The Separating Words Problem

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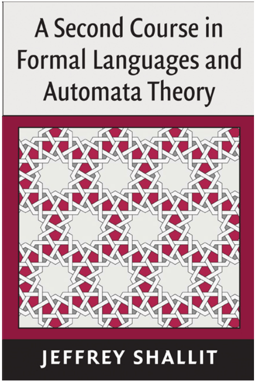

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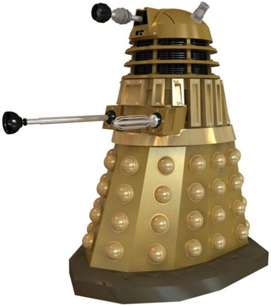
The Simplest Computational Problem?

Imagine a stupid computing device with very limited powers...



What is the simplest computational problem you could ask it to solve?

The Simplest Computational Problem?

- not the addition of two numbers

- not sorting

- it's telling two inputs apart - distinguishing them

Thanks to Gavin Rymill for letting me use his Dalek image.

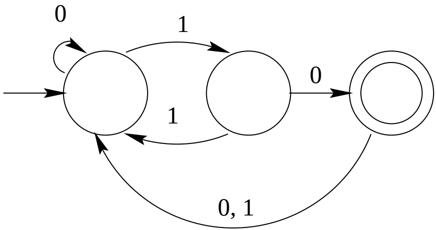
Our Computational Model: the DFA

Our computational model is the **deterministic finite automaton**, or DFA.

It consists of

- Q, a finite nonempty set of states
- q_0, an initial state
- F, a set of final states
- δ, a transition function that tells you how inputs move the machine from one state to another

An example of a DFA



– initial state has sourceless incoming arrow

– final states are denoted by double circles

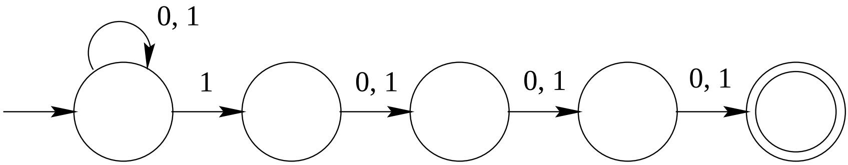
– a word is **accepted** if it labels a path from the initial state to a final state; otherwise it is **rejected**

DFA versus NFA

An automaton is **deterministic** if, for each state and input symbol, there is only one possible state that can be entered next. We call this a DFA.

Otherwise it is **nondeterministic**. We call this an NFA.

Example of an NFA



This NFA accepts all words having a 1 in a position that is 4 spots from the right end.

Motivation

We want to know how many states suffice to tell one length-n input from another.

On average, it's easy — but how about in the worst case?

Motivation: a classical problem from the early days of automata theory:

Given two automata, how big a word do we need to distinguish them?

Motivation

More precisely, given two DFA's M_1 and M_2, with m and n states, respectively, with $L(M_1) \neq L(M_2)$, what is a good bound on the length of the shortest word accepted by one but not the other?

- The cross-product construction gives an upper bound of $mn - 1$ (make a DFA for $L(M_1) \cap \overline{L(M_2)}$)
- But an upper bound of $m + n - 2$ follows from the usual algorithm for minimizing automata
- Furthermore, this bound is best possible.
- For NFA's the bound is exponential in m and n

Our problem is the inverse problem: given two distinct *words*, how big an automaton do we need to separate them?

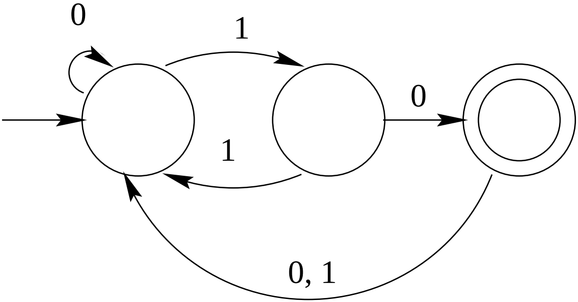
That is, given two words w and x of length $\leq n$, what is the smallest number of states in any DFA that accepts one word, but not the other?

Call this number $\mathrm{sep}(w, x)$.

Separation

A machine M **separates** the word w from the word x if M accepts w and rejects x, or vice versa.

For example, the machine below separates 0010 from 1000.



However, no 2-state DFA can separate these two words. So $\mathrm{sep}(1000, 0010) = 3$.

Separating Words of Different Length

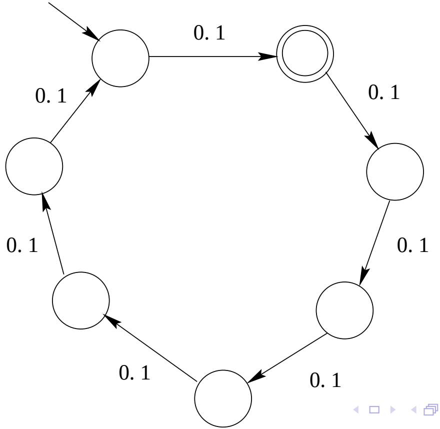
Easy case: if the two words are of different lengths, both $\leq n$, we can separate them with a DFA of size $O(\log n)$.

For by the prime number theorem, if $k \neq m$, and $k, m \leq n$ then there is a prime $p = O(\log n)$ such that $k \not\equiv m \pmod{p}$.

So we can accept one word and reject the other by using a cycle mod p, and the appropriate residue class.

Separating Words of Different Length

Example: suppose $|w| = 22$ and $|x| = 52$. Then $|w| \equiv 1 \pmod 7$ and $|x| \equiv 3 \pmod 7$. So we can accept w and reject x with a DFA that uses a cycle of size 7, as follows:



Separating Words with Different Prefix

For the remainder of the talk, then, we only consider the case where $|w| = |x|$.

We can separate w from x using $d + O(1)$ states if they differ in some position d from the start, since we can build a DFA to accept words with a particular prefix of length d.

Separating Words with Different Prefix

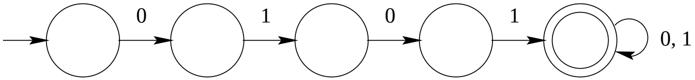
For example, to separate

010100111011001100000

from

010011111010111001011

we can build a DFA to recognize words that begin with 0101:



(Transitions to a dead state omitted.)

Separating Words With Different Suffix

Similarly, we can separate w from x using $d + O(1)$ states if they differ in some position d from the end.

The idea is to build a pattern-recognizer for the suffix of w of length d, ending in an accepting state if the suffix is recognized.

Separating Words With Different Suffix

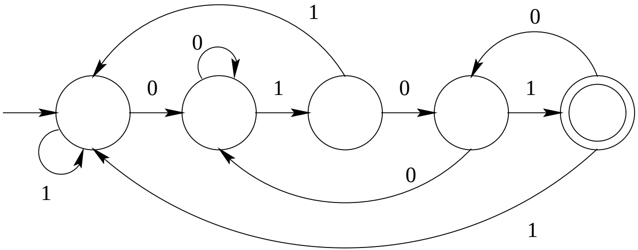
For example, to separate

$$1111101001100101010 1$$

from

$$11111011010010101101$$

we can build a DFA to recognize those words that end in 0101:



Separating Words With Differing Number of 1's

Can we separate two words having differing numbers of 1's?

Yes. By the prime number theorem, if $|w|, |x| = n$, and w and x have k and m 1's, respectively, then there is a prime $p = O(\log n)$ such that $k \not\equiv m \pmod{n}$.

So we can separate w from x just by counting the number of 1's, modulo p.

Separating Words with Differing Number of Patterns

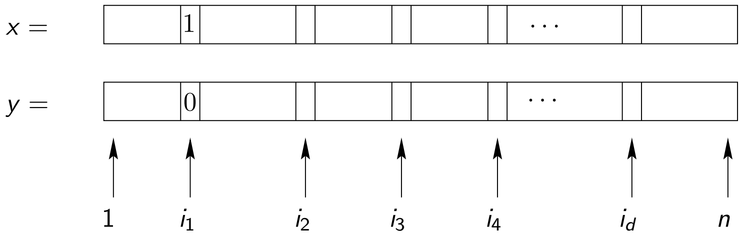
Similarly, we can separate two length-n words w, x using $O(d \log n)$ states if there is a pattern of length d occurring a differing number of times in w and x.

Separation of Very Similar Words

The *Hamming distance* between w and x is the number of positions where they differ.

If the Hamming distance between w and x is small, say $< d$, we can separate two length-n words using $O(d \log n)$ states.

The idea is as follows:



Let $i_1, i_2, \ldots, i_d$ be the positions where x and y differ.

Separation of Very Similar Words

Now consider $N = (i_2 - i_1)(i_3 - i_1) \cdots (i_d - i_1)$. Then $N < n^{d-1}$.

So N is not divisible by some prime $p = O(\log N) = O(d \log n)$.

So $i_j \not\equiv i_1 \pmod{p}$ for $2 \leq j \leq d$.

Now count the number, modulo 2, of 1's occurring in positions congruent to i_1 (mod p).

These positions do not include any of $i_2, i_2, \ldots, i_d$, by the way we chose p, and the two words agree in all other positions.

So x contains exactly one more 1 in these positions than w does, and hence we can separate the two words using $O(d \log n)$ states.

The Separation Number

- Let

$$S(n) := \max_{\substack{|w|=|x|=n \\ w \neq x}} \mathrm{sep}(w, x),$$

the smallest number of states required to separate any two words of length n.

- The separation problem was first studied by Goralcik and Koubek 1986, who proved $S(n) = o(n)$.

- In 1989 Robson obtained the best known upper bound: $S(n) = O(n^{2/5}(\log n)^{3/5})$.

Dependence on Alphabet Size

For equal-length words, $S(n)$ doesn't depend on alphabet size (provided it is at least 2).

To see this, let $S_k(n)$ be the maximum number of states needed to separate two length-n words over an alphabet of size k.

Suppose x, y are distinct words of length n over an alphabet Σ of size $k > 2$.

Then x and y must differ in some position, say for $a \neq b$,

$$\begin{aligned} x &= x' \; a \; x'' \\ y &= y' \; b \; y''. \end{aligned}$$

Dependence on Alphabet Size

$$\begin{aligned} x &= x' \; a \; x'' \\ y &= y' \; b \; y''. \end{aligned}$$

Map a to 0, b to 1 and assign all other letters arbitrarily to either 0 or 1.

This gives two new distinct binary words X and Y of the same length.

If X and Y can be separated by an m-state DFA, then so can x and y, by renaming transitions to be over Σ instead of 0 and 1.

Thus $S_k(n) \le S_2(n)$. But clearly $S_2(n) \le S_k(n)$, since every binary word can be considered as a word over a larger alphabet. So $S_k(n) = S_2(n)$.

Robson's Upper Bound

Robson's upper bound of $O(n^{2/5}(\log n)^{3/5})$ is hard to explain. But he also proved:

Theorem (Robson, 1996). We can separate words by computing the parity of the number of 1's occurring in positions congruent to $i \pmod{j}$, for $i, j = O(\sqrt{n})$.

This gives the bound $S(n) = O(n^{1/2})$.

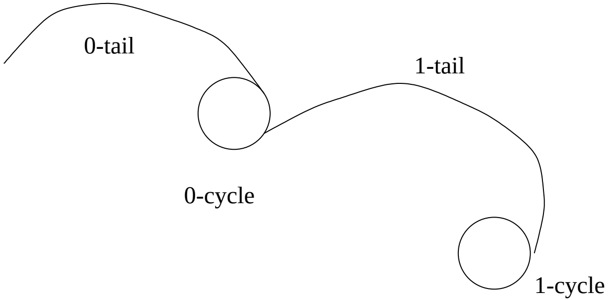
Open Problem 1: Improve Robson's bound of $O(n^{2/5}(\log n)^{3/5})$ on $S(n)$.

Lower Bounds

- Claim: $S(n) = \Omega(\log n)$.
- To see this, consider the two words

$$0^{t-1+\mathrm{lcm}(1,2,\ldots,t)}1^{t-1} \quad \text{and} \quad 0^{t-1}1^{t-1+\mathrm{lcm}(1,2,\ldots,t)}.$$

Proof in pictures:



Lower Bounds

So no t-state machine can distinguish these words.

Now $\text{lcm}(1, 2, \ldots, t) = e^{t+o(t)}$ by the prime number theorem, and the lower bound $S(n) = \Omega(\log n)$ follows.

Some data:

n	$S(n)$	n	$S(n)$
1	2	10	4
2	2	11	4
3	2	12	4
4	3	13	4
5	3	14	4
6	3	15	4
7	3	16	4
8	3	17	4
9	3	18	5

Separating a Word from Its Reverse

Maybe it's easier to separate a word w from its reverse w^R, than the general case of two words.

However, no better upper bound is known.

We still have a lower bound of $\Omega(\log n)$ for this problem:

Consider separating

$$w = 0^{t-1}10^{t-1+\mathrm{lcm}(1,2,\ldots t)}$$

from

$$w^R = 0^{t-1+\mathrm{lcm}(1,2,\ldots t)}10^{t-1}.$$

Then no DFA with $\leq t$ states can separate w from w^R.

Reverses of Two Words

- Must $\mathrm{sep}(w^R, x^R) = \mathrm{sep}(w, x)$?
- No, for $w = 1000$, $x = 0010$, we have

$$\mathrm{sep}(w, x) = 3$$

- but

$$\mathrm{sep}(w^R, x^R) = 2.$$

Open Problem 2: Is

$$\left|\mathrm{sep}(x, w) - \mathrm{sep}(x^R, w^R)\right|$$

unbounded?

Separating a Word from Its Conjugates

- Two words w, w' are *conjugates* if one is a cyclic shift of the other.
- For example, the English words `enlist` and `listen` are conjugates
- Is the separating words problem any easier if restricted to pairs of conjugates?
- There is still a lower bound of $\Omega(\log n)$ for this problem, as given by

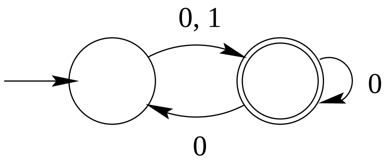
$$w = 0^{t-1} 1 0^{t-1+\mathrm{lcm}(1,2,\ldots t)} 1$$

and

$$w' = 0^{t-1+\mathrm{lcm}(1,2,\ldots t)} 1 0^{t-1} 1.$$

Separation by NFA's

- We can define $\mathrm{nsep}(w, x)$ in analogy with sep: the number of states in the smallest NFA accepting w but rejecting x.
- Now there is an asymmetry in the inputs: $\mathrm{nsep}(w, x)$ need not equal $\mathrm{nsep}(x, w)$.
- For example, the following 2-state NFA accepts $w = 000100$ and rejects $x = 010000$, so $\mathrm{nsep}(w, x) \leq 2$.



- But there is no 2-state NFA accepting x and rejecting w, so $\mathrm{nsep}(x, w) \geq 3$.

Separation by NFA's

- Do NFA's give more power?
- Yes,

$$\text{sep}(0001, 0111) = 3$$

but

$$\text{nsep}(0001, 0111) = 2.$$

More Variations on Separating Words

Is

$$\mathrm{sep}(x, w)/\mathrm{nsep}(x, w)$$

unbounded?

Yes.

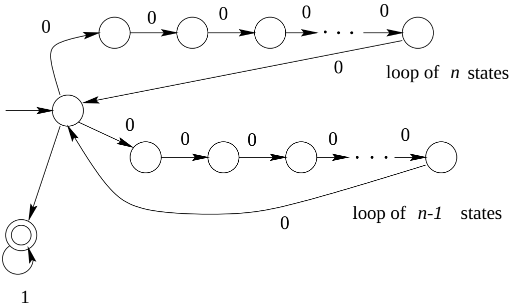
Consider once again the words

$$w = 0^{t-1+\mathrm{lcm}(1,2,\ldots,t)}1^{t-1} \quad \text{and} \quad x = 0^{t-1}1^{t-1+\mathrm{lcm}(1,2,\ldots,t)}$$

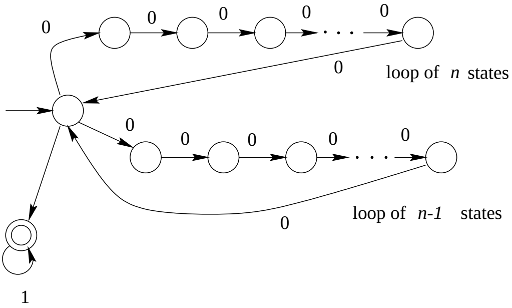
where $t = n^2 - 3n + 2$, $n \geq 4$.

We know from before that any DFA separating these words must have at least $t + 1 = n^2 - 3n + 3$ states.

Now consider the following NFA M:



The language accepted by this NFA is $\{0^a \ : \ a \in A\}1^*$, where A is the set of all integers representable by a non-negative integer linear combination of n and $n - 1$.



But $t - 1 = n^2 - 3n + 1 \notin A$ (compute mod $n - 1$ and mod n).

On the other hand, every integer $\geq t$ is in A. Hence
$w = 0^{t-1+\mathrm{lcm}(1,2,\ldots,t)}1^{t-1}$ is accepted by M but
$x = 0^{t-1}1^{t-1+\mathrm{lcm}(1,2,\ldots,t)}$ is not.

M has $2n = \Theta(\sqrt{t})$ states, so
$\mathrm{sep}(x, w)/\mathrm{nsep}(x, w) \geq \sqrt{t} = \Omega(\sqrt{\log |x|})$, which is unbounded.

Lower Bound for Nondeterministic Separation

Theorem. No NFA of n states can separate

$$0^{n^2} 1^{n^2+\mathrm{lcm}(1,2,\ldots,n)}$$

from

$$0^{n^2+\mathrm{lcm}(1,2,\ldots,n)} 1^{n^2}.$$

Proof. We use the same argument as for DFA's, and the fact (Chrobak) that any unary n-state NFA can be simulated by a DFA with with a "tail" of at most n^2 states and a cycle of size dividing $\mathrm{lcm}(1, 2, \ldots, n)$.

This gives a lower bound of $\Omega(\log n)$ on nondeterministic separation.

Nondeterministic Separation of Reversed Words

A result of Sarah Eisenstat (MIT), May 2010:

Theorem. We have $\mathrm{nsep}(w, x) = \mathrm{nsep}(w^R, x^R)$.

Proof. Let M be an NFA with the smallest number of states accepting w and rejecting x. Now make a new NFA M' with initial state equal to any one element of $\delta(q_0, w)$ and final state q_0, and all other transitions of M reversed. Then M' accepts w^R. But M' rejects x^R. For if it accepted x^R then M would also accept x, which it doesn't.

Open Questions about Nondeterministic Separation

Open Problem 3: Find better bounds on $\mathrm{nsep}(w, x)$ for $|w| = |x| = n$, as a function of n.

Open Problem 4: Find better bounds on $\mathrm{sep}(w, x)/\mathrm{nsep}(w, x)$.

Permutation Automata

Instead of arbitrary automata, we could restrict our attention to automata where each letter induces a permutation of the states ("permutation automata").

For an n-state automaton, the action of each letter can be viewed as an element of S_n, the symmetric group on n elements.

Turning the problem around, then, we could ask: what is the shortest pair of distinct equal-length binary words w, x, such that for all morphisms $\sigma : \{0,1\}^* \rightarrow S_n$ we have $\sigma(w) = \sigma(x)$?

You might suspect that the answer is $\mathrm{lcm}(1, 2, \ldots, n)$.

But for $n = 4$, here is a shorter pair (of length 11): 00000011011 and 11011000000.

A Problem in Groups

Now if $\sigma(w) = \sigma(x)$ for all σ, then (if we define $\sigma(x^{-1}) = \sigma(x)^{-1}$) $\sigma(wx^{-1})$ = the identity permutation for all σ.

Call any nonempty word y over the letters $0, 1, 0^{-1}, 1^{-1}$ an *identical relation* if $\sigma(y)$ = the identity for all morphisms σ.

We say y is *nontrivial* if y contains no occurrences of 00^{-1} and 11^{-1}.

What is the length ℓ of the shortest nontrivial identical relation over S_n?

Recently Gimadeev and Vyalyi proved $\ell = 2^{O(\sqrt{n} \log n)}$.

Separation by Context-Free Grammars

- Given two words w, x, what's the smallest CFG generating w but not x?
- Size of grammar is measured by number of productions
- Problem: right-hand sides can be arbitrarily complicated
- Solution: Use CFG's in Chomsky normal form (CNF), where all productions are of the form $A \longrightarrow BC$ or $A \longrightarrow a$.

Separation by Context-Free Grammars

In 1999 Currie, Petersen, Robson and JOS proved:

- If $|w| \neq |x|$ then there is a CFG in CNF with $O(\log \log n)$ productions separating w from x. Furthermore, this bound is optimal.
- Idea: again, if w and x are of different lengths, both $\leq n$, there is a prime $p = O(\log n)$ such that $i = |w| \not\equiv |x| \pmod{p}$.
- We can generate Σ^p in $O(\log p)$ productions in a CFG.
- So we can generate $(\Sigma^p)^*\Sigma^i$ in $O(\log \log n)$ productions.
- There is a matching lower bound.

More on Context-Free Separation

- If $|w| = |x|$ there is a CFG in CNF with $O(\log n)$ productions separating w from x.
- There is a lower bound of $\Omega(\frac{\log n}{\log \log n})$.
- Upper bound is similar to before
- For the lower bound, we use a counting argument.

Open Problem 5: Find matching upper and lower bounds in the case $|w| = |x|$.

Suppose you have regular languages R_1, R_2 with $R_1 \subseteq R_2$ and $R_2 - R_1$ infinite.

Then it is easy to see that there is a regular language R_3 such that $R_1 \subseteq R_3 \subseteq R_2$ such that $R_2 - R_3$ and $R_3 - R_1$ are both infinite.

This is a kind of topological separation property.

Another Kind of Separation

In 1980, Bucher asked:

Open Problem 6: Is the same true for context-free languages?

That is, given context-free languages L_1, L_2 with $L_1 \subseteq L_2$ and $L_2 - L_1$ infinite, need there be a context-free language L_3 such that $L_1 \subseteq L_3 \subseteq L_2$ such that $L_2 - L_3$ and $L_3 - L_1$ are both infinite?

For Further Reading

- J. M. Robson, Separating words with machines and groups, *RAIRO Info. Theor. Appl.* **30** (1996), 81–86.
- J. Currie, H. Petersen, J. M. Robson, and J. Shallit, Separating words with small grammars, *J. Autom. Lang. Combin.* **4** (1999), 101–110.