Lazy Ostrowski Numeration and Sturmian Words

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Periods of a word

An integer p, with $1 \leq p \leq |x|$, is called a *period* of a finite word x if $x[i] = x[i + p]$ for $1 \leq i \leq |x| - p$.

Example: `alfalfa` has period 3.

A period p of x is *nontrivial* if $p < |x|$.

The least period of a word x is called *the* period, and is written $\mathrm{per}(x)$.

The number of nontrivial periods of a word x is denoted $\mathrm{nnp}(x)$. For example, $\mathrm{nnp}($`adoradora`$) = 2$.

Exponent and critical exponent

The *exponent* of a finite nonempty word x is defined to be $\exp(x) := |x| / \operatorname{per}(x)$.

For example, $\exp(\texttt{entente}) = 7/3$.

The *critical exponent* $\operatorname{ce}(x)$ of a finite or infinite word x is defined to be

$$\operatorname{ce}(x) := \sup\{\exp(p) \; : \; p \text{ is a nonempty factor of } x\}.$$

Motivation for the talk

The original motivation for this research was to answer the following question:

When does a word have lots of periods?

Obviously, one way a word can have lots of periods is if it is periodic: 0^n has n periods. So a word with high exponent will have lots of periods.

On the other hand, $0^n 1^{n^2} 0^n$ has lots of periods, but very small exponent $(n^2 + 2n)/(n^2 + n) \approx 1 + 1/n$. So exponent alone can't be the whole story. Maybe critical exponent?

No! A word like 01^n0 has only one period, but has high critical exponent.

So what should we do?

Initial critical exponent

Instead we'll consider the initial critical exponent.

The *initial critical exponent* $\text{ice}(x)$ of a finite or infinite word x is defined to be

$$\text{ice}(x) := \sup\{\exp(p) \ : \ p \text{ is a nonempty prefix of } x\}.$$

For example, $\text{ice}(\texttt{phosphorus}) = 7/4$.

This concept was (essentially) introduced by Berthé, Holton, and Zamboni in 2006.

Digression: borders of a word

A word w is a *border* of a word x if w is both a prefix and suffix of x.

For example, `ionization` has the border `ion`.

Borders are allowed to overlap, but we generally rule out borders w where $w = \epsilon$ or $w = x$.

A border w of x is *short* if $|w| < |x|/2$.

Basic observation: A word has a nontrivial period t iff it has a border of length $n - t$.

Example: `abracadabra` has nontrivial periods 7 and 10, and borders of length 4 and 1.

An inequality for the number of periods

Now, back to counting periods. Here is our main result #1, relating periods to ice:

Theorem. Let x be a bordered word of length $n \geq 1$. Let $e = \text{ice}(x)$. Then

$$\text{nnp}(x) \leq \frac{e}{2} + 1 + \frac{\ln(n/2)}{\ln(e/(e-1))}.$$

Proof.
Break the bound up into two pieces, by considering the periods of size $\leq n/2$ and $> n/2$. Call these the *short* and *long* periods.

Proof of the period inequality

Let $p = \text{per}(x)$, the shortest period of x.

If p is short, then x has short periods $p, 2p, 3p, \ldots, \lfloor n/(2p) \rfloor p$.

Clearly $\text{ice}(x) \geq n/p$, so we get at most $e/2$ short periods from this list.

To see that there are no other short periods, let q be some short period not on this list. Then $p < q \leq n/2$ by assumption.

By the Fine-Wilf theorem, if a word of length n has two periods p, q with $n \geq p + q - \gcd(p, q)$, then it also has period $\gcd(p, q)$.

Since $\gcd(p, q) \leq p$, either $\gcd(p, q) < p$, which is a contradiction, or $\gcd(p, q) = p$, which means q is a multiple of p, another contradiction.

Proof of the period inequality

Next, let's consider the long periods or, alternatively, the short borders (those of length $< n/2$).

Suppose x has borders y, z of length q and r respectively, with $q < r < n/2$.

Then $x = yy'y = zz'z$ for words y' and z'. Hence $z = yt = t'y$ for some nonempty words t and t'.

Then by the Lyndon-Schützenberger theorem we know there exist words u, v with u nonempty, and an integer $d \geq 0$, such that $t' = uv$, $t = vu$, and $y = (uv)^d u$.

Hence x has the prefix $z = yt = (uv)^{d+1}u$, which means $e = \text{ice}(x) \geq |z|/|uv| = r/(r-q)$.

Proof of the period inequality

The inequality $r/(r-q) \leq e$ is equivalent to $r/q \geq e/(e-1)$.

If $b_1 < b_2 < \cdots < b_t$ are the lengths of all the short borders of x then

$$\begin{aligned} b_1 &\geq 1 \\ b_2 &\geq (e/(e-1))b_1 \geq e/(e-1), \end{aligned}$$

and so forth, and hence $b_t \geq (e/(e-1))^{t-1}$.

All these borders are of length at most $n/2$, so $n/2 > b_t \geq (e/(e-1))^{t-1}$.

Hence

$$t \leq 1 + \frac{\ln(n/2)}{\ln(e/(e-1))},$$

and the result follows. ■

Expected value of initial critical exponent

Theorem. Let $k \geq 2$. Over a k-letter alphabet, the expected number of borders (equivalently, the number of nontrival periods) of a length-n word is $k^{-1} + k^{-2} + \cdots + k^{1-n} \leq \frac{1}{k-1}$.

Proof. By the linearity of expectation, the expected number of borders is the sum, from $i = 1$ to $n - 1$, of the expected value of the indicator random variable B_i taking the value 1 if there is a border of length i, and 0 otherwise.

Once the left border of length i is chosen arbitrarily, the i bits of the right border are fixed, and so there are $n - i$ free choices of symbols.

This means that $E[B_i] = k^{n-i}/k^n = k^{-i}$.

Expected value of initial critical exponent

Theorem. The expected value of $\text{ice}(x)$, for finite or infinite words x, is $\Theta(1)$.

Proof. Let's count the fraction H_j of words having at least a j'th power prefix. Count the number of words having a j'th power prefix with period 1, 2, 3, etc. This double counts, but shows that $H_j \leq k^{1-j} + k^{2(1-j)} + \cdots = 1/(k^{j-1} - 1)$ for $j \geq 2$. Clearly $H_1 = 1$. Then $H_{j-1} - H_j$ is the fraction of words having a $(j-1)$th power prefix but no jth power prefix. These words will have an ice at most j. So the expected value of ice is bounded above by

$$2(H_1 - H_2) + 3(H_2 - H_3) + 4(H_3 - H_4) + \cdots$$

$$= 2H_1 + H_2 + H_3 + H_4 + \cdots = 2 + H_2 + H_3 + H_4 + \cdots$$

$$= 2 + \sum_{j \geq 2} 1/(k^{j-1} - 1) = 2 + \sum_{j \geq 1} 1/(k^j - 1).$$

Characteristic Sturmian words

Let $0 < \alpha < 1$ be an irrational real number with continued fraction expansion $[0, a_1, a_2, \ldots]$.

The *characteristic Sturmian word* $\mathbf{x}_\alpha$ is an infinite word

$$x_1 x_2 x_3 \cdots$$

defined by

$$x_i = \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor.$$

For example, for $\alpha = \sqrt{2} - 1$ the characteristic Sturmian word $\mathbf{x}_\alpha$ is

$$0101001010010101001010010100 \cdots .$$

The Ostrowski α-numeration system

You were waiting patiently for the numeration systems. Here they are.

With every real irrational α, $0 < \alpha < 1$, we associate a numeration system based on the continued fraction expansion $\alpha = [0, a_1, a_2, a_3, \ldots]$ This is called the *Ostrowski α-numeration system*.

Define $p_i/q_i = [0, a_1, \ldots, a_i]$ to be the i'th convergent. In the (ordinary) Ostrowski α-numeration system, we write

$$n = \sum_{0 \leq i \leq t} d_i q_i$$

where $d_t > 0$ and the d_i satisfy certain inequalities.

Alexander Ostrowski (1893-1986)

Photo courtesy of Archives of the Mathematisches Forschungsinstitut Oberwolfach

The lazy Ostrowski numeration system

But we're going to be more concerned with the *lazy Ostrowski system* (Epifanio et al., 2012, 2016).

This representation is again defined through the sum $n = \sum_{0 \leq i \leq t} d_i q_i$ but with slightly different conditions:

(a) $0 \leq d_0 < a_1$;
(b) $0 \leq d_i \leq a_{i+1}$ for $i \geq 1$;
(c) For $i \geq 2$, if $d_i = 0$, then $d_{i-1} = a_i$;
(d) If $d_1 = 0$, then $d_0 = a_1 - 1$.

By convention, we write it as a finite word $d_t d_{t-1} \cdots d_1 d_0$, starting with the most significant digit.

Main result #2

Here it is in words:

From the lazy Ostrowski α-representation of n, one can directly read off all the periods of the length-n prefix X_n of the Sturmian characteristic word $\mathbf{x}_\alpha$.

More precisely,

Main result #2

Let Y_n for $n \geq 1$ be the prefix of $\mathbf{x}_\alpha$ of length n.

Let $\mathrm{PER}(n)$ denote the set of all periods of Y_n (including the trivial period n).

Theorem. (a) The number of periods of Y_n (including the trivial period n) is equal to the sum of the digits in the lazy Ostrowski representation of n.

(b) Suppose the lazy Ostrowski representation of n is $\sum_{0 \leq i \leq t} d_i q_i$. Define

$$A(n) = \left\{ e q_j + \sum_{j < i \leq t} d_i q_i : 1 \leq e \leq d_j \text{ and } 0 \leq j \leq t \right\}.$$

Then $\mathrm{PER}(n) = A(n)$.

Example of the theorem

As an example of the theorem, suppose $\alpha = \sqrt{2} - 1$.

Write $n = 23$ in lazy Ostrowski: $12 + 2 \cdot 5 + 1$.

Then the periods are
$12, 12 + 5 = 17, 12 + 5 + 5 = 22, 12 + 5 + 5 + 1 = 23$.

So the nonempty borders are size $11, 6, 1$.

Take $Y_{23} = 01010010100101010010100$.

Here are the borders:

01010010100101010010100

01010010100101010010100

01010010100101010010100

Brief sketch of the proof

Let $X_i = Y_{q_i}$.

Frid (2018) defined two kinds of Ostrowski representations.

A representation $n = \sum_{0 \leq i \leq t} d_i q_i$ is *legal* if $0 \leq d_i \leq a_{i+1}$.

A representation $n = \sum_{0 \leq i \leq t} d_i q_i$ is *valid* if $Y_n = X_t^{d_t} \cdots X_0^{d_0}$.

She proved the very nice result: **every legal representation is valid.**

Brief sketch of the proof

Let $n = \sum_{0 \le i \le t} d_i q_i$ be the lazy Ostrowski representation of n. It's legal, hence valid, hence $Y_n = X_t^{d_t} X_{t-1}^{d_{t-1}} \cdots X_0^{d_0}$.

What we want to show is that each of the following is a period of Y_n:

$X_t,\ X_t^2,\ \ldots,\ X_t^{d_t},$
$X_t^{d_t} X_{t-1},\ X_t^{d_t} X_{t-1}^2,\ \ldots,\ X_t^{d_t} X_{t-1}^{d_{t-1}}, \ldots,$
$X_t^{d_t} X_{t-1}^{d_{t-1}} \cdots X_1^{d_1} X_0,\ X_t^{d_t} X_{t-1}^{d_{t-1}} \cdots X_1^{d_1} X_0^2,\ \ldots,\ X_t^{d_t} X_{t-1}^{d_{t-1}} \cdots X_1^{d_1} X_0^{d_0}.$

To show $A(n) \subseteq \mathrm{PER}(n)$, we let U be one of the words above. Then by Frid's theorem $Y_n = UY_{n'}$ for an appropriate n'.

But $Y_{n'}$ is a prefix of Y_n, so Y_n is a prefix of UY_n.

So U is a period of Y_n, as desired. That proves one direction of our theorem. For the other direction, we use an induction.

News flash!

Philipp Hieronymi and his group at Illinois have implemented a prover for Sturmian characteristic words.

With this prover they were able to prove our Main Result #2 above just by stating it in first-order logic!

Special case of the Fibonacci word

In the special case of the Fibonacci word **f**, we have $\alpha = (\sqrt{5} - 1)/2$.

To get the periods of the length-n prefix Y_n of **f**, write n in "lazy Fibonacci" representation:

$$n = F_{a_t} + F_{a_{t-1}} + \cdots + F_{a_1}$$

where $a_t > a_{t-1} > \cdots > a_1$.

Then the periods are

$$\begin{aligned} &F_{a_t}, \\ &F_{a_t} + F_{a_{t-1}}, \\ &\ldots, \\ &F_{a_t} + F_{a_{t-1}} + \cdots + F_{a_1}. \end{aligned}$$

Special case of the Fibonacci word

More results on the Fibonacci word:

The shortest prefix of $\mathbf{f}$ having exactly n periods (including the trivial period) is of length $F_{n+3} - 2$, for $n \geq 1$.

The longest prefix of $\mathbf{f}$ having exactly n periods (including the trivial period) is of length $F_{2n+2} - 1$, for $n \geq 1$.

The least period of $\mathbf{f}[0..m-1]$ is F_n for $F_{n+1} - 1 \leq m \leq F_{n+2} - 2$ and $n \geq 2$.

Tightness of the inequality on periods

Let g_s, for $s \geq 1$, be the prefix of length $F_{s+2} - 2$ of $\mathbf{f}$. Thus, for example, $g_1 = \epsilon$, $g_2 = 0$, $g_3 = 010$, $g_4 = 010010$, and so forth.

In our period inequality

$$\text{nnp}(x) \leq \frac{e}{2} + 1 + \frac{\ln(n/2)}{\ln(e/(e-1))}$$

the bound is tight, up to an additive factor, for the words g_s.

Let $\tau = (1 + \sqrt{5})/2$, the golden ratio.

Theorem. Take $x = g_s$ for $s \geq 4$. Then the left-hand side of the inequality is $s - 2$, while the right-hand side is asymptotically $s + c$ for $c = 3 + \tau^2/2 - (\ln 2\sqrt{5})/(\ln \tau) \doteq 1.19632$.

Measures of periodicity for infinite words

What we have seen suggests exploring

$$M(x) := \frac{\mathrm{nnp}(x)}{\mathrm{ice}(x) \ln |x|}$$

as a measure of periodicity for finite words x. It also suggests studying the following measures of periodicity for infinite words $\mathbf{x}$.

For $n \geq 2$ let Y_n be the prefix of length n of $\mathbf{x}$. Then define

$$P(\mathbf{x}) := \limsup_{n \to \infty} \; M(Y_n)$$
$$p(\mathbf{x}) := \liminf_{n \to \infty} \; M(Y_n)$$

For the “typical” infinite word $\mathbf{x}$ we have $P(\mathbf{x}) = p(\mathbf{x}) = 0$.

Thus it is of interest to find words $\mathbf{x}$ where $P(\mathbf{x})$ and $p(\mathbf{x})$ are large.

An example: the period-doubling word

The *period-doubling word* **d** is defined to be the fixed point of the morphism sending $1 \to 10$ and $0 \to 11$.

Theorem. $P(\mathbf{d}) = \frac{1}{2 \ln 2} \doteq 0.7213$ and $p(\mathbf{d}) = \frac{1}{4 \ln 2} \doteq 0.36067$.

An example: the period-doubling word

Proof. Let $r(n)$ denote the number of periods (including the trivial period) in the length-n prefix of $\mathbf{d}$. We can use the theorem-proving software `Walnut` to calculate the periods of prefixes of $\mathbf{d}$.

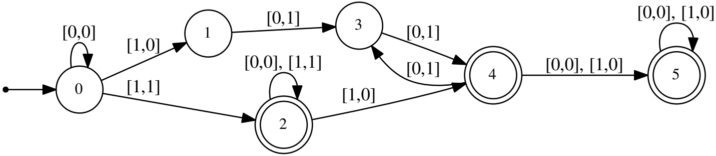
We write a first-order logical formula $\mathrm{pdp}(m, p)$ stating that the prefix of length $m \geq 1$ of $\mathbf{d}$ has period p, $1 \leq p \leq m$:

$$\begin{aligned}\mathrm{pdp}(m, p) &:= (1 \leq p \leq m) \;\wedge\; \mathbf{d}[0..m-p-1] = \mathbf{d}[p..m-1] \\ &= (1 \leq p \leq m) \;\wedge\; \forall t \; (0 \leq t < m-p) \implies \mathbf{d}[t] = \mathbf{d}[t+p].\end{aligned}$$

An example: the period-doubling word

Such a formula can be automatically translated, using `Walnut`, to an automaton that recognizes the language

$$\{(n, p)_2 \ : \ \text{the length-}n\text{ prefix of } \mathbf{d} \text{ has period } p\}.$$



An example: the period-doubling word

Such an automaton can be automatically converted by `Walnut` to a linear representation for $r(n)$. This is a triple (v, ρ, w) where v, w are vectors, and ρ is a matrix-valued morphism, such that $r(n) = v \cdot \rho((n)_2) \cdot w$.

The values are given below:

$$v = [1\,0\,0\,0\,0\,0] \quad \rho(0) = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&1 \\ 0&0&0&0&0&1 \end{bmatrix} \quad \rho(1) = \begin{bmatrix} 0&1&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

An example: the period-doubling word

From this we can easily compute the relations

$$
\begin{aligned}
r(0) &= 0 \\
r(2n+1) &= r(n)+1, \quad n \geq 0 \\
r(4n) &= r(n)+1, \quad n \geq 1 \\
r(4n+2) &= r(n)+1, \quad n \geq 0.
\end{aligned}
$$

Reinterpreting this definition for r, we see that $r(n)$ is equal to the length of the (unique) factorization of $(n)_2$ into the factors 1, 00, and 10.

It now follows that

(a) The smallest m such that $r(m) = n$ is $m = 2^n - 1$;

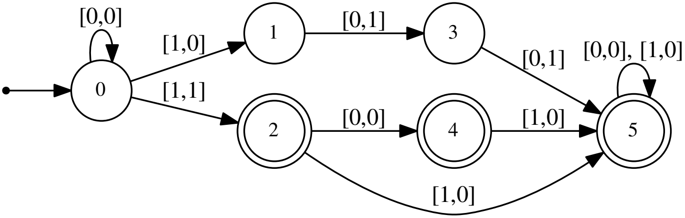
(b) The largest m such that $r(m) = n$ is $m = \lfloor 2^{2n+1}/3 \rfloor$, with $(m)_2 = (10)^n$.

An example: the period-doubling word

Similarly, we can use `Walnut` to determine the smallest period p of every length-n prefix of **d**. We use the predicate

$$\mathrm{pdlp}(n,p) := \mathrm{pdp}(n,p) \;\wedge\; \forall q\ (1 \leq q < p) \implies \mathrm{pdp}(n,q).$$

This gives the automaton



Inspection of this automaton shows that least period of the prefix of length n is, for $s \geq 2$, equal to $3 \cdot 2^{s-2}$ for $2^s \leq n < 5 \cdot 2^{s-2}$ and 2^s for $5 \cdot 2^{s-2} \leq n < 2^{s+1}$. So the ice of every length-n prefix of **d** for $2^t - 1 \leq n \leq 2^{t+1} - 2$, is $2 - 2^{1-t}$.

The result now follows.

Shortest overlap-free binary word with p periods

Recall that an *overlap* is a word of the form $axaxa$, where a is a single letter and x is a (possibly empty) word. An example in English is the word `alfalfa`. We say a word is *overlap-free* if no finite factor is an overlap.

Define $f(p)$ to be the length of the shortest overlap-free binary word having p nontrivial periods.

Theorem. We have $f(1) = 2$, $f(2) = 5$, and

$$f(p) \leq \frac{17}{6} \cdot 4^{p-2} + \frac{2}{3} \quad \text{for } p \geq 3 \; .$$

Shortest overlap-free binary word with p periods

Proof sketch. Define $\mu(0) = 01$ and $\mu(1) = 10$. If $w = axa$ for a single letter a, define $\gamma(w) = a^{-1}\mu^2(w)a^{-1}$. Furthermore define

$$A_n = \begin{cases} 001001100100, & \text{if } n = 3; \\ \gamma(A_{n-1}), & \text{if } n \geq 4. \end{cases}$$

Then we can prove by induction that A_n is a overlap-free palindrome with n nontrivial periods for $n \geq 3$. ■

Shortest squarefree ternary word with p periods

Recall that a *square* is a word of the form xx, where x is a nonempty word. An example in English is the word `murmur`. We say a word is *squarefree* if no finite factor is a square.

Define $g(p)$ to be the length of the shortest squarefree ternary word having p nontrivial periods.

Theorem. We have $g(1) = 3$, $g(2) = 7$, and

$$g(p) \leq \frac{17}{12} \cdot 4^{p-1} + \frac{1}{3} \quad \text{for } p \geq 3 \; .$$

Open problems

1. Prove that the bound for binary overlap-free words $f(p)$ obtained above is optimal.

2. For ternary squarefree words, determine the asymptotic behavior of $g(p)$.

3. Find an exact expression for the limit, as $n \to \infty$, of the expected value of ice of the length-n words over a k-letter alphabet. For example, for $k = 2$, this seems to be about 2.494.