value of the inner product $\mathbf{x} \cdot \mathbf{a}$ – this is a plane perpendicular to $\mathbf{a}$. The set of all vectors with the same value of the outer product $\mathbf{x} \wedge \mathbf{a}$ is also indicated – this is the line of all points that span the same directed area with $\mathbf{a}$ (since for the position vector of any point $\mathbf{p} = \mathbf{x} + \lambda\mathbf{a}$ on that line, we have $\mathbf{p} \wedge \mathbf{a} = \mathbf{x} \wedge \mathbf{a} + \lambda\mathbf{a} \wedge \mathbf{a} = \mathbf{x} \wedge \mathbf{a}$ by the anti-symmetry property). Neither of these sets is a singleton (in spaces of more than 1 dimension), so the inner and outer products are not fully invertible. The geometric product provides both the plane and the line, and therefore permits determining their unique intersection $\mathbf{x}$, as illustrated in the figure. Therefore it is invertible: from $\mathbf{x}\,\mathbf{a}$ and $\mathbf{a}$, we can retrieve $\mathbf{x}$.

Eq.(5) defines the geometric product *only for vectors*. For arbitrary elements of our algebra it is defined using *linearity, associativity and distributivity over addition*; and we make it coincide with the usual scalar product in the vector space, as the notation already suggests. That gives the following axioms (where α and β are scalars, $\mathbf{x}$ is a vector, A, B, C are general elements of the algebra):

$$\begin{array}{rll} \text{scalars} & \alpha\,\beta \text{ and } \alpha\,\mathbf{x} \text{ have their usual meaning in } V^m & (6) \\ \text{scalars commute} & \alpha\,A = A\,\alpha & (7) \\ \text{vectors} & \mathbf{x}\,\mathbf{a} = \mathbf{x}\cdot\mathbf{a} + \mathbf{x}\wedge\mathbf{a} & (8) \\ \text{associativity} & A\,(B\,C) = (A\,B)\,C & (9) \end{array}$$

We have thus defined the geometric product in terms of inner and outer product; yet we claimed that it is more fundamental than either. Mathematically, it is more elegant to replace eq.(8) by 'the square of a vector $\mathbf{x}$ is a scalar $Q(\mathbf{x})$'. This function Q can then actually be interpreted as the metric of the space, the same as the one used in the inner product, and it gives the same geometric algebra [5]. Our choice for eq.(8) was to define the new product in terms of more familiar quantities, to aid your intuitive understanding of it.

Let us show by example how these rules can be used to develop the geometric algebra of 3-dimensional Euclidean space. We introduce, for convenience only, an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^3$. Since this implies that $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$, we get the commutation rules

$$\mathbf{e}_i\mathbf{e}_j = \begin{cases} -\mathbf{e}_j\mathbf{e}_i & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{10}$$

In fact, the former is equal to $\mathbf{e}_i \wedge \mathbf{e}_j$, whereas the latter equals $\mathbf{e}_i \cdot \mathbf{e}_i$. Considering the unit 2-blade $\mathbf{e}_i \wedge \mathbf{e}_j$, we find its square:

$$\begin{aligned} (\mathbf{e}_i \wedge \mathbf{e}_j)^2 &= (\mathbf{e}_i \wedge \mathbf{e}_j)\,(\mathbf{e}_i \wedge \mathbf{e}_j) = (\mathbf{e}_i\,\mathbf{e}_j)\,(\mathbf{e}_i\,\mathbf{e}_j) \\ &= \mathbf{e}_i\,\mathbf{e}_j\,\mathbf{e}_i\,\mathbf{e}_j = -\mathbf{e}_i\,\mathbf{e}_i\,\mathbf{e}_j\,\mathbf{e}_j = -1 \end{aligned} \tag{11}$$

So a unit 2-blade squares to -1. Continued application of eq.(10) gives the full multiplication for all basis elements in the 'Clifford algebra' of 3-dimensional space. The resulting multiplication table is given in Figure 3. Arbitrary elements are expressible as a linear combination of these basis elements, so this table determines the full algebra.

3.1.1 Exponential representation

Note that the geometric product is sensitive to the relative directions of the vectors: for parallel vectors $\mathbf{a}$ and $\mathbf{b}$, the outer product contribution is zero, and $\mathbf{a}\,\mathbf{b}$ is a scalar and commutative in its factors; for perpendicular vectors, $\mathbf{a}\,\mathbf{b}$ is a 2-blade, and anticommutative. In general, if the angle between $\mathbf{a}$ and $\mathbf{b}$ is ϕ in their common plane with unit 2-blade $\mathbf{I}$, we can write (in a Euclidean space)

$$\mathbf{a}\,\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,(\cos\phi + \mathbf{I}\,\sin\phi) \tag{12}$$

using a common rewriting of the inner product, and eq.(3). We have seen above that $\mathbf{I}\,\mathbf{I} = -1$, and this permits the shorthand of the exponential notation (by the usual definition of the exponential as a converging series of terms)

$$\mathbf{a}\,\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,(\cos\phi + \mathbf{I}\,\sin\phi) = |\mathbf{a}|\,|\mathbf{b}|\,e^{\mathbf{I}\phi}. \tag{13}$$

$\mathcal{C}\ell_3$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{e}_{123}$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$-\mathbf{e}_1$	$\mathbf{e}_{123}$	$\mathbf{e}_3$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{e}_{123}$	$\mathbf{e}_1$	$-\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	-1	$\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_3$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$-\mathbf{e}_{23}$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	-1	$-\mathbf{e}_1$
$\mathbf{e}_{123}$	$\mathbf{e}_{123}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	-1

Figure 3: The multiplication table of the geometric algebra of 3-dimensional Euclidean space, on an orthonormal basis. Shorthand: $\mathbf{e}_{12} \equiv \mathbf{e}_1 \wedge \mathbf{e}_2$, etcetera.

All this is reminiscent of complex numbers, but it really is different. Firstly, geometric algebra has given a straightforward real geometrical interpretation to all elements occurring in this equation, notably of $\mathbf{I}$ as the unit area element of the common plane of $\mathbf{a}$ and $\mathbf{b}$. Secondly, the math differs: if $\mathbf{I}$ were a complex scalar, it would have to commute with *all* elements of the algebra by eq.(7), but instead it satisfies $\mathbf{a}\,\mathbf{I} = -\mathbf{I}\,\mathbf{a}$ for vectors $\mathbf{a}$ in the $\mathbf{I}$-plane. We will use the exponential notation a lot when we study rotations, in our follow-up paper.

3.1.2 Many grades in the geometric product

It is a consequence of the definition of the geometric product that 'a vector squares to a scalar': the geometric product of a vector with itself is a scalar. Therefore when you multiply two blades, the vectors in them may multiply to a scalar (if they are parallel) or to a 2-blade (if they are not). As a consequence, when you multiply two blades of grade k and ℓ using the geometric product, the result potentially contains parts of all grades $(k+\ell), (k+\ell-2), \cdots, (|k-\ell|+2), |k-\ell|$, just depending on how their factors align. This series of terms contains *all* the information about the geometrical relationships of the blades: their span, intersection, relative orientation, etc.

3.2 An inner product of blades

In geometric algebra, the standard inner product of two vectors can be seen as the symmetrical part of their geometric product:

$$\mathbf{a} \cdot \mathbf{b} = \tfrac{1}{2}(\mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a})$$

Just as in the usual definition, this embodies the metric of the vector space and can be used to define distances. It also codifies the perpendicularity required in projection operators. Now that vectors are viewed as representatives of 1-dimensional subspaces, we of course want to extend this metric capability to arbitrary subspaces. The inner product can be generalized to general subspaces in several ways. The tidiest method mathematically is explained in [2] [5]; this leads to the *contraction* inner product (denoted by ‘ $\rfloor$ ’), which has a clean geometric meaning. In this intuitive introduction, we prefer to give the geometric meaning first.

> $\mathbf{A}\rfloor\mathbf{B}$ is a blade representing the complement (within the subspace $\mathbf{B}$) of the orthogonal projection of $\mathbf{A}$ onto $\mathbf{B}$; it is linear in $\mathbf{A}$ and $\mathbf{B}$; and it coincides with the usual inner product $\mathbf{a}\cdot\mathbf{b}$ of V^m when computed for vectors $\mathbf{a}$ and $\mathbf{b}$.

The above determines our inner product uniquely.[1] It turns out not to be symmetrical (as one would expect since the definition is asymmetrical) and also not associative. But we do demand linearity, to make it computable between any two elements in our linear space (not just blades). Note that earlier on we used only the inner product between vectors $\mathbf{a}\cdot\mathbf{b}$, which we would now write as $\mathbf{a}\rfloor\mathbf{b}$.

We will just give the rules by which to compute the resulting inner product for arbitrary blades, omitting their derivation (essentially as in [5]). In the following α, β are scalars, $\mathbf{a}$ and $\mathbf{b}$ vectors and A, B, C general elements of the algebra.

$$\begin{aligned}
\text{scalars} &\quad \alpha\rfloor\beta = \alpha\,\beta && (14)\\
\text{vector and scalar} &\quad \mathbf{a}\rfloor\beta = 0 && (15)\\
\text{scalar and vector} &\quad \alpha\rfloor\mathbf{b} = \alpha\,\mathbf{b} && (16)\\
\text{vectors} &\quad \mathbf{a}\rfloor\mathbf{b} \text{ is the usual inner product } \mathbf{a}\cdot\mathbf{b} \text{ in } V^m && (17)\\
\text{vector and element} &\quad \mathbf{a}\rfloor(\mathbf{b}\wedge B) = (\mathbf{a}\rfloor\mathbf{b})\wedge B - \mathbf{b}\wedge(\mathbf{a}\rfloor B) && (18)\\
\text{distribution} &\quad (A\wedge B)\rfloor C = A\rfloor(B\rfloor C) && (19)
\end{aligned}$$

As we said, linearity and distributivity over $+$ also hold, but the inner product is *not* associative. The inner product of two blades is again a blade [1] (as one would hope, since they represent subspaces and so should the result). It is easy to see that the grade of this blade is

$$\mathsf{grade}\,(\mathbf{A}\rfloor\mathbf{B}) = \mathsf{grade}\,(\mathbf{B}) - \mathsf{grade}\,(\mathbf{A})\,, \tag{20}$$

since the projection of $\mathbf{A}$ onto $\mathbf{B}$ has the same grade as $\mathbf{A}$, and its complement in $\mathbf{B}$ the ‘co-dimension’ of this projection in the subspace spanned by $\mathbf{B}$. Since no subspace has a negative dimension, the contraction $\mathbf{A}\rfloor\mathbf{B}$ is zero when $\mathsf{grade}\,(\mathbf{A}) > \mathsf{grade}\,(\mathbf{B})$ (and this is the main difference between the contraction and the other inner product).

When used on blades as $(\mathbf{A}\wedge\mathbf{B})\rfloor\mathbf{C} = \mathbf{A}\rfloor(\mathbf{B}\rfloor\mathbf{C})$, rule eq.(19) gives the inner product its meaning of being the perpendicular part of one subspace inside another. In words it would read something like: ‘to get the part of $\mathbf{C}$ perpendicular to the subspace that is the span of $\mathbf{A}$ and $\mathbf{B}$, take the part of $\mathbf{C}$ perpendicular to $\mathbf{B}$; then of that, take the part perpendicular to $\mathbf{A}$’.

Figure 4 gives an example: the inner product of a vector $\mathbf{a}$ and a 2-blade $\mathbf{B}$, producing the vector $\mathbf{a}\rfloor\mathbf{B}$. Note that the usual inner product for vectors $\mathbf{a}$ and $\mathbf{b}$ has the

[1] The resulting contraction inner product differs slightly from the inner product commonly used in the geometric algebra literature. The contraction inner product has a cleaner geometric semantics, and more compact mathematical properties, and that makes it better suited to computer science. The two inner products can be expressed in terms of each other, so this is not a severely divisive issue. They ‘algebraify’ the same geometric concepts, in just slightly different ways. See also [2].

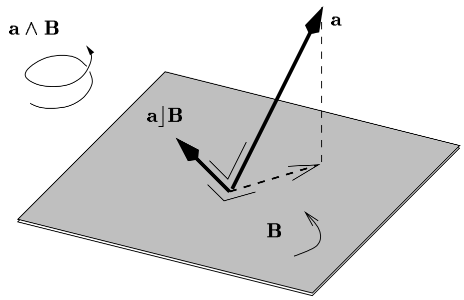

Figure 4: *The inner product of blades. The corkscrew denotes the orientation of the trivector.*

right semantics: the subspace that is the orthogonal complement (in the space spanned by $\mathbf{b}$) of the projection of $\mathbf{a}$ onto $\mathbf{b}$ contains only the point at their common origin, and is therefore represented by a *scalar* (0-blade) linear in $\mathbf{a}$ and $\mathbf{b}$.

With the definition of the inner product for blades, we can generalize the relationship eq.(8) between a geometric product and its inner and outer product parts. For a vector $\mathbf{x}$ and a blade $\mathbf{A}$, we have:

$$\mathbf{x}\,\mathbf{A} = \mathbf{x} \rfloor \mathbf{A} + \mathbf{x} \wedge \mathbf{A}. \tag{21}$$

Note that if the first argument is not a vector, this formula does not apply. In general, the geometric product of two blades contains many more terms, which may be written as interleavings of the inner and outer product of the vectors spanning the blades.

3.3 The outer product

We have already seen the outer product in Section 2, where it was used to construct the subspaces of the algebra. Once we have the geometric product, it is better to see the outer product as its anti-symmetric part:

$$\mathbf{a} \wedge \mathbf{b} = \tfrac{1}{2}(\mathbf{a}\,\mathbf{b} - \mathbf{b}\,\mathbf{a})$$

and, slightly more general, if the second factor is a blade:

$$\mathbf{a} \wedge \mathbf{B} = \tfrac{1}{2}\left(\mathbf{a}\,\mathbf{B} + (-1)^{\mathsf{grade}(\mathbf{B})}\mathbf{B}\,\mathbf{a}\right), \tag{22}$$

This leads to the defining properties we saw before (as before, α, β are scalars, $\mathbf{a}, \mathbf{b}$ are vectors, A, B, C : are general elements):

$$\begin{aligned}
\text{scalars} \qquad & \alpha \wedge \beta = \alpha\,\beta && (23)\\
\text{scalar and vector} \qquad & \alpha \wedge \mathbf{b} = \alpha\,\mathbf{b} && (24)\\
\text{anti-symmetry for vectors} \qquad & \mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} && (25)\\
\text{associativity} \qquad & (A \wedge B) \wedge C = A \wedge (B \wedge C) && (26)
\end{aligned}$$

Linearity and distributivity over $+$ also hold.

The grade of a k-blade is the number of vector factors that span it. Therefore the grade of an outer product of two blades is simply

$$\mathsf{grade}\,(\mathbf{A} \wedge \mathbf{B}) = \mathsf{grade}\,(\mathbf{A}) + \mathsf{grade}\,(\mathbf{B})\,. \tag{27}$$

Of course the outcome may be 0, so this zero element of the algebra should be seen as an element of arbitrary grade. There is then no need to distinguish separate zero scalars, zero vectors, zero 2-blades, etcetera.

3.3.1 Subspace objects without shape

We reiterate that the outer product of k-vectors gives a 'bit of k-space', in a manner that includes the attitude of the space element, its orientation (or 'handedness') and its magnitude. For a 2-blade $\mathbf{a} \wedge \mathbf{b}$, this was conveyed in eq.(3).

Yet $\mathbf{a}\wedge\mathbf{b}$ is not an area element with well-defined shape, even though one is tempted to draw it as a parallelogram (as in Figure 1c). For instance, by the properties of the outer product, $\mathbf{a}\wedge\mathbf{b} = \mathbf{a}\wedge(\mathbf{b}+\lambda\mathbf{a})$, for any λ, so $\mathbf{a}\wedge\mathbf{b}$ is just as much the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}+\lambda\mathbf{a}$. Playing around, you find that you can move around pieces of the area elements while still maintaining the same product $\mathbf{a} \wedge \mathbf{b}$; so really, a bivector does not have any fixed shape or position, it is just a chunk of a precisely defined amount of 2-dimensional directed area in a well-defined plane. It follows that the 2-blades have an existence of their own, independent of any vectors that one might use to define them.

We will take these non-specific shapes made by the outer product and 'force them into shape' by carefully chosen geometric products; this will turn out to be a powerful and flexible technique to get closed coordinate-free computational expressions for geometrical constructions.

3.3.2 Linear (in)dependence

Note that if three vectors are linearly dependent, they satisfy

$$\mathbf{a},\mathbf{b},\mathbf{c} \text{ linearly dependent} \quad \Longleftrightarrow \quad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0.$$

We interpret the latter immediately as the geometric statement that the vectors span a zero volume. This makes linear dependence a computational property rather than a predicate: three vectors can be 'almost linearly dependent'. The magnitude of $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$ obviously involves the determinant of the matrix $(\mathbf{a}\;\mathbf{b}\;\mathbf{c})$, so this view corresponds with the usual computation of determinants to check degeneracy.

4 Solving geometric equations

The geometric product is invertible, so 'dividing by a vector' has a unique meaning. We usually do this through 'multiplication by the inverse of the vector'. Since multiplication is not necessarily commutative, we have to be a bit careful: there is a 'left division' and a 'right division'. As you may verify, the unique inverse of a vector $\mathbf{a}$ is

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{a} \rfloor \mathbf{a}}$$

since that is the unique element that satisfies: $\mathbf{a}^{-1}\,\mathbf{a} = 1 = \mathbf{a}\,\mathbf{a}^{-1}$. In general, a blade $\mathbf{A}$ has the inverse

$$\mathbf{A}^{-1} = \frac{\widetilde{\mathbf{A}}}{\mathbf{A} \rfloor \widetilde{\mathbf{A}}}$$

where $\widetilde{\mathbf{A}}$ is the *reverse* of $\mathbf{A}$, obtained by switching its spanning factors: if $\mathbf{A} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k$, then $\widetilde{\mathbf{A}} = \mathbf{a}_k \wedge \cdots \wedge \mathbf{a}_2 \wedge \mathbf{a}_1$. The reverse of $\mathbf{A}$ differs from $\mathbf{A}$ by a sign $(-1)^{\frac{1}{2}k(k-1)}$. You may verify that $\mathbf{A} \rfloor \widetilde{\mathbf{A}}$ is a scalar (and in Euclidean space, even a positive scalar, which can be considered as the 'norm squared' of $\mathbf{A}$; if it is zero, the blade $\mathbf{A}$ has no inverse, but this does not happen in Euclidean vector spaces).

Invertibility is a great help in solving geometric problems in a closed coordinate-free computational form. The common procedure is as follows: we know certain defining properties of objects in the usual terms of perpendicularity, spanning, rotations etcetera. These give equations typically expressed using the derived products. You combine these equations algebraically, with the goal of finding an expression for the unknown object involving only the geometric product; then division (permitted by the invertibility of the geometric product) should provide the result.

Let us illustrate this by an example. Suppose we want to find the component $\mathbf{x}_\perp$ of a vector $\mathbf{x}$ perpendicular to a vector $\mathbf{a}$. The perpendicularity demand is clearly

$$\mathbf{x}_\perp \rfloor \mathbf{a} = 0.$$

A second demand is required to relate the magnitude of $\mathbf{x}_\perp$ to that of $\mathbf{x}$. Some practice in 'seeing subspaces' in geometrical problems reveals that the area spanned by $\mathbf{x}$ and $\mathbf{a}$ is the same as the area spanned by $\mathbf{x}_\perp$ and $\mathbf{a}$, seee Figure 5a. This is expressed using the outer product:

$$\mathbf{x}_\perp \wedge \mathbf{a} = \mathbf{x} \wedge \mathbf{a}.$$

These two equations should be combined to form a geometric product. In this example, it is clear that just adding them works, yielding

$$\mathbf{x}_\perp \rfloor \mathbf{a} + \mathbf{x}_\perp \wedge \mathbf{a} = \mathbf{x}_\perp\,\mathbf{a} = \mathbf{x} \wedge \mathbf{a}.$$

This one equation contains the full geometric relationship between $\mathbf{x}$, $\mathbf{a}$ and the unknown $\mathbf{x}_\perp$. Geometric algebra solves this equation through division on the right by $\mathbf{a}$:

$$\mathbf{x}_\perp = (\mathbf{x} \wedge \mathbf{a})/\mathbf{a} = (\mathbf{x} \wedge \mathbf{a})\,\mathbf{a}^{-1}. \tag{28}$$

We rewrote the division by $\mathbf{a}$ as multiplication by the subspace $\mathbf{a}^{-1}$ to show clearly that we mean 'division on the right'.

This is an example of how the indefinite shape $\mathbf{x} \wedge \mathbf{a}$ spanned by the outer product is just the right element to generate a perpendicular to a vector $\mathbf{a}$ in its plane, through

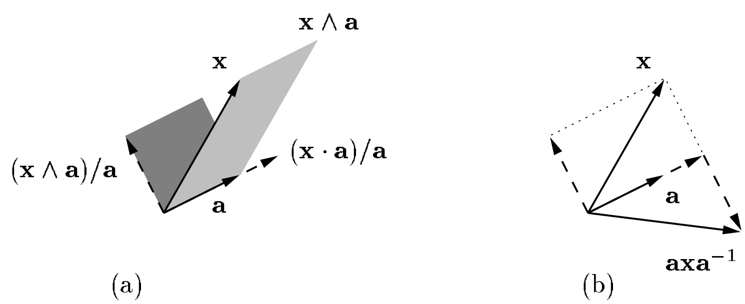

Figure 5: *(a) Projection and rejection of* $\mathbf{x}$ *relative to* $\mathbf{a}$*. (b) Reflection of* $\mathbf{x}$ *in* $\mathbf{a}$*.*

the geometric product. Note that eq.(28) agrees with the well-known expression of $\mathbf{x}_\perp$ using the inner product of vectors:

$$\mathbf{x}_\perp = (\mathbf{x} \wedge \mathbf{a})\,\mathbf{a}^{-1} = (\mathbf{x}\,\mathbf{a} - \mathbf{x} \cdot \mathbf{a})\,\mathbf{a}^{-1} = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\,\mathbf{a}. \tag{29}$$

The geometric algebra expression using outer product and inverse generalizes immediately to arbitrary subspaces $\mathbf{A}$.

4.1 Projection of subspaces

We generalize the above as the decomposition of a vector to an arbitrary blade $\mathbf{A}$, using the geometric product decomposition of eq.(21):

$$\mathbf{x} = (\mathbf{x}\,\mathbf{A})\,\mathbf{A}^{-1} = (\mathbf{x} \rfloor \mathbf{A})\,\mathbf{A}^{-1} + (\mathbf{x} \wedge \mathbf{A})\,\mathbf{A}^{-1} \tag{30}$$

It can be shown that the first term is a blade fully inside $\mathbf{A}$: it is the *projection* of $\mathbf{x}$ onto $\mathbf{A}$. Likewise, it can be shown that the second term is a vector perpendicular to $\mathbf{A}$, sometimes called the *rejection* of $\mathbf{x}$ by $\mathbf{A}$. The projection of a blade $\mathbf{X}$ of arbitrary dimensionality (grade) onto a blade $\mathbf{A}$ is given by the extension of the above, as

$$\text{projection of } \mathbf{X} \text{ onto } \mathbf{A}\text{:} \quad \mathbf{X} \;\mapsto\; (\mathbf{X} \rfloor \mathbf{A})\,\mathbf{A}^{-1}$$

Geometric algebra often allows such a straightforward extension to arbitrary dimensions of subspaces, without additional computational complexity. We will see why when we treat linear mappings in our follow-up paper.

4.2 Reflection of subspaces

The *reflection* of a vector $\mathbf{x}$ relative to a fixed vector $\mathbf{a}$ can be constructed from the decomposition of eq.(30) (used for a vector $\mathbf{a}$), by changing the sign of the rejection (see Figure 5b). This can be rewritten in terms of the geometric product:

$$(\mathbf{x} \rfloor \mathbf{a})\,\mathbf{a}^{-1} - (\mathbf{x} \wedge \mathbf{a})\,\mathbf{a}^{-1} = (\mathbf{a} \rfloor \mathbf{x} + \mathbf{a} \wedge \mathbf{x})\,\mathbf{a}^{-1} = \mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}. \tag{31}$$

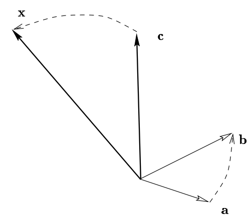

Figure 6: *Ratios of vectors*

So the reflection of $\mathbf{x}$ in $\mathbf{a}$ is the expression $\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}$, see Figure 5b; the reflection in a plane perpendicular to $\mathbf{a}$ is then $-\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}$. (We will see this 'sandwiching' operator in more detail in our follow-up paper.)

We can extend this formula to the reflection of a blade $\mathbf{X}$ relative to the vector $\mathbf{a}$, this is simply

$$\text{reflection in vector } \mathbf{a}\text{:}\quad \mathbf{X} \;\mapsto\; \mathbf{a}\,\mathbf{X}\,\mathbf{a}^{-1}.$$

and even to the reflection of a blade $\mathbf{X}$ in a k-blade $\mathbf{A}$, which turns out to be

$$\text{general reflection:}\quad \mathbf{X} \;\mapsto\; -(-1)^k\,\mathbf{A}\,\mathbf{X}\,\mathbf{A}^{-1}.$$

Note that these formulas permit you to do reflections of subspaces without first decomposing them in constituent vectors. It gives the possibility of reflecting a polyhedral object by directly using a facet representation, rather than acting on individual vertices.

4.3 Vector division

With subspaces as basic elements of computation, we can directly solve equations in similarity problems such as indicated in Figure 6:

> Given two vectors $\mathbf{a}$ and $\mathbf{b}$, and a third vector $\mathbf{c}$, determine $\mathbf{x}$ so that $\mathbf{x}$ is to $\mathbf{c}$ as $\mathbf{b}$ is to $\mathbf{a}$, i.e. solve $\mathbf{x} : \mathbf{c} = \mathbf{b} : \mathbf{a}$.

In geometric algebra the problem reads $\mathbf{x}\,\mathbf{c}^{-1} = \mathbf{b}\,\mathbf{a}^{-1}$, and through right-multiplication by $\mathbf{c}$, the solution is immediately

$$\mathbf{x} = (\mathbf{b}\,\mathbf{a}^{-1})\,\mathbf{c}. \tag{32}$$

This is a computable expression. For instance, with $\mathbf{a} = \mathbf{e}_1$, $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{c} = \mathbf{e}_2$ in the standard orthonormal basis, we obtain $\mathbf{x} = ((\mathbf{e}_1 + \mathbf{e}_2)\,\mathbf{e}_1^{-1})\mathbf{e}_2 = (1 - \mathbf{e}_1\,\mathbf{e}_2)\mathbf{e}_2 = \mathbf{e}_2 - \mathbf{e}_1$. In the follow-up paper, we'll develop this into a method to handle rotations.

5 Summary

In this paper, we have introduced blades and three products of geometric algebra. The geometric product is the most important: it is the only one that is invertible. All three products can operate directly on blades, which represent subspaces of arbitrary dimension. We hope that this introduction has given you a hint of the structure of geometric algebra. In the next paper, we will show how to wield these products to construct operations like rotations, and we will look at more advanced topics such as differentiation, linear algebra, and homogeneous representations.

References

[1] T.A. Bouma, L. Dorst, H. Pijls, *Geometric Algebra for Subspace Operations*, submitted to Applicandae Mathematicae, preprint available at `http://xxx.lanl.gov/abs/math.LA/0104159`.

[2] L. Dorst, *Honing geometric algebra for its use in the computer sciences*, in: Geometric Computing with Clifford Algebra, G. Sommer, editor;, Springer ISBN 3-540-41198-4, 2001. Preprint available at `http://www.wins.uva.nl/˜leo/clifford/`

[3] R. Goldman, *Illicit Expressions in Vector Algebra*, ACM Transactions of Graphics, vol.4, no.3, 1985, pp. 223—243.

[4] R. Goldman, *The Ambient Spaces of Computer Graphics and Geometric Modeling*, IEEE Computer Graphics and Applications, vol.20, pp. 76–84, 2000.

[5] P. Lounesto, *Marcel Riesz's work on Clifford algebras*, in: Clifford numbers and spinors, E.F. Bolinder and P. Lounesto, eds., Kluwer Academic Publishers, 1993, pp.119–241.

[6] S. Mann, N. Litke, and T. DeRose, *A Coordinate Free Geometry ADT*, Research Report CS-97-15, Computer Science Department, University of Waterloo, June, 1997. Available at: `ftp://cs-archive.uwaterloo.ca/cs-archive/CS-97-15/`