Coordinate Free Perspective Projection of Points in the Conformal Model Using Transversions

Stephen Mann

Abstract Goldman presented a method for computing a versor form of the perspective projection of points in the conformal model. His method starts with a view direction and distance to the projection plane, and constructs the eye point from that information. He then uses a rotor to rotate the eye to the origin. In this paper, an alternate construction using transversion for perspective projection is given that allows for the eye point, view direction and projection plane to be placed arbitrarily in space.

1 Introduction

In the computer graphics rendering pipeline, a series of affine transformations are applied to a model, followed by a perspective transformations. These 3D transformations can all be represented with 4×4 matrices that can be multiplied together, resulting in a single 4×4 matrix that is used to transform points. The result is that the composition of all the transformations is obtained with a single matrix multiply. After the matrix multiply, a division is required to complete the perspective transformation.

While Doran et al. [3] have shown that projective geometry can be modeled in the algebra $\mathscr{R}_{n,n}$, recently a specific conformal model of geometry, which has fewer dimensions, has gained popularity. The geometric algebra formulation of the conformal model allows for the representation of translation, uniform scale, rotation, and spherical inversion as rotors. However, operations such as shear, non-uniform scale, and perspective projection are not easily performed in the conformal model, as they are not angle preserving. Regardless, when using the conformal model for computer graphics, ideally we would have a rotor formulation of perspective transformation.

Stephen Mann
University of Waterloo, e-mail: smann@uwaterloo.ca

Recently, Goldman [6] has done just that: he has shown a way to map points through a perspective transformation in the conformal model using rotors. Goldman's idea is to map the point to project, p, to a plane through the origin, n_p. He then applies a constructed versor to n_p (a type of rotation/spherical inversion), giving a sphere s whose center s_c is the projection of p. Further, the weight of s can be used for hidden surface removal. While it may seem inconsistent to represent the non-conformal perspective transformation as a rotor in a conformal model, realize that the normalization of the sphere by its weight is the non-conformal step, and is similar to what is done in computer graphics, where a division is required to complete the perspective transformation.

Goldman's construction starts with a view direction and the distance between the view point and the projection plane, and computes fixed locations for the eye point and for the projection plane used for the perspective projection (although by using translation and scale, he can essentially locate them anywhere).

In this paper, I give a new construction for perspective projection of points in the conformal model. Like Goldman's construction, I associate a plane n_p with the point to project p, and I construct a versor to map n_p to a sphere whose center is the projection of p. My solution differs from Goldman's in the particular versor used. Further, my construction starts with the eye at an arbitrary location, while Goldman's construction uses translation and scale to get to the more general setting. More fundamentally, Goldman's construction has its basis in rotations, while mine is based on spherical inversion.

2 Background

I assume the reader is familiar with geometric algebra; for an introduction to geometric algebra, see [4, 9, 2]. This article uses the following notation. We will be working with the conformal model [1, 10, 2] built on top of R^3 with an origin n_o and a point at infinity n_∞. Recall that $\mathbf{v}\cdot n_o = \mathbf{v}\cdot n_\infty = 0$ for any vector $\mathbf{v}\in R^3$, and that $n_o^2 = n_\infty^2 = 0$ and $n_o\cdot n_\infty = -1$. In describing Goldman's construction, we will need $\bar{e} = n_o - \frac{1}{2}n_\infty$.

A point p offset from n_o by a vector $\mathbf{v}_p$ is given by

$$pt(\mathbf{v}_p) = p = n_o + \mathbf{v}_p + \frac{1}{2}||\mathbf{v}_p||^2 n_\infty.$$

A plane n with normal $\mathbf{n}$ is represented as

$$n = \mathbf{n} + d n_\infty,$$

where d is the closest distance from n to n_o. Alternatively, if the plane is known to pass through the point f, then its representation is

$$n = f\cdot(\mathbf{n}\wedge n_\infty).$$

A sphere s is represented as

$$s = s_c - \frac{1}{2} r^2 n_\infty,$$

where s_c is a point and r is the radius of the sphere. Note that a sphere of radius d with its center at $d\mathbf{v}$ has the form

$$s = n_o + d\mathbf{v},$$

where $\mathbf{v}$ is a unit vector.

A versor v is the product of 1-blades, and is used in a "sandwiching" operation or versor product to transform elements of the geometric algebra. The application of an even versor (the only versors used in this paper) is

$$x \mapsto v x v^{-1}$$

A normal n, when used as a versor, represents a reflection in the plane n; likewise, a sphere s represents a spherical inversion through s when used as a versor. The composition of a mirror reflection through a plane n and a spherical inversion through a sphere s, where n and s share a point, is a form of a *transversion*.

3 Goldman's Construction

Goldman constructs a perspective projection in the conformal model as follows. Given a unit view direction $\mathbf{v}_d$ and a distance $d/2 = \csc(\theta) \geq 1$ representing the distance between the eye point and the perspective plane, Goldman constructs a versor

$$v = \cos(\theta/2)\mathbf{v}_d + \sin(\theta/2)\bar{e},$$

which can be interpreted as a sphere. He then constructs the versor

$$R_{nv} = v\,\mathbf{v}_d.$$

Goldman sets the projection plane to $\mathbf{v}_d + \cot(\theta) n_\infty$, and locates the eye point at

$$f = n_o + (\cot(\theta) - \csc(\theta))\mathbf{v}_d + \frac{1}{2}(\cot(\theta) - \csc(\theta))^2 n_\infty. \tag{1}$$

Goldman then shows that for any point $p = pt(\mathbf{p})$ that $s = R_{nv}(\mathbf{p} - \mathbf{f})R_{nv}^{-1}$ is a sphere whose center is the projection of p onto the projection plane, where $\mathbf{f}$ is the offset of f from n_o. Goldman further proves that the weight of s can be used for hidden surface removal.

Note the order of construction: you start with $\csc(\theta)$ and a view direction, and then construct the eye point from this information, which is backwards from how things are specified in computer graphics. Further note the restriction that $d/2 = \csc(\theta) \geq 1$. This restriction is needed because Goldman's construction is rotating the eye to the n_o, with the rotation being around a subspace that is a distance 1 from the n_o. The rotation's appearance as a spherical inversion/mirror is more of an interpretation of the transformation in the conformal model; the same equations applied in the homogeneous model (where Goldman's construction first appeared) are a rotation in that model.

4 Eye point at n_o

For perspective projection, we would like to begin with an eye position, view direction, and distance to the projection plane, and then derive a projection that maps a point p onto the projection plane as a perspective projection. In this section, I observe that a spherical inversion is a type of projection, and then give a construction using a spherical inversion for perspective projection for the eye point restricted to be n_o; in the next section, I will prove its correctness, and then in Section 6, I will give the details showing how to apply the construction when the eye point is at an arbitrary location in space.

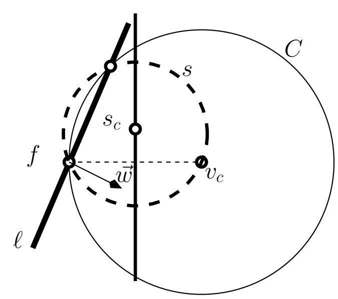

Fig. 1 Spherical inversion of a line ℓ into the dashed sphere.

To begin, consider the 2D spherical inversion in Figure 1. Given a line ℓ that passes through a point f, and given a circle C with center v_c that contains f, then the spherical inversion of ℓ in C results in a circle s that contains the two intersections of ℓ and C as well as v_c (the image of n_∞ under the inversion). The center s_c of s will lie along the perpendicular bisector of $\overline{fv_c}$, a chord of s. Further, note that for any line ℓ through f, $\overline{fv_c}$ will be a chord of the circle resulting from the spherical inversion of ℓ in C.

Thus, spherical inversion in C of a pencil of lines through the point f on C results in a pencil of circles whose centers lie on the perpendicular bisector of the segment from f to the center of C. The perpendicular bisector will be our projection plane, and the centers of the sphere s will be the desired projections.

Our real interest is in direction vectors $\mathbf{w}$, which are perpendicular to the lines ℓ, where we want to compute the intersection of $\mathbf{w}$ with the projection plane. We would like s_c to be this intersection, but we see that from the figure that s_c is on the wrong side of $\overline{fv_c}$ to be the desired intersection, and thus the need for the mirror reflection. What follows are the details of scale, etc., to make this construction into a perspective projection.

Figure 2, left, illustrates my construction in the 2D setting. Given an eye point $f = n_o$, view direction $\mathbf{v}_d$ (where $|\mathbf{v}_d| = 1$) and distance to the projection plane $d/2$, we will construct a plane n_v containing n_o as our mirror plane and a sphere v tangent to n_v at n_o in which to do the spherical inversion.

The mirror reflection in a plane through n_o perpendicular to $\mathbf{v}_d$ is easily accomplished by the plane $n_v = \mathbf{v}_d$. We construct a point v_c and a sphere v centered at v_c of radius d as follows:

$$\begin{aligned} v_c &= n_o + d\mathbf{v}_d + \tfrac{1}{2}||d\mathbf{v}_p||^2 n_\infty \\ &= n_o + d\mathbf{v}_d + \tfrac{1}{2}d^2 n_\infty \end{aligned} \tag{2}$$

$$\begin{aligned} v &= v_c - \tfrac{1}{2}d^2 n_\infty \\ &= n_o + d\mathbf{v}_d. \end{aligned} \tag{3}$$

Now construct the transversion

$$\begin{aligned} R_{nv} &= v\, n_v \\ &= (n_o + d\mathbf{v}_d)\mathbf{v}_d \\ &= d - \mathbf{v}_d\, n_o. \end{aligned} \tag{4}$$

Associate the plane $n_p = \mathbf{p}$ with the point $p = pt(\mathbf{p})$. To project the point p, we apply R_{nv} to n_p, giving the sphere s:

$$s = R_{nv}\, n_p\, R_{nv}^{-1}. \tag{5}$$

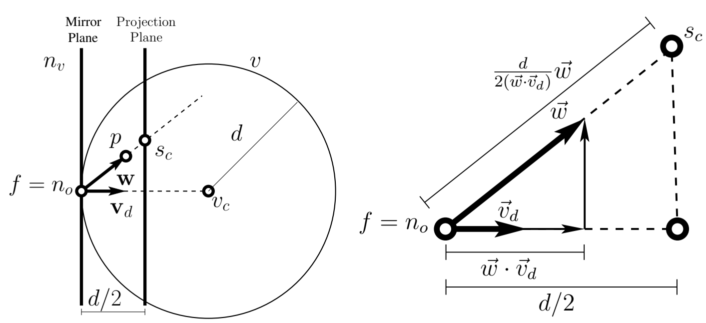

Fig. 2 Left: Projection with eye at n_o. Right: Scaling of $\mathbf{w}$ under projection.

In the next section, I will show that the center s_c of sphere s is the projection of p and that the weight s can be used for hidden surface removal.

5 Proof of Projection

The following theorem shows that the construction in the previous section acts as a perspective projection of a point $p = pt(\mathbf{p})$.

Theorem 1. *Given the eye point $f = n_o$, a view direction $\mathbf{v}_d$, and a distance to the projection plane $d/2$.*

Construct the plane $n_v = \mathbf{v}_d$, a versor v using (2) and (3) and a transversion $R_{nv} = v\,n_v$.

Then for any point $p = pt(\mathbf{v}_p)$, and associated plane $n_p = \mathbf{v}_p$, the application of R_{nv} to n_p,

$$s = R_{nv}\,n_p\,R_{nv}^{-1},$$

is a sphere whose center s_c is the projection of p onto the plane $\mathbf{v}_d + \frac{d}{2}n_\infty$ relative to the eye point n_o. Further, the weight of s is proportional to $\mathbf{v}_p$.

Proof. Let $\mathbf{w} = \mathbf{p}$ (to match the notation of the next section). We know $v = d\mathbf{v}_d + n_o$ and $R_{nv} = v\,n_v = d - \mathbf{v}_d\,n_o$. Then $R_{nv}^{-1} = \frac{1}{d} + \frac{\mathbf{v}_d\,n_o}{d^2}$. Applying the transversion R_{nv} to $\mathbf{w}$ yields

$$\begin{aligned}
R_{nv}\,\mathbf{w}\,R_{nv}^{-1} &= (d - \mathbf{v}_d\,n_o)\,\mathbf{w}\left(\frac{1}{d} + \frac{\mathbf{v}_d\,n_o}{d^2}\right) \\
&= \mathbf{w} + \frac{d\mathbf{w}(\mathbf{v}_d\,n_o)}{d^2} - \frac{1}{d}(\mathbf{v}_d\,n_o)\mathbf{w} - \frac{(\mathbf{v}_d\,n_o)\mathbf{w}(\mathbf{v}_d\,n_o)}{d^2} \\
&= \mathbf{w} + \frac{1}{d}(\mathbf{w}\mathbf{v}_d + \mathbf{v}_d\mathbf{w})n_o \\
&= \mathbf{w} + \frac{2}{d}(\mathbf{w}\cdot\mathbf{v}_d)n_o
\end{aligned}$$

Normalizing with respect to n_o, we see that

$$s = R_{nv}\mathbf{w}R_{nv}^{-1} = \frac{d}{2(\mathbf{w}\cdot\mathbf{v}_d)}\mathbf{w} + n_o$$

is a sphere whose center s_c lies in the plane $\mathbf{v}_d + \frac{d}{2}n_\infty$ (the projection plane), that this versor essentially scales $\mathbf{w}$ (and so s_c is the projection of p; see Figure 2, right), and further, the weight of the unnormalized sphere is proportional to the $\mathbf{v}_d$ component of $\mathbf{w}$ and thus proportional to $\mathbf{v}_p$. □

6 Arbitrary eye point

Rather than place f at n_o, we want to place the eye point f at an arbitrary location. Since the rotor is constructed as a mirror reflection and a spherical inversion, the perspective projection construction of Section 4 generalizes to f at an arbitrary point by constructing a plane through f and a sphere with the appropriate center that passes through f. The following gives the formulas for this more general setting.

A plane passing through f perpendicular to the view direction $\mathbf{v}_d$ is given by

$$n_v = f \cdot (\mathbf{v}_d n_\infty).$$

A sphere of radius d centered a distance d from f along the view direction $\mathbf{v}_d$ is given by

$$\begin{aligned} v_c &= n_o + (\mathbf{f} + d\mathbf{v}_d) + \tfrac{1}{2}||\mathbf{f} + d\mathbf{v}_p||^2 n_\infty \\ v &= v_c - \tfrac{1}{2} d^2 n_\infty \end{aligned}$$

Our perspective transformation versor R_{nv} is again given by

$$R_{nv} = v n_v.$$

We now need to map the vector from the eye point f to the point p to a plane through p. We first construct the vector $\mathbf{w} = \mathbf{p} - \mathbf{f}$, and then the plane to map is

$$n_w = f \cdot (\mathbf{w} n_\infty).$$

Applying the versor R_{nv} to n_w results in a sphere who center is the desired projection of p.

This method was implemented in GAViewer [5], which was very helpful in devising this construction; a screen shot is shown in Figure 3. In this figure, the red vectors are based at n_o; the blue vectors are based at f. Also shown in the figure are the sphere v with center v_c, the projection plane, and the plane n_w that gets mapped to a sphere centered at s_c.

7 Summary and Future Work

In this paper, I have presented a versor for performing the perspective projection of points in the conformal model. My result is a related to Goldman's construction, but unlike his, my method directly allows for an arbitrary eye point. Further, my construction does not require a minimum distance to the projection plane.

Beyond this, there are more fundamental differences in the two constructions. In particular, Goldman's construction is based on rotating the eye point to n_o, and thus the precise positioning of the projection plane and eye point with trigonometric

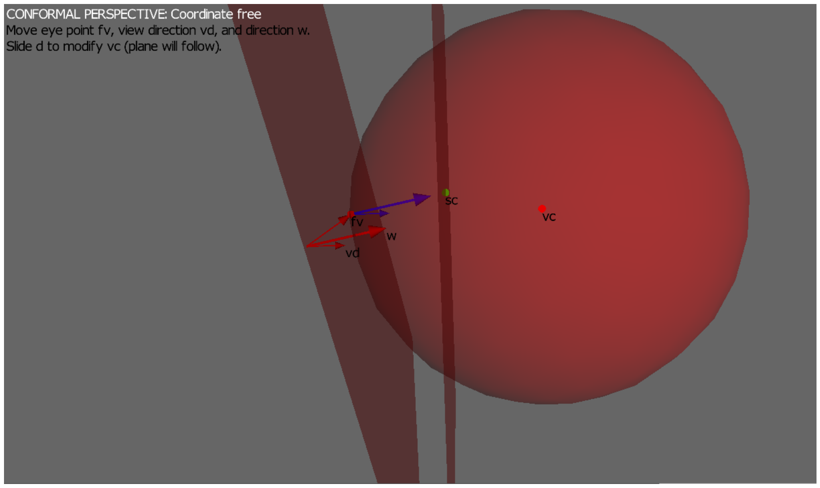

Fig. 3 GAViewer screen shot showing the projection of `w` onto `sc`.

functions seen in Section 3. That his construction results in a spherical inversion in the conformal model is mostly a result of interpretation, since the same equations in the homogeneous model are a rotation [11].

In comparison, my construction is based on the idea that a spherical inversion acts as a form of projective transformation (Figure 2). Combining a spherical inversion with a mirror reflection allows us to obtain a perspective projection at an arbitrary eye point, view direction, etc. Further, the two methods give different rotors when using the same view position, etc. These different rotors transform Euclidean vectors (planes through the origin) to the same spheres, but give different results when applied to n_o and n_∞. See the appendix for a specific example of this.

Note also that the idea of using a transversion for perspective projection can also be used in a dual quaternion model [12] and in Gunn's model of Euclidean geometry [8].

A variety of questions remain:

1. The construction is for points. Can it be extended to other objects in the conformal model?
2. Although perspective projection is modeled as a versor, this versor can not be composed with other versor transformations of the conformal model, since we have to map the points to transform into planes. Is there a construction of a rotor for perspective projection that can be applied directly to points rather than to vectors (planes)? Such a rotor would allow us to compose the modeling and perspective transformations into one transformation.
3. Although the weight of the resulting sphere can be used for a form of hidden surface removal, we can not effectively use the resulting spheres to do scan conversion of polygons in screen space, as the mapping of the weight is inconsistent

(perspectively) with perspective mapping of the points. Instead, the scan conversion would have to be done before the perspective division is applied.

8 Appendix: Example of differing rotors

This section gives an example of where Goldman and my constructions give the same result on Euclidean vectors but different results on n_o and n_∞.

Place the eye f at $f = n_o$. Let the view direction $\mathbf{v}_d$ be e_1. Let the distance to the view plane be $d = 2$ (this results in $\theta = \pi/6$ for Goldman's method). Then the rotor constructed by my method is $R_{nv} = 4 - e_1 n_o$, while the rotor constructed by Goldman's method (composed with a translation to move the eye to the origin) is $R_{nv} = 3.85 - e_1 \wedge n_o + 0.13 n_o \wedge n_\infty$. When either versor is applied to e_1, both methods give $R_{nv} e_1 R_{nv}^{-1} = e_1 + 0.5 n_o$. When applied to n_o, my method results in $R_{nv} n_o R_{nv}^{-1} = n_o$ while Goldman's gives $R_{nv} n_o R_{nv}^{-1} = 0.93 n_o$, and when applied to n_∞ my method results in $R_{nv} n_\infty R_{nv}^{-1} = 0.5 e_1 + 0.13 n_o + n_\infty$ while Goldman's gives $R_{nv} n_\infty R_{nv}^{-1} = 0.54 e_1 + 0.13 n_o + 1.07 n_\infty$.

References

1. P. Anglès. Construction de revêtements du group conform d'un espace vectorial muni d'une "métrique" de type (p,q). *Annales de l'Institut Henri Poincaré*, Section A, Vol. XXXIII:33–51, 1980.
2. Eduardo Bayro-Corrochano. *Geometric Computing: for Wavelet Transforms, Robot Vision, Learning, Control and Action*. Springer, 2010.
3. C. Doran, D. Hestenes, F. Sommen, and N. Van Acker. Lie groups as spin groups. *Journal of Mathematical Physics*, 34(8), 1993.
4. Leo Dorst, Daniel Fontijne, and Stephen Mann. *Geometric Algebra for Computer Science*. Morgan-Kaufmann, 2007.
5. Daniel Fontijne and Leo Dorst. GAviewer, interactive vizualiation software for geometric algebra, 2002-2010. Downloadable at www.geometricalgebra.net/downloads.
6. Ron Goldman. Computing perspective projections in 3-dimensions using rotors in the conformal model of clifford algebra. *Submitted for publication*.
7. Ron Goldman. A homogeneous model for 3-dimensional computer graphics based on clifford algebra for R^3. In L. Dorst and J. Lasenby, editors, *Guide to Geometric Algebra in Practice*. Springer-Verlag, 2011.
8. Charles Gunn. On the homogeneous model of euclidean geometry. In *Guide to Geometric Algebra in Practice*, pages 297–328. Springer, 2011.
9. D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus*. Reidel, 1984.
10. David Hestenes, Hongbo Li, and Alyn Rockwood. A unified algebraic framework for classical geometry. In *Geometric Computing with Clifford Algebra*. Springer, 1999.
11. Xiaohong Jia and Ron Goldman. Representing perspective projections as rotors in the homogeneous model of clifford algebra for 3-dimensional euclidean space. *Submitted for publication*.
12. Ladislav Kavan, Steven Collins, Jiri Zara, and Carol O'Sullivan. Geometric skinning with approximate dual quaternion blending. *ACM TOG*, 27(4), 2008.