Robbins and Ardila meet Berstel

Jeffrey Shallit
School of Computer Science, University of Waterloo, Canada
shallit@uwaterloo.ca
https://cs.uwaterloo.ca/~shallit

Neville Robbins

Federico Ardila

Jean Berstel

An infinite product

The story starts with an infinite product:

$$\prod_{i \geq 0} (1 + X^{2^i}) = (1+X)(1+X^2)(1+X^4)(1+X^8)\cdots$$
$$= 1 + X + X^2 + X^3 + X^4 + X^5 + \cdots$$
$$= \frac{1}{1-X}.$$

Proof. Every positive integer has a unique representation as a sum of distinct powers of 2.

Another infinite product

What if we change the plus signs to minus signs? Then we get

$$\begin{aligned}\prod_{i\geq 0}(1-X^{2^i}) &= (1-X)(1-X^2)(1-X^4)(1-X^8)\cdots \\ &= 1 - X - X^2 + X^3 - X^4 + X^5 + \cdots \\ &= \sum_{n\geq 0}(-1)^{t_n}X^n,\end{aligned}$$

where $(t_n)_{n\geq 0}$ is the famous *Thue-Morse sequence*, counting the parity of the number of 1's in the base-2 representation of n.

All coefficients of this power series are (obviously) 1 or -1.

Yet another infinite product

What if we change the powers of 2 to the Fibonacci numbers?
(Recall $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.)

$$\begin{aligned}\prod_{i\geq 2}(1 - X^{F_i}) &= (1-X)(1-X^2)(1-X^3)(1-X^5)(1-X^8)\cdots \\ &= 1 - X - X^2 + X^4 + X^7 - X^8 + X^{11} - X^{12} + \cdots \\ &= \sum_{n\geq 0} a(n)X^n \\ &= ???\end{aligned}$$

We want to understand $a(n)$.

Notice that the coefficients $a(n)$ seem to be either $1, 0$, or -1.

Is that true? Is it *obviously* true?

A remark

Remark: if instead we use the Tribonacci numbers (where each is the sum of the *three* previous) as exponents, then the coefficients of the corresponding power series are unbounded.

Robbins' theorem

Neville Robbins (1996) proved that $a(n)$ is always either $-1, 0$, or 1.

His proof is not so easy and is largely case based.

Here's a picture of just part of the proof:

is true, except possibly when $k_1 - k_2 \equiv 0 \pmod 4$. In this case, we have $a(n) = -a(n_1) - a(n_2)$. Again by Theorem 6 we have:

$$a(n_1) = \begin{cases} -a(n_2) - a(n_3) & \text{if } k_2 - k_3 \equiv 0 \pmod 4, \\ -a(n_2) & \text{if } k_2 - k_3 \equiv 1 \pmod 4, \\ a(n_3) & \text{if } k_2 - k_3 \equiv 2 \pmod 4, \\ 0 & \text{if } k_2 - k_3 \equiv 3 \pmod 4. \end{cases}$$

Therefore, we have

$$a(n) = \begin{cases} a(n_3) & \text{if } k_2 - k_3 \equiv 0 \pmod 4, \\ 0 & \text{if } k_2 - k_3 \equiv 1 \pmod 4, \\ -a(n_2) - a(n_3) & \text{if } k_2 - k_3 \equiv 2 \pmod 4, \\ -a(n_2) & \text{if } k_2 - k_3 \equiv 3 \pmod 4. \end{cases}$$

Thus, $|a(n)| \le 1$ except, possibly, when $k_2 - k_3 \equiv 2 \pmod 4$. In the latter case, we evaluate $a(n_2)$ using Theorem 6. We then see that $|a(n)| \le 1$ except, possibly, when $k_3 - k_4 \equiv 2 \pmod 4$, in which case $a(n) = -a(n_3) - a(n_4)$. If $|a(n)| > 1$, then we would have an infinite sequence: $n > n_1 > n_2 > n_3 > \cdots$. This is impossible, so we must have $|a(n)| \le 1$ for all n.

Ardila's proof

Federico Ardila (2004) revisited the question and found a simpler proof...but one that still requires some complications and an induction. For example ...

Proposition 1: Let $n \geq 5$ be an integer. Consider the coefficients $a(m)$ for m in the interval $[F_n, F_{n+1})$. Split this interval into the three subintervals $[F_n, F_n + F_{n-3} - 2]$, $[F_n + F_{n-3} - 1, F_n + F_{n-2} - 1]$ and $[F_n + F_{n-2}, F_{n+1} - 1]$.

1. The numbers $a(F_n), a(F_n + 1), \ldots, a(F_n + F_{n-3} - 2)$ are equal to the numbers $(-1)^{n-1}a(F_{n-3} - 2), (-1)^{n-1}a(F_{n-3} - 3), \ldots, (-1)^{n-1}a(0)$ in that order.
2. The numbers $a(F_n + F_{n-3} - 1), a(F_n + F_{n-3}), \ldots, a(F_n + F_{n-2} - 1)$ are equal to 0.
3. The numbers $a(F_n + F_{n-2}), a(F_n + F_{n-2} + 1), \ldots, a(F_{n+1} - 1)$ are equal to the numbers $a(0), a(1), \ldots, a(F_{n-3} - 1)$ in that order.

And neither Robbins' nor Ardila's proofs suggest how to prove any generalization.

Could there be some other approach?

Sketch of another approach

1. Obtain an expression for $a(n)$ in terms of Fibonacci representations.
2. Create a DFA based on a transducer of Berstel that "normalizes" non-canonical Fibonacci representations.
3. Obtain a "linear representation" for $a(n)$ from the DFA.
4. Show that the space of possible values of this linear representation is finite.

Steps 3 and 4 are purely computational and can be carried out by existing software.

Step 1: Connection with Fibonacci representations

Robbins already noticed the connection between the coefficients $a(n)$ of our series, and the total number of representations of n as sums of distinct Fibonacci numbers, denoted $r(n)$.
Let

- $r_e(n)$ denote the number of representations of n as the sum of an even number of distinct Fibonacci numbers;
- $r_o(n)$ denote the number of representations of n as the sum of an odd number of distinct Fibonacci numbers.

Then clearly $r(n) = r_e(n) + r_o(n)$.

Fibonacci representations

Furthermore, every representation of n as the sum of an even number of distinct Fibonacci numbers contributes 1 to $a(n)$, and every representation of n as the sum of an odd number of distinct Fibonacci numbers contributes -1 to $a(n)$. So

$$a(n) = r_e(n) - r_o(n).$$

When combined with $r(n) = r_e(n) + r_o(n)$ we get

$$a(n) = 2r_e(n) - r(n).$$

So to compute $a(n)$ we just have to compute $r(n)$ and $r_e(n)$.

Fibonacci representations

In Fibonacci representation we write n as a sum of distinct Fibonacci numbers as follows:

$$n = \sum_{2 \le i \le t} e_i F_i,$$

where each $e_i \in \{0, 1\}$. For example, $23 = 21 + 2 = F_8 + F_3$.

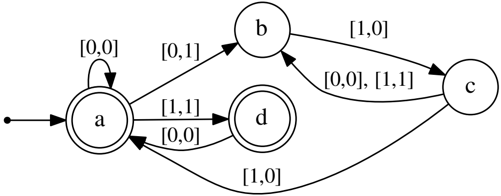
To get a *unique* representation we must impose an additional condition, such as $e_i e_{i+1} = 0$ for all i (no two consecutive Fibonacci numbers can be used). This is the *canonical* or *Zeckendorf* representation.

For a word $w \in \{0, 1\}^t$, we define $[w]_F$ to be the number denoted by the sum $\sum_{1 \le i \le t} w[i] F_{t+2-i}$. And we define $(n)_F$ to be the canonical Fibonacci representation of n, starting with the most significant digit.

Step 2: Enter Berstel

In a 2001 paper, Jean Berstel created a 4-state transducer that can convert any Fibonacci representation to its canonical representation. We can rewrite his transducer as a DFA as follows: it takes two Fibonacci representations in parallel, and accepts iff the second is the "normalized" version of the first.

Here it is:



For example, 23 can be represented by both 0101110 (unnormalized) and 1000010 (normalized), and if we feed in $[0,1][1,0][0,0][1,0][1,0][1,1][0,0]$ then we visit states *abcbcada* and accept.

The number of representations as sums of Fibonacci numbers

With Berstel's automaton A, we can find a formula for $r(n)$, the number of representations of n as a sum of Fibonacci numbers. Namely, if y is the canonical Fibonacci representation of n , then

$$\begin{aligned} r(n) &= \#\{x \in \{0,1\}^* \,:\, |x| = |y| \text{ and } [x]_F = [y]_F\} \\ &= \#\{x \in \{0,1\}^* \,:\, A \text{ accepts the pair } (x,y)\}. \end{aligned}$$

And this in turn corresponds to the number of length-$|y|$ paths in A whose second component is labeled with y. But this can be expressed by the *linear representation* $v\mu(y)w$ where

$$v = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}; \quad \mu(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad \mu(1) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So $r(n) = v\mu((n)_F)w$.

Number of Fibonacci representations with even number of parts

Now we want to do the same thing for $r_e(n)$, the number of Fibonacci representations for n, having an even number of parts.

To do this we simply form a cross-product with Berstel's automaton, demanding that x in the input (x, y) have an even number of 1's.

This gives us another linear representation, (v', μ', w'), as follows:

$$v' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T ; \quad \mu'(0) = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&1&1&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{bmatrix} ; \quad \mu'(1) = \begin{bmatrix} 0&0&1&0&0&0&0&1 \\ 0&0&0&1&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix} ; \quad w' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} .$$

Then $r_e(n) = v'\mu'((n)_F)w'$.

Step 3: Linear representation for $a(n)$

Now that we have linear representations for $r(n)$ and $r_e(n)$, we can obtain a linear representation for $a(n)$ using the equation $a(n) = 2r_e(n) - r(n)$ obtained previously:

$$v'' = \left[\begin{smallmatrix} 2v \\ -v' \end{smallmatrix}\right]^T; \quad \mu''(0) = \left[\begin{smallmatrix} \mu(0) & 0 \\ 0 & \mu'(0) \end{smallmatrix}\right]; \quad \mu''(1) = \left[\begin{smallmatrix} \mu(1) & 0 \\ 1 & \mu'(1) \end{smallmatrix}\right]; \quad w'' = \left[\begin{smallmatrix} w \\ w' \end{smallmatrix}\right].$$

This gives us a linear representation of rank 12 for $a(n)$.

It can be minimized using a well-known algorithm to one of rank 4, namely (y, γ, z):

$$y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T; \quad \gamma(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}; \quad \gamma(1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad z = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}.$$

Step 4: Finishing the proof

With the linear representation (y, γ, z) for $a(n)$ we can now complete the proof of the Robbins-Ardila theorem:

We just use breadth-first search to explore the space of vectors of the form

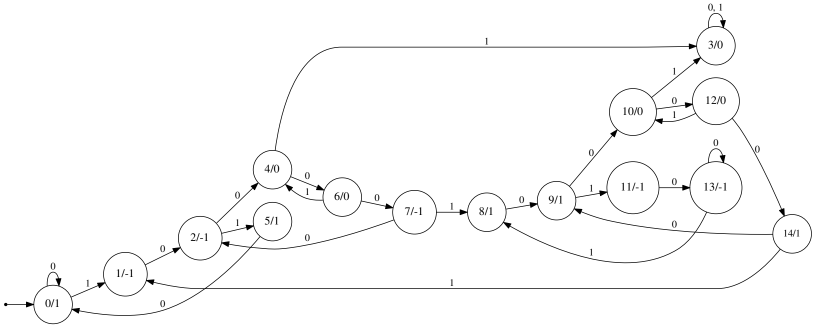
$$S = \{y\gamma(x) \,:\, x \in \{0,1\}^*\}.$$

We can easily check that $|S| = 15$ and further that $sz \in \{-1, 0, 1\}$ for $s \in S$.

This completes the proof...

An automaton for computing $a(n)$

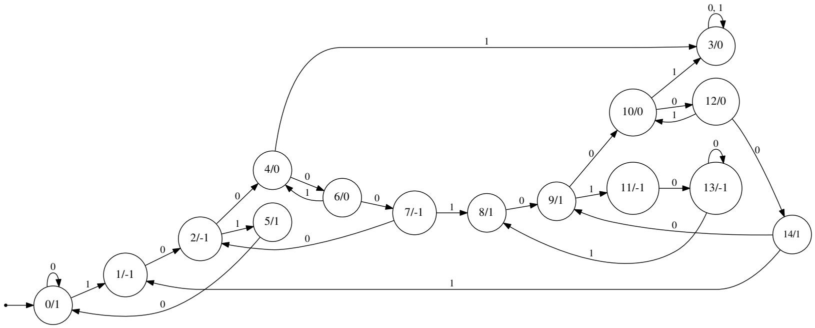
...but also gives something more: an automaton that, on input $(n)_F$, computes $a(n)$. So the sequence $(a(n))_{n\geq 0}$ is Fibonacci-automatic!



(Input is n in canonical Fibonacci representation, output is the number after the slash in the last state reached.)

Theorem (Ardila). Almost all n satisfy $a(n) = 0$.

Proof. Almost all n have an occurrence of the factor 01001001 in their Fibonacci representation. But 01001001 is a synchronizing word for our automaton, taking every state to state 3, from which it can never leave.



New results

The same kinds of computational techniques can be used to prove the following, with basically no effort:

Theorem. Let $r_{m,t}(n)$ denote the number of Fibonacci representations of n whose number of parts is congruent to $t \pmod{m}$. Then

(a) $r_{3,i}(n) - r_{3,i+1}(n) \in \{-1, 0, 1\}$ for $i = 0, 1, 2$ and $n \geq 0$.

(b) (Weinstein)

$$(r_{3,0}(n) - r_{3,1}(n))(r_{3,0}(n) - r_{3,2}(n))(r_{3,1}(n) - r_{3,2}(n)) = 0$$

for all $n \geq 0$.

(c) $r_{4,0}(n) - r_{4,2}(n)$ is unbounded as $n \to \infty$.

For further reading

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