$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cdots}}}}}$$

Alf van der Poorten and Continued Fractions

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Why Focus on Continued Fractions?

"It is notorious that it is generally damnably difficult to explicitly compute the continued fraction of a quantity presented in some other form..."

— VDP (1992)

- VDP loved continued fractions
- They feature in at least 1/4 of all his published papers...
  - ...including his most-cited papers
- He obtained many interesting and novel results about them

VDP's Continued Fraction Themes

- Use the 2×2 matrix formulation of continued fractions
- Work with continued fractions of formal Laurent series; then "specialize" by setting $X = c$
  - or reduce mod p
- Use tools to manipulate expansions; e.g., Raney's theorem
- Use the "folding lemma" of Mendès France

Basics of Continued Fractions

- a continued fraction is an object of the form

$$\alpha \quad = \quad a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cdots}}$$

- A continued fraction is said to be *simple* if the numerators b_i are all 1. In this case we abbreviate it by $[a_0, a_1, a_2, \ldots]$.
- For real numbers α, the simple continued fraction expansion is essentially unique if the *partial quotients* a_i are integers ≥ 1 for $i \geq 1$;
- rational numbers have terminating expansions;
  irrational numbers have infinite, nonterminating expansions;

Expansions for quadratic numbers

- an expansion is ultimately periodic iff it is the root of a quadratic equation.
- Examples:
  - $\sqrt{2} = [1, 2, 2, 2, \ldots];$
  - $\sqrt{7} = [2, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \ldots];$
  - $\frac{9+2\sqrt{39}}{15} = [1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \ldots].$

Expansions of some famous numbers

Some familiar numbers have expansions with an obvious pattern:

- $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$;
- $e^{1/2} = [1, 1, 1, 1, 5, 1, 1, 9, 1, 1, 13, \ldots]$;
- $\tan 1 = [1, 1, 1, 3, 1, 5, 1, 7, 19, 1, 11, \ldots]$;
- $\tanh 1 = [0, 1, 3, 5, 7, 9, 11, 13, \ldots]$;
- $e^2 = [7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, \ldots]$

Expansions of some famous numbers

But for some numbers no one knows an easy-to-describe pattern:

- $\pi = [3, 7, 15, 1, 292, \ldots]$;
- $\sqrt[3]{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, \ldots]$;
- $\gamma = [0, 1, 1, 2, 1, 2, 1, 4, 3, 13, \ldots]$;
- $\zeta(3) = [1, 4, 1, 18, 1, 1, 1, 4, 1, 9, \ldots]$
- $\log 2 = [0, 1, 2, 3, 1, 6, 3, 1, 1, 2, 1, 1, 1, \ldots]$

My introduction to VDP: Apéry and $\zeta(3)$

"A proof that Euler missed ... Apéry's proof of the irrationality of $\zeta(3)$" *Math. Intelligencer* **1** (1979), 195–203.

- VDP's most cited paper (60 citations on MathSciNet; 295 on Google Scholar)

- a breathless account of Apéry's June 1978 lecture at the *Journées Arithmétiques* announcing a proof the irrationality of $\zeta(3)$

- typical VDP style, e.g.,

"Though there had been earlier rumours of his [Apéry] claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions."

In that paper:

$$\zeta(3) = \cfrac{6}{5 - \cfrac{1}{117 - \cfrac{64}{535 - \cfrac{729}{1436 - \cfrac{4096}{3105 - \cdots}}}}}$$

where the general term is

$$-\frac{n^6}{34n^3 + 51n^2 + 27n + 5}.$$

A continued fraction for $\zeta(2)$

Also in that paper:

$$\zeta(2) = \frac{\pi^2}{6} = \cfrac{5}{3 + \cfrac{1}{25 + \cfrac{16}{69 + \cfrac{81}{135 + \cfrac{256}{223 + \cdots}}}}}$$

where the general term is

$$+\frac{n^4}{11n^2 + 11n + 3}.$$

Red herring or open problem?

Convergents

- define $p_{-2} = 0$; $p_{-1} = 1$; $q_{-2} = 1$; $q_{-1} = 0$, and

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \\ q_n &= a_n q_{n-1} + q_{n-2} \end{aligned}$$

  for $n \geq 0$.
- then $p_n/q_n = [a_0, a_1, \ldots, a_n]$.
- the converse is true under suitable hypotheses.
- therefore

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{bmatrix} \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}.$$

Convergents

- It follows that

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$$

- this idea is apparently due to Hurwitz (c. 1917), Frame (1949), and Kolden (1949), independently — but popularized by VDP

- take the transpose to get

$$\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$$

- and so $q_n / q_{n-1} = [a_n, a_{n-1}, \ldots, a_1]$, a theorem due to Galois in 1829.

Convergents

- take the determinant to get

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n+1}, \tag{1}$$

  a famous identity.

- divide (1) by $q_n q_{n-1}$ to get

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n+1}}{q_{n-1} q_n}$$

  and hence

$$\sum_{1 \le i \le n} \frac{(-1)^{n+1}}{q_{n-1} q_n} = \sum_{1 \le i \le n} \left(\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}} \right) = \frac{p_n}{q_n} - \frac{p_0}{q_0};$$

- Therefore

$$\frac{p_n}{q_n} = a_0 + \sum_{1 \le i \le n} \frac{(-1)^{n+1}}{q_{n-1} q_n}$$

Convergents

- now, letting $n \to \infty$, we see that if $\theta = [a_0, a_1, a_2, \ldots]$ is irrational, then
$$\theta = a_0 + \sum_{i \geq 1} \frac{(-1)^{n+1}}{q_{n-1} q_n}.$$

Continued Fractions for Algebraic Numbers

Theorem. (Bombieri & VDP, 1995) Let $\gamma > 1$ be the unique positive zero of the polynomial $f(X)$, and suppose

$$\gamma = [a_0, a_1, \ldots].$$

Then under some technical conditions we have $a_{n+1} = \lfloor \gamma_{n+1} \rfloor$ where p_n/q_n denotes the n'th convergent and

$$\gamma_{n+1} = \frac{(-1)^{n+1}}{q_n^2} \frac{f'(p_n/q_n)}{f(p_n/q_n)} - \frac{q_{n-1}}{q_n} + \frac{(-1)^n}{q_n^2} \sum_{\beta \neq \gamma:\ f(\beta)=0} \left(\frac{p_n}{q_n} - \beta \right)^{-1}.$$

Example. Let $f(X) = X^3 - 2$. Then

$$\sqrt[3]{2} = [1, a_1, a_2, \ldots]$$

with

$$a_{n+1} = \left\lfloor \frac{(-1)^{n+1}}{q_n} \frac{3p_n^2}{p_n^3 - 2q_n^3} - \frac{q_{n-1}}{q_n} \right\rfloor$$

for $n \geq 0$.

Some Unusual Continued Fraction Expansions

$$2\sum_{n\geq 0} 2^{-2^n} = [1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, \ldots]$$

(bounded partial quotients)

$$\begin{aligned}\sum_{n\geq 2} 2^{-F_n} \quad &= \quad [0, 1, 10, 6, 1, 6, 2, 14, 4, 124, 2, 1, 2, 2039, \\ &\quad\; 1, 9, 1, 1, 1, 262111, 2, 8, 1, 1, 1, 3, 1, 536870655, \\ &\quad\; 4, 16, 3, 1, 3, 7, 1, 140737488347135, \ldots].\end{aligned}$$

(large partial quotients are close to powers of 2)

Some Unusual Continued Fraction Expansions

$$\sum_{n\geq 1} 10^{-n!} = [0, 9, 11, 99, 1, 10, 9, 999999999999, 1, 8, 10, 1, 99, 11,$$

$$9, \overbrace{999999999999999 \cdots 9999999999999999999}^{72}, \ldots]$$

(Liouville's transcendental number)

$$\sum_{i\geq 0} \frac{(-1)^i}{b_i} = [0, 1, 1, 1, 2^2, 3^2, 14^2, 129^2, 25298^2, \ldots]$$

where $b_0 = 1$, and $b_{n+1} = b_n^2 + b_n$ for $n \geq 0$. (Cahen's constant)

Each of these can be viewed as *specializations* of continued fractions for elements of $\mathbb{Q}((X^{-1}))$: formal Laurent series.

Theorem. (Artin, 1924)
Let n be an integer. Any formal power series

$$f(X) = \sum_{-\infty < i \leq n} b_i X^i \in \mathbb{Q}((X^{-1}))$$

can be expressed uniquely as a continued fraction

$$\begin{aligned} f(X) &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \\ &= [a_0, a_1, a_2, \ldots] \end{aligned}$$

where $a_i \in \mathbb{Q}[X]$ for $i \geq 0$ and $\deg a_i > 0$ for $i > 0$.

An Example

$$
\begin{aligned}
f(X) &:= \frac{1}{\sqrt{X^2-1}} = \sum_{k\geq 0} 2^{-2k}\binom{2k}{k} X^{-2k-1} \\
&= X^{-1} + \frac{1}{2}X^{-3} + \frac{3}{8}X^{-5} + \frac{5}{16}X^{-7} + \frac{35}{128}X^{-9} + \cdots \\
&= [0,\ X,\ -2X,\ 2X,\ -2X,\ 2X,\ -2X,\ \ldots]
\end{aligned}
$$

Here are the first few convergents to the continued fraction for f:

n	0	1	2	3	4
a_n	0	X	$-2X$	$2X$	$-2X$
p_n	0	1	$-2X$	$-4X^2+1$	$8X^3-4X$
q_n	1	X	$-2X^2+1$	$-4X^3+3X$	$8X^4-8X^2+1$

An Example (continued)

It can be proved that

- $p_n(X) = (-1)^{\lfloor n/2 \rfloor}\ U_{n-1}(X)$;
- $q_n(X) = (-1)^{\lfloor n/2 \rfloor}\ T_n(X)$;

where T and U are the Chebyshev polynomials of the first and second kinds, respectively.

The Folding Lemma

Lemma. Let

- w denote the word $a_1, a_2, \ldots, a_n$,
- let $-w$ denote the word w with all terms negated, and
- let w^R denote the reversed word.

Then

$$\frac{p_n}{q_n} + \frac{(-1)^n}{xq_n^2} = [a_0,\ w,\ x - \frac{q_{n-1}}{q_n}] = [a_0,\ w,\ x,\ -w^R].$$

Proof of the Folding Lemma

Proof. We have

$$
\begin{aligned}
[a_0, w, x \quad - \quad q_{n-1}/q_n] &\leftrightarrow \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} x - q_{n-1}/q_n & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} xp_n - (p_n q_{n-1} - p_{n-1} q_n)/q_n & p_n \\ xq_n & q_n \end{bmatrix} \\
&\leftrightarrow \frac{p_n}{q_n} + \frac{(-1)^n}{xq_n^2}.
\end{aligned}
$$

∎

Application of the folding lemma

Define

$$h_n(X) = X \sum_{0 \le i \le n} X^{-2^i} \in \mathbb{Q}(X^{-1}).$$

We find

$$\begin{aligned}
h_1(X) &= 1 + X^{-1} = [1, X] \\
h_2(X) &= 1 + X^{-1} + X^{-3} = [1, X, -X, -X] \\
h_3(X) &= [1, X, -X, -X, -X, X, X, -X] \\
h_4(X) &= [1, X, -X, -X, -X, X, X, -X, \\
&\quad -X, X, -X, -X, X, X, X, -X].
\end{aligned}$$

In general, by the Folding Lemma, if

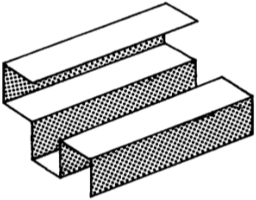
$$h_i(X) = [1,\ Y],$$

then

$$h_{i+1}(X) = [1,\ Y,\ -X,\ -Y^R].$$

Continued fractions and folding paper

So — as explained in VDP's 2nd most cited paper, entitled FOLDS! — the terms of this continued fraction are given by folding a piece of paper repeatedly, then unfolding and reading the sequence of folds as "up" or "down".



Symmetry in continued fractions

FOLDS! also contains the following beautiful proof of H. J. S. Smith of Fermat's two-square theorem:

Let $p = 4k + 1$ be a prime. Consider the continued fraction expansion of p/n for $2 \leq n \leq (p-1)/2$:

$$\frac{p}{n} = [a_1, \ldots, a_r].$$

Then $a_1 \geq 2$ and $a_r \geq 2$. By the result mentioned previously,

$$[a_r, \ldots, a_1] = \frac{p}{n'}$$

for some integer n' with $2 \leq n' \leq (p-1)/2$.

This gives an involution of the $(p-1)/2 - 1 = 2k - 1$ numbers from 2 to $(p-1)/2$; so some integer x is paired with itself.

Then

$$\frac{p}{x} = [a_1, \ldots, a_r] = [a_r, \ldots, a_1].$$

Symmetry in continued fractions

If it's a palindrome of odd length, say

$$[b_1, \ldots, b_s, b, b_s, \ldots, b_1]$$

then, using the pairing

$$[b_1, \ldots, b_s] \leftrightarrow \begin{bmatrix} a & a' \\ b & b' \end{bmatrix}$$

we see

$$\begin{aligned} \frac{p}{x} &= [b_1, \ldots, b_s, c, b_s, \ldots, b_1] \leftrightarrow \begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \begin{bmatrix} b & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \\ &= \begin{bmatrix} a^2c + 2aa' & abc + ab' + a'b' \\ abc + ab' + a'b & b^2c + 2bb' \end{bmatrix}, \end{aligned}$$

a contradiction, since we cannot have
$p = a^2c + 2aa' = a(ac + 2a')$.

Symmetry in continued fractions

So the palindrome must be of even length, say,

$$\begin{aligned}\frac{p}{x} &= [b_1, \ldots, b_s, b_s, \ldots, b_1] \leftrightarrow \begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \\ &= \begin{bmatrix} a^2 + a'^2 & ab + a'b' \\ ab + a'b' & b^2 + b'^2 \end{bmatrix},\end{aligned}$$

expressing p as the sum of two squares.

Taking the determinant, we get $p(b^2 + b'^2) - x^2 = 1$, or $x^2 \equiv -1 \pmod{p}$ — so we get a square root of $-1 \pmod{p}$, for free!

Specialization

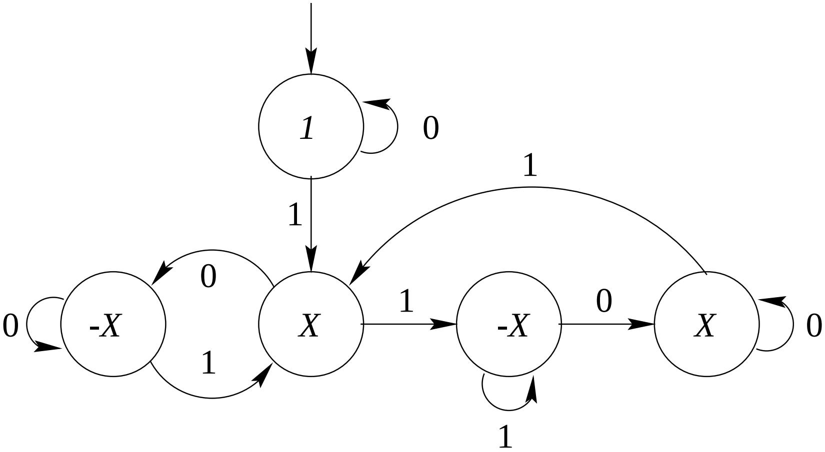
The expansion

$$X \sum_{0 \leq i \leq n} X^{-2^i} = [1, X, -X, -X, -X, X, X, -X, -X, X, -X, -X, \ldots$$
$$= [a_0, a_1, a_2, \ldots]$$

is atypical in several respects:

- the partial quotients a_i have integer coefficients;
- the coefficients lie in a finite set;
- the partial quotients (except the first) are all linear monomials in X.
- the partial quotients are given by a finite automaton

A MSD-first automaton for the partial quotients



Specialization

Now "specialize", setting $X = 2$. We get

$$2 \sum_{i \geq 0} 2^{-2^i} = [1, 2, -2, -2, -2, 2, 2, -2, -2, 2, \cdots],$$

which is an "illegal" expansion ... so we need to "remove" the -2's somehow.

This can be done with the following

Lemma.

$$\begin{aligned} [a, -b, c] &= [a-1,\ 1,\ b-2,\ 1,\ c-1]; \\ [a, 0, b] &= [a+b]. \end{aligned}$$

Application of the Folding Lemma

Thus

$$\begin{aligned}
2\sum_{i\geq 0} 2^{-2^i} &= [1,2,-2,-2,-2,2,2,-2,-2,2,-2,\cdots] \\
&= [1,1,1,0,1,-3,-2,2,2,-2,-2,2,-2,\cdots] \\
&= [1,1,2,-3,-2,2,2,-2,-2,2,-2,\cdots] \\
&= [1,1,1,1,1,1,-3,2,2,-2,-2,2,-2,\cdots] \\
&= [1,1,1,1,1,0,1,1,1,1,2,-2,-2,2,-2,\cdots] \\
&= [1,1,1,1,2,1,1,1,2,-2,-2,2,-2,\cdots] \\
&= [1,1,1,1,2,1,1,1,1,1,0,1,-3,2,-2,\cdots] \\
&= [1,1,1,1,2,1,1,1,1,2,-3,2,-2,\cdots] \\
&= [1,1,1,1,2,1,1,1,1,1,1,1,1,1,-2,\cdots] \\
&\cdots
\end{aligned}$$

Application of the Folding Lemma

More generally, we have the following

Theorem. (VDP and JOS, 1990)

Let $a_0 = 1$, $a_i = \pm 1$ for $i \geq 1$.

Then the number

$$2 \sum_{i \geq 0} a_i 2^{-2^i}$$

is transcendental and its continued fraction expansion consists solely of 1's and 2's.

Liouville's Transcendental Number

Let $f_n(X) = \sum_{1 \le k \le n} X^{-k!}$, and define $f(X) = f_\infty(X)$.

In 1851, Liouville proved that $f(b)$ is transcendental for all integers $b \ge 2$.

We can apply the folding Lemma to this number; we get

$$\begin{aligned}
f_1(X) &= [0,\ X] \\
f_2(X) &= [0,\ X,\ -1,\ -X] = [0,\ X-1,\ X+1] \\
f_3(X) &= [0,\ X-1,\ X+1,\ X^2,\ -X-1,\ -X+1] \\
f_4(X) &= [0,\ X-1,\ X+1,\ X^2,\ -X-1,\ -X+1,\ -X^{12}, \\
&\qquad X-1,\ X+1,\ -X^2,\ -X-1,\ -X+1] \\
&\ \vdots
\end{aligned}$$

Liouville's Transcendental Number

Hence, we get

$$\begin{aligned} f(X) &= [0,\ X-1,\ X+1,\ X^2,\ -X-1,\ -X+1,\ -X^{12}, \\ &\quad X-1,\ X+1,\ -X^2,\ -X-1,\ -X+1,\ -X^{72},\ldots] \end{aligned}$$

where the large partial quotients are

$$n! - 2(n-1)! = (n-2)(n-1)!.$$

An Unusual Continued Fraction

$$
\begin{aligned}
2^{-1}+2^{-2} \quad + & \quad 2^{-3}+2^{-5}+2^{-8}+2^{-13}+\cdots+2^{-F_n}+\cdots \\
= & \quad [0,1,10,6,1,6,2,14,4,124,2,1,2,2039, \\
& \quad 1,9,1,1,1,262111,2,8,1,1,1,3,1,536870655, \\
& \quad 4,16,3,1,3,7,1,140737488347135,\ldots].
\end{aligned}
$$

A Fibonacci Power Series

Similarly

$$\begin{aligned} X^{-1} \quad + & \quad X^{-2} + X^{-3} + X^{-5} + X^{-8} + \cdots + X^{-F_n} + \cdots + \\ = & \quad [0,\ X-1,\ X^2+2X+2,\ X^3-X^2+2X-1, \\ & \quad -X^3+X-1,\ -X,\ -X^4+X,\ -X^2, \\ & \quad -X^7+X^2,\ -X-1,\ X^2-X+1,\ X^{11}-X^3, \\ & \quad -X^3-X,\ -X,\ X,\ X^{18}-X^5,\ -X,\ X^3+1,\ X, \\ & \quad -X,\ -X-1,\ -X+1,\ -X^{29}+X^8,\ X-1,\ldots] \end{aligned}$$

What's going on here?

"We remark that to our surprise, and horror, continued fraction expansion of formal power series appears to adhere to the cult of Fibonacci." – VDP (1998)

Theorem. (VDP & JOS, 1993). Let (F_n) be the sequence of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Let (L_n) be the sequence of Lucas numbers, obeying the same recurrence relation, but with $L_0 = 2$ and $L_1 = 1$.
Define

$$s_n = X^{-1} + X^{-2} + X^{-3} + X^{-5} + \cdots + X^{-F_n} = [0, f_n].$$

Then for $n \geq 11$ we have s_{n+1}

$$\begin{aligned}
&= [0, f_n, 0, -f_{n-4}, -X^{L_{n-4}}, f_{n-4}^R, 0, -f_{n-3}, X^{F_{n-4}}, f_{n-3}^R] \\
&= [0, g_n, X^{F_{n-5}}, f_{n-4}^R, 0, -f_{n-4}, -X^{L_{n-4}}, f_{n-4}^R, 0, -f_{n-3}, X^{F_{n-4}}, f_{n-3}^R] \\
&= [0, g_n, X^{F_{n-5}} - X^{L_{n-4}}, f_{n-4}^R, 0, -f_{n-3}, X^{F_{n-4}}, f_{n-4}^R] \\
&= [0, g_{n+1}, X^{F_{n-4}}, f_{n-3}^R].
\end{aligned}$$

and

$$s_\infty = X^{-1} + X^{-2} + \cdots + X^{-F_n} + \cdots = \lim_{n \to \infty} [0, g_h],$$

where

$$g_n = f_{n-1}, 0, -f_{n-5}, -X^{L_{n-5}}, f_{n-5}^R, 0, -f_{n-4}.$$

The Fibonacci power series

Corollary. A polynomial p is a partial quotient in the expansion of the Fibonacci power series if and only if p or $-p$ occurs in the following list:

$$X + 1;$$
$$X^2 \pm X + 1;$$
$$X^2 + 2X + 2;$$
$$X^3 + 1;$$
$$X^3 + X;$$
$$X^3 - X + 1;$$
$$X^3 - X^2 + 2X - 1;$$
$$X^{F_n};$$
$$X^{L_{n+2}};$$
$$X^{L_{n+1}} - X^{F_n}$$

for $n \geq 1$

The Fibonacci power series

By specialization we get

Corollary. The large partial quotients

$$2039, 262111, 536870655, 140737488347135, \ldots$$

in the continued fraction expansion of

$$2^{-1} + 2^{-2} + 2^{-3} + 2^{-5} + 2^{-8} + \cdots$$

differ by 1 from the numbers

$$2^{L_{h+1}} - 2^{F_h}$$

for $h \geq 4$.

An open question

Let $f(X) = \sum_{n \geq 0} a_n X^{-n} \in \mathbb{Q}[[X^{-1}]]$, where $a_n \in \{0, 1\}$.

Give necessary and sufficient conditions for the continued fraction of $f(X)$ to have only polynomials with integer coefficients in its continued fraction expansion.

Now let's go back to $h(X) = X \sum_{i \geq 0} X^{-2^i}$ ...

Convergents to $h(X)$

Here is a table of the first few convergents to $h(X)$:

n	a_n	$p_n(X)$	$q_n(X)$
0	1	1	1
1	X	$X+1$	X
2	$-X$	$-X^2-X+1$	$-X^2+1$
3	$-X$	X^3+X^2+1	X^3
4	$-X$	$-X^4-X^3-X^2-2X+1$	$-X^4-X^2+1$
5	X	$-X^5-X^4-X^2+X+1$	$-X^5+X$
6	X	$-X^6-X^5-X^4-2X^3$ $-X+1$	$-X^6-X^4+1$
7	$-X$	$X^7+X^6+X^4+1$	X^7
8	$-X$	$-X^8-X^7-X^6-2X^5$ $-X^4-2X^3-2X+1$	$-X^8-X^6-X^4+1$
9	X	$-X^9-X^8-X^6-X^5$ $-X^4-2X^2+X+1$	$-X^9-X^5+X$

Denominators of the Convergents

n	$a_n(X)$	$q_n(X)$
0	1	1
1	X	X
2	$-X$	$-X^2+1$
3	$-X$	X^3
4	$-X$	$-X^4-X^2+1$
5	X	$-X^5+X$
6	X	$-X^6-X^4+1$
7	$-X$	X^7
8	$-X$	$-X^8-X^6-X^4+1$
9	X	$-X^9-X^5+X$
10	$-X$	$X^{10}-X^8-X^4-X^2+1$
11	$-X$	$-X^{11}+X^3$
12	X	$-X^{12}+X^{10}-X^8-X^2+1$
13	X	$-X^{13}-X^9+X$
14	X	$-X^{14}-X^{12}-X^8+1$

Properties of the Denominators of the Convergents

Theorem. (Allouche, Lubiw, Mendès France, VDP, JOS)
All of the coefficients of the denominators of the convergents to $X \sum_{i \geq 0} X^{-2^i}$ lie in $\{0, \pm 1\}$.

Proof. (Sketch)

- The low-order terms of $q_{2^k+n-1}(X)$ (i.e., those of degree $< 2^k$) are exactly the same as those of $q_{2^k-n-1}(X)$;
- The high-order terms of $q_{2^k+n-1}(X)$ are, up to a change of signs of individual terms, equal to $X^{2^k} q_{n-1}(X)$;

A Converse: The Continuant Tree

We can obtain a converse to the preceding theorem.

Define an infinite labeled binary tree T with root r and node n labeled $L(n)$, as follows:

- $L(r) = 1$;
- $L(\mathrm{left}(n)) = XL(n) + L(\mathrm{parent}(n))$;
- $L(\mathrm{right}(n)) = -XL(n) + L(\mathrm{parent}(n))$.

The paths in this tree consist of the consecutive denominators of the convergents to the continued fraction

$$[1, \pm X, \pm X, \pm X, \ldots].$$

The convergents

Theorem.
If a path in T consists entirely of polynomials with coefficients in $\{0, \pm 1\}$, then it is the sequence of denominators of convergents to a formal power series of the form

$$X \sum_{i \geq 0} \pm X^{-2^i}.$$

The convergents for $h(X)$

Theorem. (A, L, MF, vdP, S)
Let

$$\begin{aligned} h(X) &= X \sum_{i \geq 0} X^{-2^i} \\ &= [a_0, a_1, a_2, \ldots] \end{aligned}$$

and set $p_n/q_n = [a_0, a_1, a_2, \ldots, a_n]$. Then

(a) $q_{2n+1}(X) = Xq_n(X^2)$;
(b) $q_{2n}(X) = (-1)^n(q_n(X^2) - q_{n-1}(X^2))$;
(c) The polynomial $q_{2n+1}(X)$ is odd;
(d) The polynomial $q_{2n}(X)$ is even.

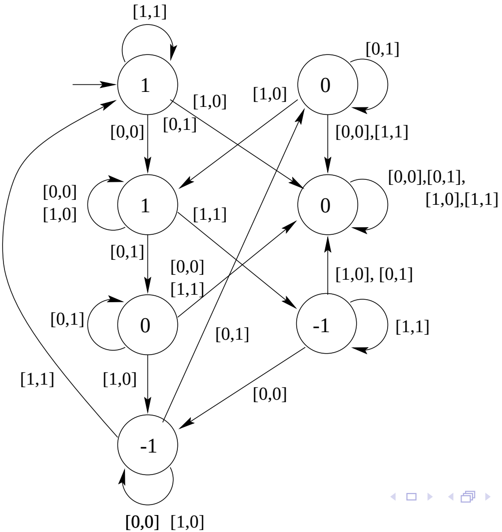
The Coefficient Table is Automatic

Theorem. (Allouche, Lubiw, Mendès France, VDP, JOS)
Define $c_{m,n} = [X^n]q_m(X)$, the coefficient of the X^n term in the polynomial $q_m(X)$. Then the double sequence (table) $(c_{m,n})_{m,n \geq 0}$ is automatic.

Here is what a small portion of this infinite table looks like:

$m \backslash n$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	-1	0	0	0	0	0	0	0	0
2	1	0	-1	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0
4	1	0	-1	0	-1	0	0	0	0	0
5	0	1	0	0	0	-1	0	0	0	0
6	1	0	0	0	-1	0	-1	0	0	0
7	0	0	0	0	0	0	0	1	0	0
8	1	0	0	0	-1	0	-1	0	-1	0
9	0	1	0	0	0	-1	0	0	0	-1

A LSD-First Automaton for the Coefficients of the Denominators of the Convergents



We can also consider the formal power series

$$g_\epsilon(X) := \sum_{i\geq 0} \epsilon_i X^{-2^i} \qquad h_\epsilon(X) := Xg_\epsilon(X)$$

where $\epsilon_i = \pm 1$.

Nearly everything we proved for $h(X)$ also holds for these series. In particular

- The partial quotients for the continued fractions for $g_\epsilon(X)$ and $h_\epsilon(X)$ lie in a finite set;
- The sign sequence $(\epsilon_i)_{i\geq 0}$ is ultimately periodic iff the corresponding sequence of partial quotients is 2-automatic;
- The denominators of the convergents to $g_\epsilon(X)$ and $h_\epsilon(X)$ have all their coefficients in the set $\{0, \pm 1\}$;
- The sign sequence $(\epsilon_i)_{i\geq 0}$ is ultimately periodic iff the double sequence of coefficients of the denominators of the convergents is 2-automatic.

An Open Question

Let the Fibonacci numbers be defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. VDP and JOS considered the continued fraction for the formal power series

$$r(X) = \sum_{i \geq 2} X^{-F_i} = X^{-1} + X^{-2} + X^{-3} + X^{-5} + X^{-8} + \cdots .$$

We proved the partial quotients all have integer coefficients.

Numerical experiments suggest that the denominators of the convergents have all their coefficients in $\{0, \pm 1\}$.

Can this be proved?

VDP's Hadamard Quotient Theorem

Theorem. Let K be a field of characteristic 0. Suppose $\sum_{n\geq 0} b_n X^n$ and $\sum_{n\geq 0} c_n X^n$ in $K[[X]]$ are the expansions of rational functions with $c_n \neq 0$ for all $n \geq n_0$. If the quotients b_n/c_n all belong to a finitely generated ring over $\mathbb{Z}$, then $\sum_{n\geq n_0} \frac{b_n}{c_n} X^n$ is the expansion of a rational function.

Proof outline by vdp in 4 papers (1982–84).

Full details written down by Robert Rumely (68 pages!) in 1986–87.

Application of the Hadamard Quotient Theorem

Theorem. (H. W. Lenstra, Jr., and JOS)
Let θ be an irrational real number with simple continued fraction expansion $\theta = [a_0, a_1, \ldots]$ and convergents p_n/q_n for $n \geq 0$. Then the following four conditions are equivalent:
(a) $(p_n)_{n\geq 0}$ satisfies a linear recurrence with constant coefficients;
(b) $(q_n)_{n\geq 0}$ satisfies a linear recurrence with constant coefficients;
(c) $(a_n)_{n\geq 0}$ is ultimately periodic;
(d) θ is a quadratic irrational.

Since then, a better proof was found by Andrew Granville that does not depend on HQT, and generalizations by Bézivin.

Some VDP-isms

"This appears to raise a metaphysical problem until continued fractions come to the rescue."

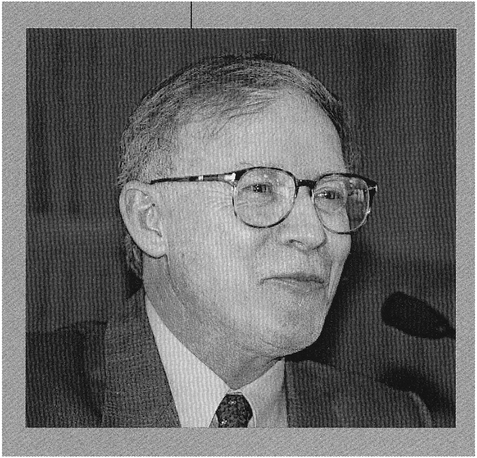
"In this report I sanitise (in the sense of 'bring some sanity to') the arguments of earlier reports..."

(in a paper about Schneider's continued fraction expansions) "Let it be clearly said that Schneider's continued fraction barely warrants even the present limited effort."

"This example is likely to have first been noticed by persons excessively interested in Fibonacci numbers."

(about refereeing) "Happily, here there is no tradition that it is wrong to be scathing when that is appropriate."

Farewell VDP



For Further Reading

1. J.-P. Allouche, A. Lubiw, M. Mendès France, A. J. van der Poorten, and J. Shallit, Convergents of folded continued fractions, *Acta Arithmetica* **77** (1996), 77–96.
2. M. Mendès France, A. J. van der Poorten, and J. Shallit, On lacunary formal power series and their continued fraction exapnsion, in *Number Theory in Progress*, Walter de Gruyter, 1999, pp. 321–326.
3. E. Bombieri and A. J. van der Poorten, Continued fractions of algebraic numbers, in *Computational Algebra and Number Theory*, Kluwer, 1995, pp. 137–152.
4. A. J. van der Poorten and J. Shallit, Folded continued fractions, *J. Number Theory* **40** (1992), 237–250.
5. A. J. van der Poorten and J. Shallit, A specialised continued fraction, *Canad. J. Math.* **45** (1993), 1067–1079.