Continued Fractions – New and Old Results

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Continued fractions

By a *continued fraction* we mean an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}}$$

or

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cdots}}}$$

where $a_1, a_2, \dots,$ are positive integers and a_0 is an integer. To save space, as usual, we write $[a_0, a_1, \dots, a_n]$ for the first expression and $[a_0, a_1, \dots,]$ for the second.

What's known

- Euler, Lagrange: a real number has an ultimately periodic continued fraction expansion if and only if it is a quadratic irrational
- Other interesting numbers:

$$
\begin{aligned}
e &= [2, (1, 2n, 1)_{n=1}^{\infty}] = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots] \\
e^2 &= [7, (3n-1, 1, 1, 3n, 12n+6)_{n=1}^{\infty}] \\
&= [7, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, \ldots] \\
\tan 1 &= [(1, 2n-1)_{n=1}^{\infty}] = [1, 1, 1, 3, 1, 5, 1, 7, \ldots]
\end{aligned}
$$

  These three are examples of "Hurwitz continued fractions", where there is a "quasiperiod" of terms that grow linearly.
- No simple pattern is currently known for the expansions of π or e^3 or e^4.

Irrationality measure

The *irrationality measure* $\mu(x)$ of a real number x is the supremum of the real numbers μ such that the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

has infinitely many solutions in rational numbers p/q.

So it measures how well x is approximated by rationals.

Liouville's theorem: if x is algebraic of degree n then $\mu(x) \leq n$.

Roth's theorem: if x is an algebraic irrational number then $\mu(x) = 2$.

If the continued fraction for x has *bounded partial quotients* then $\mu(x) = 2$.

Irrationality measures of some famous numbers

- $\mu(e) = 2$
- $\mu(\pi) \leq 7.6063$ (Salikhov)
- $\mu(\pi^2) \leq 5.441243$ (Rhin & Viola)
  – by studying the integral

$$\int_0^1 \int_0^1 \left(\frac{x^h (1-x)^i y^k (1-y)^j}{(1-xy)^{i+j-l}} \right)^n \frac{dxdy}{1-xy} = a_n - b_n \zeta(2),$$

  in particular for $(h, i, j, k, l) = (12, 12, 14, 14, 13)$.
- $\mu(\log 2) \leq 3.57455391$ (Marcovecchio)
- $\mu(\zeta(3)) \leq 5.513891$ (Rhin & Viola)

The usual conventions

Given a real irrational number x with partial quotients

$$x = [a_0, a_1, a_2, \ldots]$$

we define the sequence of convergents by

$$\begin{aligned} p_{-2} &= 0 \qquad & p_{-1} &= 1 \qquad & p_n &= a_n p_{n-1} + p_{n-2} \quad (n \geq 0) \\ q_{-2} &= 1 \qquad & q_{-1} &= 0 \qquad & q_n &= a_n q_{n-1} + q_{n-2} \quad (n \geq 0) \end{aligned}$$

and then

$$[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}.$$

Furthermore, we use the Hurwitz-Kolden-Frame representation of continued fractions via 2×2 matrices, as follows:

$$M(a_0, \ldots, a_n) := \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix}.$$

The matrix approach

By taking determinants we immediately deduce the classical identity

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$$

for $n \geq 0$.

Given a finite sequence $z = (a_0, \ldots, a_n)$ we let z^R denote the reversed sequence $(a_n, \ldots, a_0)$. A sequence is a *palindrome* if $z = z^R$. By taking the transpose of the equation above, we get

$$M(a_n, \ldots, a_0) := \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix}.$$

Hence if $[a_0, a_1, \ldots, a_n] = p_n / q_n$

then $[a_n, \ldots, a_1, a_0] = p_n / p_{n-1}$.

Ultimately periodic continued fractions

By an expression of the form $[x, \overline{w}]$, where x and w are finite strings, we mean the continued fraction $[x, w, w, w, \ldots]$, where the overbar or "vinculum" denotes the repeating portion.

Thus, for example,

$$\sqrt{7} = [2, \overline{1, 1, 1, 4}\,] = [2, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \ldots].$$

Numbers with "easy to describe" continued fractions

- We want to understand the characteristics and Diophantine properties of numbers whose continued fraction expansion is generated by a simple computational model, such as a finite automaton.
- Famous example: the Thue-Morse sequence on the symbols $\{a, b\}$ where $\overline{a} = b$ and $\overline{b} = a$, given by

$$\mathbf{t} = t_0 t_1 t_2 \cdots = abbabaab \cdots$$

and defined by

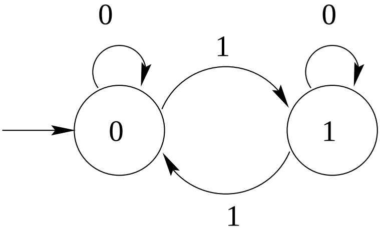
$$t_n = \begin{cases} a, & \text{if } n = 0; \\ t_{n/2}, & \text{if } n \text{ even}; \\ \overline{t_{n-1}}, & \text{if } n \text{ odd}. \end{cases}$$

Automatic sequences

A continued fraction like $[\mathbf{t}] = [a, b, b, a, b, a, a, b, \ldots]$ is called *automatic* because its terms are generated by a finite automaton.

Basic idea:

- input to the automaton is n expressed in base 2
- each symbol read causes a *transition* to the appropriate state
- output is associated with the last state reached



Previous results

- Queffélec proved that if a, b are distinct positive integers, then the real number $[\mathbf{t}] = [t_0, t_1, t_2, \ldots]$ is transcendental.
- Later, a simpler proof was found by Adamczewski and Bugeaud.
- Queffélec also proved the transcendence of a much wider class of automatic continued fractions.
- Many new recent results about properties of automatic, morphic, and Sturmian continued fractions (Allouche, Davis, Quéffelec, Zamboni, Adamczewski, Bugeaud, ...)
- Finally, Bugeaud proved (2012) that any automatic continued fraction is either a quadratic number or is transcendental.

Another automatic continued fraction

Consider the number

$$\alpha_k = \sum_{n \geq 0} k^{-2^n} = k^{-1} + k^{-2} + k^{-4} + k^{-8} + \cdots$$

for an integer $k \geq 2$.

These numbers, sometimes mistakenly called the Fredholm numbers, are all known to be transcendental.

Their continued fraction is (for $k \geq 3$):

$$\alpha_k = [0, k-1, k+2, k, k, k-2, k, k+2, k, k-2, k+2, k, k-2, \ldots]$$

where the partial quotients all lie in the set

$$\{0, k-2, k-1, k, k+2\}.$$

(JOS and Kmošek, independently).

Another automatic continued fraction

There is an automaton that generates this sequence:

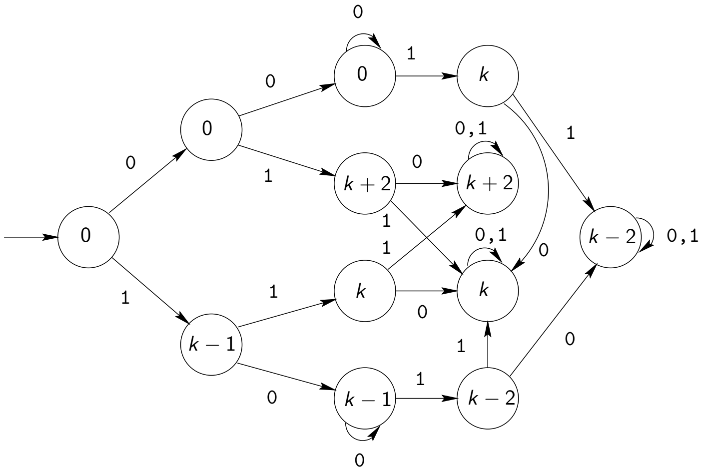


Figure: Automaton generating the continued fraction for α_k

k-regular sequences

- Drawback: all automatic sequences (and the more general class of morphic sequences) take only finitely many distinct values
- A more general class, allowing unbounded terms, is the k-regular sequences of integers, for integer $k \geq 2$.
- These are sequences $(a_n)_{n \geq 0}$ where the k-kernel, defined by

$$\{(a_{k^e n+i})_{n \geq 0} \ : e \geq 0, \ 0 \leq i < k^e\},$$

  is contained in a finitely generated module.
- Example: $s_k(n)$, the function counting the sum of the base-k digits of n. It satisfies $s_k(k^e n + i) = s_k(n) + s_k(i)$ and hence each sequence $(s_k(k^e n + i))_{n \geq 0}$ can be written as a linear combination of the sequence $(s_k(n))_{n \geq 0}$ and the constant sequence 1.

A conjecture

Conjecture

Every continued fraction where the terms form a k-regular sequence of positive integers is either transcendental or quadratic.

A particular 2-regular non-automatic sequence

- Let
$$\mathbf{s} = s_0 s_1 s_2 \cdots = (1, 2, 1, 4, 1, 2, 1, 8, \ldots)$$
where $s_i = 2^{\nu_2(i+1)}$ and $\nu_p(x)$ is the p-adic valuation of x (the exponent of the largest power of p dividing x).
- To see that $\mathbf{s}$ is 2-regular, notice that every sequence in the 2-kernel is a linear combination of $\mathbf{s}$ itself and the constant sequence $(1, 1, 1, \ldots)$.
- Define the real number σ to have continued fraction expansion $\sigma = [\mathbf{s}] = [s_0, s_1, s_2, \ldots]$
$$= [1, 2, 1, 4, 1, 2, 1, 8, \ldots] = 1.35387112842988237438889 \cdots .$$
- The sequence $\mathbf{s}$ is sometimes called the "ruler sequence".

Properties of σ

- σ has slowly growing partial quotients (indeed, $s_i \leq i + 1$ for all i)
- But empirical calculation for $\sigma^2 = 1.8329670323960030544272195442104173\,24\cdots$ demonstrates the appearance of some exceptionally large partial quotients!
- Here are the first few terms:

$$\sigma^2 = [1, 1, 4, 1, 74, 1, 8457, 1, 186282390, 1, 1, 1, 2, 1, 430917181166219, 11, 37, 1, 4, 2, 41151315877490090952542206046, 11, 5, 3, 12, 2, 34, 2, 9, 8, 1, 1, 2, 7, 13991468824374967392702752173757116934238293984253807017, \ldots$$

- First noticed by Paul D. Hanna and Robert G. Wilson in November 2004.

The basic idea

To prove that the partial quotients of σ^2 truly do grow doubly exponentially, we will rely on the following classical estimate:

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}.$$

So to show that some partial quotients of x are huge, it is sufficient to find convergents p_n/q_n of x such that $|x - p_n/q_n|$ is much smaller than q_n^{-2}.

An Old Result

We now recall a classical result:

Lemma

Let a_0 be a positive integer and w denote a finite palindrome of positive integers. Then

$$[a_0, \overline{w, 2a_0}\,] = \sqrt{\frac{p}{q}},$$

where

$$M(w) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$p = a_0^2\alpha + 2a_0\beta + \delta$$

$$q = \alpha.$$

Two sequences

We now define two related sequences for $n \geq 2$:

$$\begin{aligned} u(n) &= (s_1, s_2, \ldots, s_{2^n-3}) \\ v(n) &= (s_1, s_2, \ldots, s_{2^n-2}) = (u(n), 1) \end{aligned}$$

Recall that $s_n = 2^{\nu_2(n+1)}$ is defined in the introduction. The following table gives the first few values of these quantities:

n	$u(n)$	$v(n)$
2	2	21
3	21412	214121
4	2141218121412	21412181214121

A proposition

The following proposition, which is easily proved by induction, gives the relationship between these sequences, for $n \geq 2$:

Proposition

(a) $u(n+1) = (v(n), 2^n, v(n)^R)$;

(b) $u(n)$ *is a palindrome;*

(c) $v(n+1) = (v(n), 2^n, 1, v(n))$.

Two matrices

Furthermore, we can define the sequence of associated matrices with $u(n)$ and $v(n)$:

$$
\begin{aligned}
M(u(n)) &:= \begin{bmatrix} c_n & e_n \\ d_n & f_n \end{bmatrix} \\
M(v(n)) &:= \begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix}.
\end{aligned}
$$

A calculation

- Now let
$$\sigma_n = [1, \overline{u(n), 2}\,]$$
- Then by the previous lemma with $a_0 = 1$ and $w = u(n)$ we get
$$\sigma_n = \sqrt{\frac{c_n + 2e_n + f_n}{c_n}}.$$
- Write $\sigma = [s_0, s_1, \ldots]$ and $[s_0, s_1, \ldots, s_n] = \frac{p_n}{q_n}$.
- Furthermore define $\hat{\sigma}_n = [1, u(n)]$.
- Notice that σ, σ_n, and $\hat{\sigma}_n$ all agree on the first $2^n - 2$ partial quotients.

An inequality

- We have

$$|\sigma - \hat{\sigma}_n| < \frac{1}{q_{2^n-3}q_{2^n-2}}$$

by a classical theorem on continued fractions.

- Furthermore, since $s_{2^n-3} = 2$, $s_{2^n-2} = 1$, we have, for $n \geq 3$, that

$$\sigma < \sigma_n < \hat{\sigma}_n.$$

- Hence

$$|\sigma - \sigma_n| < \frac{1}{q_{2^n-3}q_{2^n-2}}.$$

A good approximation

Now by considering

$$\begin{aligned} M(1)M(u(n))M(1) &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_n & e_n \\ d_n & f_n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2c_n + d_n & e_n + f_n \\ c_n + d_n & c_n \end{bmatrix}, \end{aligned}$$

we see that $q_{2^n-3} = c_n$ and $q_{2^n-2} = c_n + d_n$.

A good approximation (continued)

For simplicity write $g_n = c_n + 2e_n + f_n$. Then

$$|\sigma - \sigma_n| = \left|\sigma - \sqrt{\frac{g_n}{c_n}}\right| < \frac{1}{c_n^2},$$

and so

$$\left|\sigma^2 - \frac{g_n}{c_n}\right| = \left|\sigma - \sqrt{\frac{g_n}{c_n}}\right| \cdot \left|\sigma + \sqrt{\frac{g_n}{c_n}}\right| < \frac{3}{c_n^2}.$$

So we have already found good approximations of σ^2 by rational numbers. Next we show g_n and c_n have a large common factor, which will improve the quality of the approximation.

Improving the approximation

From previous results we get that the matrix

$$\begin{bmatrix} c_{n+1} & e_{n+1} \\ d_{n+1} & f_{n+1} \end{bmatrix},$$

associated with $u(n+1)$, is equal to the matrix associated with $(v(n), 2^n, v(n)^R)$, which is

$$\begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix} \begin{bmatrix} 2^n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_n & x_n \\ y_n & z_n \end{bmatrix}$$

$$= \begin{bmatrix} 2^n w_n^2 + 2w_n y_n & 2^n w_n x_n + x_n y_n + w_n z_n \\ 2^n w_n x_n + x_n y_n + w_n z_n & 2^n x_n^2 + 2x_n z_n \end{bmatrix}.$$

Improving the approximation

Notice that

$$c_{n+1} = (2^n w_n + 2y_n)w_n. \tag{1}$$

On the other hand, we have

$$\begin{aligned}
g_{n+1} &= c_{n+1} + 2d_{n+1} + f_{n+1} \\
&= c_{n+1} + 2(2^n w_n x_n + x_n y_n + w_n z_n) + 2^n x_n^2 + 2x_n z_n \\
&= c_{n+1} + 2(2^n w_n x_n + x_n y_n + w_n z_n) + 2^n x_n^2 + 2x_n z_n \\
&\quad +2(x_n y_n - w_n z_n + 1) \\
&= c_{n+1} + (2^n w_n + 2y_n)2x_n + 2^n x_n^2 + 2x_n z_n + 2 \\
&= (2^n w_n + 2y_n)(2x_n + w_n) + 2^n x_n^2 + 2x_n z_n + 2,
\end{aligned}$$

By Euclidean division, we get

$$\gcd(g_{n+1}, 2^n w_n + 2y_n) = \gcd(2^n w_n + 2y_n, 2^n x_n^2 + 2x_n z_n + 2).$$

The other matrix

Next we work with $v(n+1)$. We get that the matrix

$$\begin{bmatrix} w_{n+1} & y_{n+1} \\ x_{n+1} & z_{n+1} \end{bmatrix}$$

associated with $v(n+1)$ is equal to the matrix associated with $(v(n), 2^n, 1, v(n))$, which is

$$\begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix} \begin{bmatrix} 2^n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_n & y_n \\ x_n & z_n \end{bmatrix}$$

$$= \begin{bmatrix} (2^n+1)w_n^2 + 2^n w_n x_n + y_n(w_n + x_n) & (2^n+1)w_n y_n + 2^n w_n z_n + y_n(y_n + z_n) \\ (2^n+1)x_n w_n + 2^n x_n^2 + z_n(w_n + x_n) & (2^n+1)x_n y_n + 2^n x_n z_n + z_n(y_n + z_n) \end{bmatrix}.$$

The other matrix

Letting $r_n := 2^n(w_n + x_n) + w_n + y_n + z_n$, we see that

$$\begin{aligned}
2^{n+1}w_{n+1} + 2y_{n+1} &= 2^{n+1}((2^n+1)w_n^2 + 2^n w_n x_n + y_n(w_n + x_n)) + \\
&\quad 2((2^n+1)w_n y_n + 2^n w_n z_n + y_n(y_n + z_n)) \\
&= 2(2^n w_n + y_n)(2^n(w_n + x_n) + w_n + y_n + z_n) \\
&= 2(2^n w_n + y_n) r_n. \qquad (2)
\end{aligned}$$

A simplification

Now

$$\begin{aligned} x_{n+1} &= (2^n+1)x_n w_n + 2^n x_n^2 + z_n(w_n+x_n) \\ &= x_n(2^n(w_n+x_n)+w_n+y_n+z_n) + w_n z_n - x_n y_n \\ &= x_n r_n + 1 \end{aligned}$$

and

$$\begin{aligned} z_{n+1} &= (2^n+1)x_n y_n + 2^n x_n z_n + z_n(y_n+z_n) \\ &= z_n(2^n(w_n+x_n)+w_n+y_n+z_n) + (2^n+1)(x_n y_n - w_n z_n) \\ &= z_n r_n - 2^n - 1. \end{aligned}$$

More simplification

Tedious simplification now gives

$$\frac{2^n x_{n+1}^2 + x_{n+1} z_{n+1} + 1}{r_n} = (2^n + 1) w_n x_n z_n + 2^n (2^n + 1) w_n x_n^2 + z_n$$
$$+ (2^n - 1) x_n + 2^n x_n^2 y_n + 2^{n+1} x_n^2 z_n + x_n y_n z_n + 2^{2n} x_n^3 + x_n z_n^2.$$

Reindexing, we get

$$\begin{aligned} c_{n+2} &= w_{n+1}(2^{n+1} w_{n+1} + 2y_{n+1}) \\ &= 2w_{n+1}(2^n w_n + y_n) r_n, \end{aligned}$$

Refining the approximation

Also, from the argument above about gcd's, we see that $2r_n \mid g_{n+2}$.

Hence for $n \geq 2$ we have

$$\frac{g_{n+2}}{c_{n+2}} = \frac{P_{n+2}}{Q_{n+2}}$$

for integers

$$P_{n+2} := \frac{g_{n+2}}{2r_n}$$
$$Q_{n+2} := w_{n+1}(2^n w_n + y_n).$$

It remains to see that P_{n+2}/Q_{n+2} are particularly good rational approximations to σ^2.

Since w_n/x_n and y_n/z_n denote successive convergents to a continued fraction, we have $w_n \geq x_n$, $w_n \geq y_n$, and $w_n \geq z_n$. It follows that

$$\begin{aligned} Q_{n+2} &= w_{n+1}(2^n w_n + y_n) \\ &= ((2^n + 1)w_n^2 + 2^n w_n x_n + y_n(w_n + x_n))(2^n w_n + y_n) \\ &\leq (2^{n+1} + 3)w_n^2 \cdot (2^n + 1)w_n \\ &= (2^{n+1} + 3)(2^n + 1)w_n^3. \end{aligned}$$

The approximation is better than we hoped

On the other hand,

$$
\begin{aligned}
c_{n+2} &= 2Q_{n+2}r_n \\
&> 2(2^n+1)w_n^2 \cdot 2^n w_n \cdot (2^n+1)w_n = 2^{n+1}(2^n+1)^2 w_n^4 \\
&\geq Q_{n+2}^{4/3} \frac{2^{n+1}(2^n+1)^2}{((2^{n+1}+3)(2^n+1))^{4/3}} \\
&> Q_{n+2}^{4/3}.
\end{aligned}
$$

Finally, the result about σ^2

Putting this all together, we get

Theorem

$$\left|\sigma^2 - \frac{P_{n+2}}{Q_{n+2}}\right| < Q_{n+2}^{-8/3}$$

for all integers $n \geq 2$.

Irrationality measure

The result we have just shown can be nicely formulated in terms of the irrationality measure.

Theorem

The irrationality measure of σ^2 *is at least* $8/3$.

It would be interesting to find the precise value of the irrationality measure of σ_2. Numerical experiments suggest that the bound $8/3$ is sharp; however, we do not currently know how to prove that.

Remarks

- Note that the classical Khintchine theorem states that for almost all real numbers (in terms of Lebesgue measure), the irrationality measure equals two.
- Hence our theorem says that the number σ^2 belongs to a very tiny set of zero Lebesgue measure.
- From Roth's theorem we conclude that σ^2 (and hence σ) are transcendental numbers. However...
- By a 2007 result of Adamczewski and Bugeaud, we already know that if a continued fraction of a real number x starts with arbitrarily long palindromes, then x is transcendental. Clearly σ satisfies this property.

Some partial quotients are double exponential in size

We now provide a lower bound for some very large partial quotients of σ^2.

For each $n \geq 2$ we have

$$\left|\sigma^2 - \frac{P_{n+2}}{Q_{n+2}}\right| < Q_{n+2}^{-8/3} < \frac{1}{2Q_{n+2}^2}.$$

In particular this implies that the rational number P_{n+2}/Q_{n+2} is a convergent of σ^2.

Observations

- Notice that P_{n+2} and Q_{n+2} are not necessarily relatively prime.
- Let $\tilde{P}_{n+2}/\tilde{Q}_{n+2}$ denote the reduced fraction of P_{n+2}/Q_{n+2}.
- If $\tilde{P}_{n+2}/\tilde{Q}_{n+2}$ is the m'th convergent of σ^2, then define A_{n+2} to be the $(m+1)$'th partial quotient of σ^2.

An inequality

Then the estimate above implies

$$\frac{1}{(A_{n+2}+2)\tilde{Q}_{n+2}^2} < \left|\sigma^2 - \frac{\tilde{P}_{n+2}}{\tilde{Q}_{n+2}}\right| < \frac{3}{c_{n+2}^2} \leq \frac{3}{4r_n^2\tilde{Q}_{n+2}^2},$$

where we used the fact (established earlier) that
$c_{n+2} = 2r_nQ_{n+2} \geq 2r_n\tilde{Q}_{n+2}$.

Hence $A_{n+2} \geq \frac{4}{3}r_n^2 - 2$.

The last result

From the formula for r_n and the inequalities $w_n \geq x_n, w_n \geq y_n, w_n \geq z_n$ one can easily derive

$$(2^n + 1)w_n \leq r_n \leq (2^{n+1} + 3)w_n.$$

This, together with the formula for w_{n+1}, gives the estimate

$$r_{n+1} \geq (2^{n+1} + 1)w_{n+1} \geq (2^{n+1} + 1)(2^n + 1)w_n^2 > r_n^2 + 1.$$

Therefore we get

Theorem

The sequence $\frac{4}{3}r_n^2 - 2$, and in turn the sequence A_{n+2}, both grow doubly exponentially.

This explains the observation of Hanna and Wilson.

Open problems

1. Say something interesting about the continued fraction expansion of e^3.
2. What are the diophantine properties of the powers of the number $[0, 1, 1, 2, 1, 2, 2, 3, \ldots]$ where the n'th partial quotient is $s_2(n)$?