Continued Fractions and Automatic Sequences

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This talk represents joint work with J.-P. Allouche, A. Lubiw, M. Mendès France, and A. van der Poorten.

Formal Power Series and Continued Fractions

Theorem. (Artin, 1924)
Let n be an integer. Any formal power series

$$f(X) = \sum_{-\infty < i \leq n} b_i X^i \in \mathbb{Q}((X^{-1}))$$

can be expressed uniquely as a continued fraction

$$f(X) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

$$= [a_0, a_1, a_2, \ldots]$$

where $a_i \in \mathbb{Q}[X]$ for $i \geq 0$ and $\deg a_i > 0$ for $i > 0$.

- The terms a_i are called the *partial quotients*.
- Define $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, $q_{-1} = 0$, and $p_n = a_n p_{n-1} + q_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 0$.
- p_n/q_n is called the nth *convergent* to the continued fraction $[a_0, a_1, a_2, \ldots]$.
- We have $[a_0, a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n}$.

An Example

$$
\begin{aligned}
f(X) &:= \frac{1}{\sqrt{X^2-1}} = \sum_{k\geq 0} 2^{-2k}\binom{2k}{k} X^{-2k-1} \\
&= X^{-1} + \frac{1}{2}X^{-3} + \frac{3}{8}X^{-5} + \frac{5}{16}X^{-7} + \frac{35}{128}X^{-9} + \cdots \\
&= [0,\ X,\ -2X,\ 2X,\ -2X,\ 2X,\ -2X,\ \ldots]
\end{aligned}
$$

Here are the first few convergents to the continued fraction for f:

n	0	1	2	3	4
a_n	0	X	$-2X$	$2X$	$-2X$
p_n	0	1	$-2X$	$-4X^2+1$	$8X^3-4X$
q_n	1	X	$-2X^2+1$	$-4X^3+3X$	$8X^4-8X^2+1$

It can be proved that

- $p_n(X) = (-1)^{\lfloor n/2 \rfloor}\ U_{n-1}(X)$;
- $q_n(X) = (-1)^{\lfloor n/2 \rfloor}\ T_n(X)$;

where T and U are the Chebyshev polynomials of the first and second kinds, respectively.

An Interesting Formal Power Series

Consider

$$h(X) := X \sum_{k \geq 0} X^{-2^k}$$

$$= 1 + X^{-1} + X^{-3} + X^{-7} + X^{-15} + \cdots$$

- The formal power series

$$X^{-1}h(X) = \sum_{k \geq 0} X^{-2^k}$$

  is sometimes erroneously called the *Fredholm series*.

- Note that h satisfies a simple functional equation, namely $h(X^2) = X(h(X) - 1)$.

The Continued Fraction for $h(X)$

Theorem. (vdP and S, 1992)

$$h(X) = X \sum_{k \geq 0} X^{-2^k}$$

$$= [a_0, a_1, a_2, \ldots]$$

$$= [1,\ X,\ -X,\ -X,\ -X,\ X,\ X,\ -X,\ \ldots]$$

Here the signs of the partial quotients a_i are determined according to a "paperfolding" rule.

This expansion is atypical in several respects

- the partial quotients a_i have integer coefficients;
- the coefficients lie in a finite set;
- the partial quotients (except the first) are all linear in X.

Theorem: The Partial Quotient Sequence for $h(X)$ Is 2-Automatic

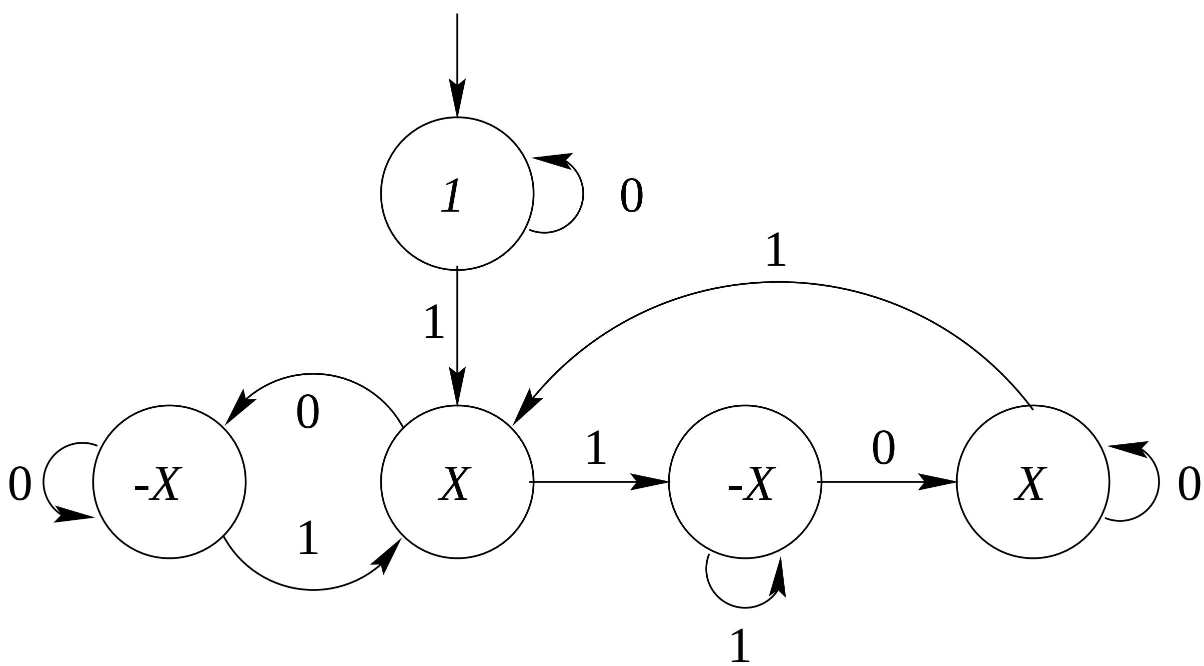
The automaton below computes the continued fraction expansion for

$$h(X) = X \sum_{k \geq 0} X^{-2^k}$$

$$= [1,\ X,\ -X,\ -X,\ -X,\ X,\ X,\ -X,\ \ldots]$$

$$= [a_0, a_1, a_2, \ldots].$$

To use this automaton, express n in base-2, and feed the bits into the automaton starting with the top state and most significant (leftmost) bit.



Convergents to $h(X)$

Here is a table of the first few convergents to $h(X)$:

n	a_n	$p_n(X)$	$q_n(X)$
0	1	1	1
1	X	$X+1$	X
2	$-X$	$-X^2-X+1$	$-X^2+1$
3	$-X$	X^3+X^2+1	X^3
4	$-X$	$-X^4-X^3-X^2-2X+1$	$-X^4-X^2+1$
5	X	$-X^5-X^4-X^2+X+1$	$-X^5+X$
6	X	$-X^6-X^5-X^4-2X^3$ $-X+1$	$-X^6-X^4+1$
7	$-X$	$X^7+X^6+X^4+1$	X^7
8	$-X$	$-X^8-X^7-X^6-2X^5$ $-X^4-2X^3-2X+1$	$-X^8-X^6-X^4+1$
9	X	$-X^9-X^8-X^6-X^5$ $-X^4-2X^2+X+1$	$-X^9-X^5+X$

Note: the coefficients of $q_n(X)$ all appear to lie in $\{0, \pm 1\}$.

Denominators of the Convergents

n	$a_n(X)$	$q_n(X)$
0	1	1
1	X	X
2	$-X$	$-X^2+1$
3	$-X$	X^3
4	$-X$	$-X^4-X^2+1$
5	X	$-X^5+X$
6	X	$-X^6-X^4+1$
7	$-X$	X^7
8	$-X$	$-X^8-X^6-X^4+1$
9	X	$-X^9-X^5+X$
10	$-X$	$X^{10}-X^8-X^4-X^2+1$
11	$-X$	$-X^{11}+X^3$
12	X	$-X^{12}+X^{10}-X^8-X^2+1$
13	X	$-X^{13}-X^9+X$
14	X	$-X^{14}-X^{12}-X^8+1$
15	$-X$	X^{15}
16	$-X$	$-X^{16}-X^{14}-X^{12}-X^8+1$
17	X	$-X^{17}-X^{13}-X^9+X$
18	X	$X^{18}-X^{16}-X^{12}+X^{10}-X^8-X^2+1$
19	X	$-X^{19}-X^{11}+X^3$

Properties of the Denominators of the Convergents

Theorem. (A, L, MF, vdp, S)
All of the coefficients of the denominators of the convergents lie in $\{0, \pm 1\}$.

Proof. (Sketch)

- The low-order terms of $q_{2^k+n-1}(X)$ (i.e., those of degree $< 2^k$) are exactly the same as those of $q_{2^k-n-1}(X)$;
- The high-order terms of $q_{2^k+n-1}(X)$ are, up to a change of signs of individual terms, equal to $X^{2^k} q_{n-1}(X)$;
- The last two observations suffice to prove the first.

Basics of Diophantine Approximation in $\mathbb{Q}((X^{-1}))$

- If $t(X) = \Sigma_{-\infty \leq k \leq n} b_k X^k$ is a nonzero formal power series, then we define

  $$\deg t = \max\{k \ : \ b_k \neq 0\},$$

  that is, the exponent of the largest nonzero term.
- If $t = 0$, we define $\deg t = -\infty$.
- Note that $\deg(f+g) \leq (\deg f)+(\deg g)$, with equality holding if $\deg f \neq \deg g$.

Theorem.

Let $t(X)$ be a formal power series with continued fraction $[a_0, a_1, a_2, \ldots]$ and convergents p_n/q_n. Then

1. $\deg(q_n t - p_n) = -\deg q_{n+1} < -\deg q_n$;
2. If $\deg(qt - p) < -\deg q$, then p/q is a convergent to t.

Diophantine Approximation for $h(X)$

Theorem. (A, L, MF, vdP, S)

Let

$$h(X) = X \sum_{k\geq 0} X^{-2^k}$$

$$= [a_0, a_1, a_2, \ldots]$$

and set $p_n/q_n = [a_0, a_1, a_2, \ldots, a_n]$. Then

(a) $q_{2n+1}(X) = Xq_n(X^2)$;

(b) $q_{2n}(X) = (-1)^n(q_n(X^2) - q_{n-1}(X^2))$;

(c) The polynomial $q_{2n+1}(X)$ is odd;

(d) The polynomial $q_{2n}(X)$ is even.

Proof.

We use the following facts:

(1) $\deg q_n = n$;

(2) $\deg(h - p_n/q_n) = -(2n+1)$.

These follow immediately from the fact that

$$h(X) = [a_0, a_1, a_2, \ldots]$$

where $\deg a_i = 1$ for $i \geq 1$.

Diophantine Approximation for $h(X)$ (II)

(2) $\deg(h - p_n/q_n) = -(2n+1)$.

Substituting X^2 for X in (2) we get

$$\deg(h(X^2) - \frac{p_n(X^2)}{q_n(X^2)}) = 2\deg(h(X) - \frac{p_n(X)}{q_n(X)}) = -(4n+2).$$

Since $h(X^2) = X(h(X) - 1)$, it follows that

$$\deg(h(X) - 1 - \frac{p_n(X^2)}{Xq_n(X^2)}) = -(4n+3). \qquad (3)$$

Now, substituting $2n$ and $2n+1$ for n in (2) we see

$$\begin{aligned} \deg\left(h(X) - \frac{p_{2n}(X)}{q_{2n}(X)}\right) &= -(4n+1); \\ \deg\left(h(X) - \frac{p_{2n+1}(X)}{q_{2n+1}(X)}\right) &= -(4n+3). \end{aligned}$$

Combine these with (3) to get

$$\begin{aligned} \deg\left(\frac{p_{2n}(X)}{q_{2n}(X)} - 1 - \frac{p_n(X^2)}{Xq_n(X^2)}\right) &= -(4n+1); \\ \deg\left(\frac{p_{2n+1}(X)}{q_{2n+1}(X)} - 1 - \frac{p_n(X^2)}{Xq_n(X^2)}\right) &\le -(4n+3). \end{aligned}$$

Diophantine Approximation for $h(X)$ (III)

$$\begin{aligned} \deg\left(\frac{p_{2n}(X)}{q_{2n}(X)} - 1 - \frac{p_n(X^2)}{Xq_n(X^2)}\right) &= -(4n+1); \\ \deg\left(\frac{p_{2n+1}(X)}{q_{2n+1}(X)} - 1 - \frac{p_n(X^2)}{Xq_n(X^2)}\right) &\leq -(4n+3). \end{aligned}$$

Now multiply through by $Xq_n(X^2)q_{2n}(X)$ and $Xq_n(X^2)q_{2n+1}(X)$ to get

$$\begin{aligned} \deg(Xp_{2n}q_n(X^2) - Xq_{2n}q_n(X^2) - p_n(X^2)q_{2n}(X)) &= 0; \\ \deg(Xp_{2n+1}q_n(X^2) - Xq_{2n+1}q_n(X^2) - p_n(X^2)q_{2n+1}(X)) &\leq -1. \end{aligned}$$

It follows that

$$Xp_{2n}q_n(X^2) - Xq_{2n}q_n(X^2) - p_n(X^2)q_{2n}(X)$$

is a constant polynomial, and by substituting 0 for X we can show that in fact it equals -1 for all n.

It also follows that

$$Xp_{2n+1}q_n(X^2) - Xq_{2n+1}q_n(X^2) - p_n(X^2)q_{2n+1}(X)$$

must equal 0, since it is a polynomial of negative degree.

Diophantine Approximation for $h(X)$ (IV)

We can rewrite the equations

$$\begin{aligned} Xp_{2n}q_n(X^2) - Xq_{2n}q_n(X^2) - p_n(X^2)q_{2n}(X) &= -1; \\ Xp_{2n+1}q_n(X^2) - Xq_{2n+1}q_n(X^2) - p_n(X^2)q_{2n+1}(X) &= 0; \end{aligned}$$

as follows:

$$\begin{bmatrix} q_{2n}(X) & X(q_{2n}(X) - p_{2n}(X)) \\ q_{2n+1}(X) & X(q_{2n+1}(X) - p_{2n+1}(X)) \end{bmatrix} \begin{bmatrix} p_n(X^2) \\ q_n(X^2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving this linear system gives

$$\begin{aligned} p_n(X^2) &= p_{2n+1}(X) - q_{2n+1}(X); \\ q_n(X^2) &= q_{2n+1}(X)/X; \end{aligned}$$

and so we get

$$\begin{aligned} p_{2n+1}(X) &= p_n(X^2) + Xq_n(X^2); \\ q_{2n+1}(X) &= Xq_n(X^2). \end{aligned}$$

This proves (a) and (c) of our theorem; parts (b) and (d) are proved similarly.

The Coefficient Table is Automatic

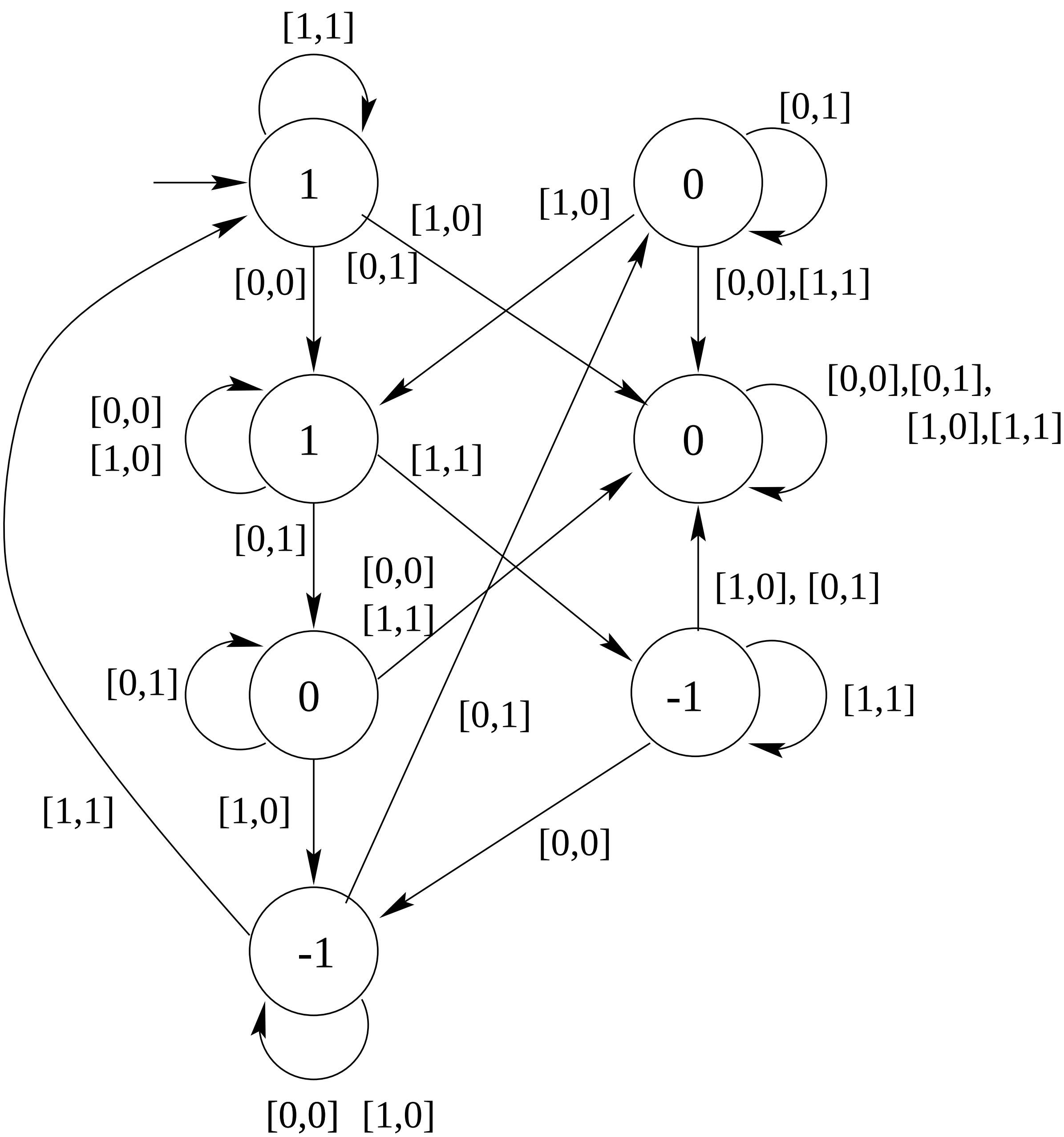
Theorem. (A, L, MF, vdP, S)

Define $c_{m,n} = [X^n]q_m(X)$, the coefficient of the X^n term in the polynomial $q_m(X)$. Then the double sequence (table) $(c_{m,n})_{m,n\geq 0}$ is automatic.

Here is what a small portion of this infinite table looks like:

$m\backslash n$	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	0	-1	0	0	0	0	0	0	0	0
2	1	0	-1	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0
4	1	0	-1	0	-1	0	0	0	0	0
5	0	1	0	0	0	-1	0	0	0	0
6	1	0	0	0	-1	0	-1	0	0	0
7	0	0	0	0	0	0	0	1	0	0
8	1	0	0	0	-1	0	-1	0	-1	0
9	0	1	0	0	0	-1	0	0	0	-1

A LSD-First Automaton for the Coefficients of the Denominators of the Convergents



More General Results

We can also consider the formal power series

$$g_\epsilon(X) := \sum_{i\geq 0} \epsilon_i X^{-2^i}$$

and

$$h_\epsilon(X) := X g_\epsilon(X)$$

where $\epsilon_i = \pm 1$.

Nearly everything we proved for $h(X)$ also holds for these series. In particular

- The partial quotients for the continued fractions for $g_\epsilon(X)$ and $h_\epsilon(X)$ lie in a finite set;
- The sign sequence $(\epsilon_i)_{i\geq 0}$ is ultimately periodic iff the corresponding sequence of partial quotients is 2-automatic;
- The denominators of the convergents to $g_\epsilon(X)$ and $h_\epsilon(X)$ have all their coefficients in the set $\{0, \pm 1\}$;
- The sign sequence $(\epsilon_i)_{i\geq 0}$ is ultimately periodic iff the double sequence of coefficients of the denominators of the convergents is 2-automatic.

An Open Question

Let the Fibonacci numbers be defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. vdP and S considered the continued fraction for the formal power series

$$r(X) = \sum_{i \geq 2} X^{-F_i} = X^{-1} + X^{-2} + X^{-3} + X^{-5} + X^{-8} + \cdots.$$

They proved the partial quotients all have integer coefficients.

Numerical experiments suggest that the denominators of the convergents have all their coefficients in $\{0, \pm 1\}$.

Can this be proved?

For Further Reading

1. J.-P. Allouche, A. Lubiw, M. Mendès France, A. van der Poorten, and J. Shallit, Convergents of folded continued fractions, to appear, *Acta Arithmetica*.

2. A. J. van der Poorten and J. Shallit, Folded continued fractions, *J. Number Theory* **40** (1992), 237–250.

3. A. J. van der Poorten and J. Shallit, A specialised continued fraction, *Canad. J. Math.* **45** (1993), 1067–1079.

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