Decidability in Automatic Sequences

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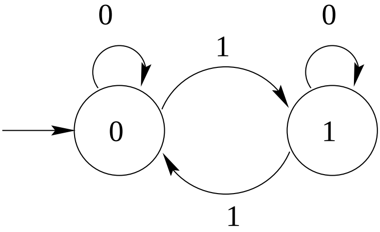
What is an automatic sequence?

- An infinite sequence

$$\mathbf{a} = a_0 a_1 a_2 \cdots$$

  over a finite alphabet of letters, generated by a finite-state machine (automaton)
- The automaton, given n as input, computes a_n as follows:
  - n is represented in some fixed integer base $k \geq 2$
  - The automaton moves from state to state according to this input
  - Each state has an output letter associated with it
  - The output on input n is the output associated with the last state reached

The canonical example: the Thue-Morse automaton



This automaton generates the Thue-Morse sequence

$$\mathbf{t} = (t_n)_{n \geq 0} = 0110100110010110 \cdots .$$

Why automatic sequences?

- A nontrivial class of self-similar sequences
- Many "naturally-occurring" sequences are automatic
- Halfway between periodic and chaotic
- Provide canonical examples for various kinds of avoidance problems

Historically interesting properties of **t**

1. **t** is not ultimately periodic.
2. **t** contains no factor that is an *overlap*, that is, a word of the form *axaxa*, where *a* is a single letter and *x* is an arbitrary finite word. (Example in English: `alfalfa`.)
3. **t** contains infinitely many distinct square factors xx, but for each such factor we have $|x| = 2^n$ or $3 \cdot 2^n$, for $n \geq 0$.
4. **t** has infinitely many distinct palindromic factors (A *palindrome* is a word equal to its reverse, like `radar`.)
5. The number $p(n)$ of distinct palindromic factors of length n in **t** is given by

$$p(n) = \begin{cases} 0, & \text{if } n \text{ odd and } n \geq 5; \\ 1, & \text{if } n = 0; \\ 2, & \text{if } 1 \leq n \leq 4, \text{ or } n \text{ even and } 3 \cdot 4^k + 2 \leq n \leq 4^{k+1}; \\ 4, & \text{if } n \text{ even and } 4^k + 2 \leq n \leq 3 \cdot 4^k. \end{cases}$$

Historically interesting properties of **t**

6. **t** is *mirror-invariant*: if x is a finite factor of **t**, then so is its reverse x^R.
7. **t** is *recurrent*, that is, every factor that occurs, occurs infinitely often.
8. **t** is *uniformly recurrent*, that is, for all factors x occurring in **t**, there is a constant $c(x)$ such that two consecutive occurrences of x are separated by at most $c(x)$ symbols.
9. **t** is *linearly recurrent*, that is, it is uniformly recurrent and furthermore there is a constant C such that $c(x) \leq C|x|$ for all factors x. In fact, the optimal bound is given by $c(1) = 3$, $c(2) = 8$, and $c(n) = 9 \cdot 2^e$ for $n \geq 3$, where $e = \lfloor \log_2(n-2) \rfloor$.

Historically interesting properties of $\mathbf{t}$

10. The lexicographically least sequence in the shift orbit closure of $\mathbf{t}$ is $\overline{t_1}\ \overline{t_2}\ \overline{t_3}\ \cdots$, which is also 2-automatic.
11. The *subword complexity* $\rho(n)$ of $\mathbf{t}$, which is the function counting the number of distinct factors of $\mathbf{t}$, is given by

$$\rho(n) = \begin{cases} 2^n, & \text{if } 0 \leq n \leq 2; \\ 2n + 2^{t+2} - 2, & \text{if } 3 \cdot 2^t \leq n \leq 2^{t+2} + 1; \\ 4n - 2^t - 4, & \text{if } 2^t + 1 \leq n \leq 3 \cdot 2^{t-1}; \end{cases}.$$

12. $\mathbf{t}$ has an unbordered factor of length n if $n \not\equiv 1 \pmod 6$ (Here by an *unbordered* word y we mean one with no expression in the form $y = uvu$ for words u, v with u nonempty.)

The claim

Claim. All of these properties can be verified (and in some case, even obtained) purely mechanically, by a machine computation.

To see how, we need to digress into...

Logic

- Let $\text{Th}(\mathbb{N}, +, 0, 1, <)$ denote the set of all true first-order sentences in the logical theory of the natural numbers with addition.
- Example: in this theory we can express the so-called "Chicken McNuggets theorem" that 43 is the largest integer that cannot be represented as a non-negative integer linear combination of $6, 9$, and 20, as follows:

$$(\forall n > 43 \ \exists x, y, z \geq 0 \text{ such that } n = 6x + 9y + 20z) \wedge \neg(\exists x, y, z \geq 0 \text{ such that } 43 = 6x + 9y + 20z). \quad (1)$$

  Here, of course, "$6x$" is shorthand for the expression "$x + x + x + x + x + x$", and similarly for $9y$ and $20z$.
- Presburger proved that $\text{Th}(\mathbb{N}, +, 0, 1, <)$ is *decidable*: that is, there exists an algorithm that, given a sentence in the theory, will decide its truth.

Decidability of Presburger arithmetic: proof sketch

- represent integers in an integer base $k \geq 2$ using the alphabet $\Sigma_k = \{0, 1, \ldots, k-1\}$.
- represent n-tuples of integers as words over the alphabet Σ_k^n, padding with leading zeroes, if necessary.
- For example, the pair $(21, 7)$ can be represented in base 2 by the word

$$[1, 0][0, 0][1, 1][0, 1][1, 1].$$

Decidability of Presburger arithmetic

- Then the relation $x + y = z$ can be checked by a simple 2-state automaton depicted below, where transitions not depicted lead to a nonaccepting "dead state".

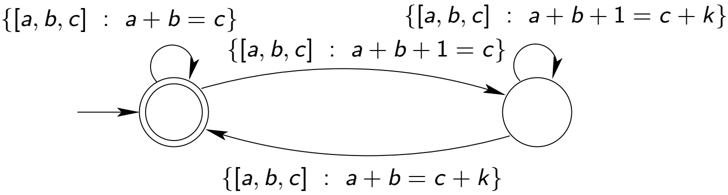


Figure: Checking addition in base k

Decidability of Presburger arithmetic: proof sketch

- Relations like $x = y$ and $x < y$ can be checked similarly.
- Given a formula with free variables $x_1, x_2, \ldots, x_n$, we construct an automaton accepting the base-k expansion of those n-tuples $(x_1, \ldots, x_n)$ for which the proposition holds.
- If a formula is of the form $\exists x_1, x_2, \ldots x_n \; p(x_1, \ldots, x_n)$, then we use nondeterminism to "guess" the x_i and check them.
- If the formula is of the form $\forall p$, we use the equivalence $\forall p \equiv \neg\exists\neg p$; this may require using the subset construction to convert an NFA to a DFA and then flipping the "finality" of states.
- Finally, the truth of a formula can be checked by using the usual depth-first search techniques to see if any final state is reachable from the start state.

Even more

- If we add the function $V_k : \mathbb{N} \to \mathbb{N}$ to our logical theory, where $V_k(x) = k^n$, and k^n is the largest power of k dividing x, it is still decidable by a similar automaton-based technique.
- By doing so, we gain the capability of deciding many questions about automatic sequences.

Theorem

There is an algorithm that, given a predicate phrased using only the universal and existential quantifiers, indexing into a given automatic sequence $\mathbf{a}$*, addition, subtraction, logical operations, and comparisons, will decide the truth of that proposition.*

We call such a predicate an *automatic predicate*.

- The worst-case running time of our algorithm is bounded above by

$$2^{2^{\cdot^{\cdot^{\cdot^{2^{p(N)}}}}}},$$

where the number of 2's in the exponent is equal to the number of quantifiers, p is a polynomial, and N is the number of states needed to describe the underlying automatic sequence.

The good news

- Nevertheless, an implementation often succeeds in verifying statements in the theory
- Many old results have been verified with this technique, and many new ones proved.

Deciding periodicity

- An infinite word $\mathbf{a}$ is *periodic* if it is of the form $x^\omega = xxx\cdots$ for a finite nonempty word x.
- It is *ultimately periodic* if it is of the form yx^ω for a (possibly empty) finite word y.
- Honkala (1986) proved that ultimate periodicity is decidable for automatic sequences.
- Using our approach: it suffices to express ultimately periodicity as an automatic predicate:

$$\exists p \geq 1, N \geq 0 \ \forall i \geq N \ \mathbf{a}[i] = \mathbf{a}[i+p].$$

- When we run this on the Thue-Morse sequence, we discover (as expected) that $\mathbf{t}$ is not ultimately periodic.

Repetitions

- Thue (1912) proved that $\mathbf{t}$ contains no overlaps; that is, $\mathbf{t}$ is overlap-free.
- Using our technique, we can express the property of having an overlap *axaxa* beginning at position N with $|ax| = p$, as follows: $\mathbf{a}[N..N+p] = \mathbf{a}[N+p..N+2p]$.
- So the corresponding automatic predicate for $\mathbf{t}$ is

$$\exists p \geq 1, N \geq 0 \ \mathbf{t}[N..N+p] = \mathbf{t}[N+p..N+2p],$$

or, in other words,

$$\exists p \geq 1, N \geq 0 \ \forall i, 0 \leq i \leq p \ \mathbf{t}[N+i] = \mathbf{t}[N+p+i].$$

- We programmed up our decision procedure and verified that indeed $\mathbf{t}$ is overlap-free.

Critical exponent

- We can define more general repetitions as follows: a word x is an α-power for $\alpha \geq 1$ if we can write $x = y^e y'$ where $e = \lfloor \alpha \rfloor$ and y' is a prefix of y and $|x| = \alpha |y|$.
- For example, `abracadabra` is an $\frac{11}{7}$-power.
- The techniques above suffice to check if a k-automatic sequence has α-powers, using the following predicate:

$$\exists N \geq 0, p, q \geq 1 \; \mathbf{a}[N..N+p-q-1] = \mathbf{a}[N+q..N+p-1] \text{ and } p = \alpha q.$$

- However, this observation alone does not suffice to compute the so-called *critical exponent* of **a**, which is the supremum over all rational α such that **a** has α-power factors.
- It turns out that the critical exponent is also computable for automatic sequences (more later).

Mirror invariance

We can express the property that **a** is mirror-invariant as follows:

$$\forall N \geq 0, \ell \geq 1 \; \exists N' \geq 0 \; \mathbf{a}[N..N+\ell-1] = \mathbf{a}[N'..N'+\ell-1]^R,$$

which is the same as

$$\forall N \geq 0, \ell \geq 1 \; \exists N' \geq 0 \; \forall i, \; 0 \leq i < \ell \; \mathbf{a}[N+i] = \mathbf{a}[N'+\ell-i-1],$$

which can be easily checked by our method.

Recurrence

- We can express the property that **a** is recurrent by saying that for each factor, and each integer M there exists a copy of that factor occurring at a position after M in **a**.
- This corresponds to the following predicate:

$$\forall N, M \geq 0, \ell \geq 1 \, \exists M' \geq M \quad \mathbf{a}[N..N{+}\ell{-}1] = \mathbf{a}[M'..M'{+}\ell{-}1].$$

- An easy argument shows that an infinite word **a** is recurrent if and only if each finite factor occurs at least twice. This means that the following simpler predicate suffices:

$$\forall N \geq 0, \ell \geq 1 \, \exists M \neq N \quad \mathbf{a}[N..N + \ell - 1] = \mathbf{a}[M..M + \ell - 1].$$

Uniform recurrence

- For uniform recurrence, we need to express the fact that two consecutive occurrences of each factor are separated by no more than C positions.
- Since there are only finitely many factors of each length, we can take C to be the maximum of the constants corresponding to each factor of that length.
- Thus uniform recurrence corresponds to the following predicate:

$$\forall \ell \geq 1 \; \exists C \geq 1 \; \forall N \geq 0 \; \exists M \text{ with } N < M \leq N + C \quad \mathbf{a}[N..N+\ell-1] = \mathbf{a}[M..M+\ell-1].$$

Orbit closure

- The *shift orbit* of a sequence $\mathbf{a} = a_0 a_1 a_2 \cdots$ is the set of all sequences under the shift, that is, the set

$$\mathcal{S} = \{a_i a_{i+1} a_{i+2} \cdots \ : \ i \geq 0\}.$$

- The *orbit closure* is the topological closure $\overline{\mathcal{S}}$ under the usual topology.
- In other words, a sequence $\mathbf{b} = b_0 b_1 b_2 \cdots$ is in $\overline{\mathcal{S}}$ if and only if, for each $j \geq 0$, the prefix $b_0 \cdots b_j$ is a factor of $\mathbf{a}$.
- Most sequences in the orbit closure of a k-automatic sequence are not automatic themselves.
- However, we can use our method to show that two distinguished sequences, the lexicographically least and lexicographically greatest sequences in the orbit closure, are indeed k-automatic.

Unbordered factors

- A word is *bordered* if it can be expressed as uvu for words u, v with u nonempty, and otherwise it is unbordered.
- Currie and Saari proved that $\mathbf{t}$ has an unbordered factor of length n if $n \not\equiv 1 \pmod 6$.
- However, these are not the only lengths with an unbordered factor; for example,

  $$0011010010110100110010110100101$$

  is an unbordered factor of length 31.
- We can express the property that $\mathbf{t}$ has an unbordered factor of length ℓ as follows:

  $$\exists N \geq 0\ \forall j, 1 \leq j \leq \ell/2\ \mathbf{t}[N..N+j-1] \neq \mathbf{t}[N+\ell-j..N+\ell-1].$$
- Using our technique, we were able to prove

Theorem

There is an unbordered factor of length ℓ in $\mathbf{t}$ if and only iff $(\ell)_2 \notin 1(01^*0)^*10^*1$.

Enumeration

- In many cases we can count the number $T(n)$ of length-n factors of an automatic sequence having a particular property P.
- Here by "count" we mean, give an algorithm A to compute $T(n)$ efficiently, that is, in time bounded by a polynomial in $\log n$.
- Although *finding* the algorithm A may not be particularly efficient, once we have it, we can compute $T(n)$ quickly.

Subword complexity

- Subword complexity counts the number of distinct length-n factors of a sequence.
- To count these factors in an automatic sequence, we create a DFA M accepting the language

$$\begin{aligned} & \{(n,\ell)_k \ : \ \mathbf{a}[n..n+\ell-1] \text{ is the first} \\ & \text{occurrence of the given factor}\} \\ = \ & \{(n,\ell)_k \ : \ \forall n' < n \ \mathbf{a}[n..n+\ell-1] \neq \mathbf{a}[n'..n'+\ell-1]\}. \end{aligned}$$

- the number of n corresponding to a given ℓ is just the number of distinct subwords of length ℓ
- this number can be expressed as the product

$$vM_{a_1} \cdots M_{a_i} w$$

for suitable vectors v, w and matrices $M_0, \ldots, M_{k-1}$, where $a_1 \cdots a_i$ is the base-k representation of ℓ, thus giving an efficient algorithm to compute it.

Enumeration

In a similar way, we can handle

- palindrome complexity (the number of distinct length-n palindromic factors)
- the number of words whose reversals are also factors;
- the number of squares of a given length;
- the number of unbordered factors

and so forth.

What other properties of automatic sequences are decidable?

- A difficult candidate: abelian properties
- We say that a nonempty word x is an *abelian square* if it of the form ww' with $|w| = |w'|$ and w' a permutation of w. (An example in English is the word `reappear`.)
- Luke Schaeffer showed that the predicate for abelian squarefreeness is indeed inexpressible in $\mathrm{Th}(\mathbb{N}, +, 0, 1, <, V_k)$

Representing rational numbers

- Represent rational number $\alpha = p/q$ by pair of integers (p, q), represented in base k; pad shorter with leading zeroes
- So representations of rationals are over the alphabet $\Sigma_k \times \Sigma_k$
- For example, if $w = [3,0][5,0][2,4][6,1]$ then $[w]_{10} = (3526, 41)$.
- Define $\text{quo}_k(x) = [\pi_1(x)]_k / [\pi_2(x)]_k$, where π_i is the projection onto the i'th coordinate
- So $\text{quo}_{10}(w) = 3526/41 = 86$.
- Canonical representations lack leading $[0,0]$'s
- Every rational has infinitely many canonical representations, e.g., as $(1,2), (2,4), (3,6), \ldots$, etc.

Automatic sets over $\mathbb{Q}^{\geq 0}$

- $\text{quo}_k(L) = \bigcup_{x \in L} \{\text{quo}_k(x)\}$

- $A \subseteq \mathbb{Q}^{\geq 0}$ is a k-automatic set of rationals if $A = \text{quo}_k(L)$ for some regular language $L \subseteq (\Sigma_k \times \Sigma_k)^*$.
- *not* the same notion as the automatic reals of Boigelot, Brusten, and Bruyère

Examples

Example 1. Let $k = 2$, $B = \{[0,0],[0,1],[1,0],[1,1]\}$, and consider

$$L_1 := B^*\{[0,1],[1,1]\}B^*.$$

Then L_1 consists of all pairs of integers where the second component has at least one nonzero digit — the point being to avoid division by 0. Then $\mathrm{quo}_k(L) = \mathbb{Q}^{\geq 0}$, the set of all non-negative rational numbers.

Example 2. Consider

$$L_2 = \{w \in (\Sigma_k^2)^* \ : \ \pi_1(w) \in 0^* C_k \text{ and } \pi_2(w) \in 0^* 1\}.$$

Then $\mathrm{quo}_k(L_2) = \mathbb{N}$.

Examples

Example 3. Let $k = 3$, and consider the language

$$L_3 := [0,1]\{[0,0],[2,0]\}^*.$$

Then $\mathrm{quo}_k(L_3)$ is the 3-*adic Cantor set*, the set of all rational numbers in the "middle-thirds" Cantor set with denominators a power of 3.

Example 4. Let $k = 2$, and consider

$$L_4 := [0,1]\{[0,0],[0,1]\}^*\{[1,0],[1,1]\}.$$

Then the numerator encodes the integer 1, while the denominator encodes all positive integers that start with 1. Hence

$$\mathrm{quo}_k(L_4) = \{\frac{1}{n} : n \geq 1\}.$$

Examples

Example 5. Let $k = 4$, and consider

$$S := \{0, 1, 3, 4, 5, 11, 12, 13, \dots\}$$

of all non-negative integers that can be represented using only the digits $0, 1, -1$ in base 4. Consider the language

$$L_5 = \{(p, q)_4 \ : \ p, q \in S\}.$$

It is not hard to see that L_5 is $(\mathbb{Q}, 4)$-automatic.
The main result in Loxton & van der Poorten [1987] can be rephrased as follows: $\mathrm{quo}_4(L_5)$ contains every odd integer.
In fact, an integer t is in $\mathrm{quo}_4(L_5)$ if and only if the exponent of the largest power of 2 dividing t is even.

Example 6. Consider

$$L_6 = \{w \in (\Sigma_k^2)^* \; : \; \pi_2(w) \in 0^*1^+0^*\}.$$

An easy exercise using the Fermat-Euler theorem shows that that $\text{quo}_k(L_6) = \mathbb{Q}^{\geq 0}$.

Example 7. For a word x and letter a let $|x|_a$ denote the number of occurrences of a in x. Consider the regular language

$$L_7 = \{w \in (\Sigma_2^2) \ : \ |\pi_1(w)|_1 \text{ is even and } |\pi_2(w)|_1 \text{ is odd}\}.$$

Then it follows from a result of Schmid [1984] that

$$\mathrm{quo}_2(L_7) = \mathbb{Q}^{\geq 0} - \{2^n \ : \ n \in \mathbb{Z}\}.$$

Basic decidability properties

Given a DFA M accepting a language L representing a set of rationals S, can decide

- if $S = \emptyset$
- given $\alpha \in \mathbb{Q}^{\geq 0}$, whether there exists $x \in S$ with $x = \alpha$ (resp., $x < \alpha$, $x \leq \alpha$, $x > \alpha$, $x \geq \alpha$, $x \neq \alpha$, etc.)
- if $|S| = \infty$
- given a finite set $F \subseteq \mathbb{Q}^{\geq 0}$, if $F \subseteq S$ or if $S \subseteq F$
- given $\alpha \in \mathbb{Q}^{\geq 0}$, if α is an accumulation point of S

sup A is rational or infinite

Given a DFA M accepting $L \subseteq (\Sigma_k \times \Sigma_k)^*$ representing a set of rationals $A \subseteq \mathbb{Q}^{\geq 0}$, what can we say about $\sup A$?

Theorem. $\sup A$ is rational or infinite, and is computable.

Proof ideas: $\mathrm{quo}_k(uv^i w)$ forms a monotonic sequence. Defining

$$\gamma(u, v) := \frac{[\pi_1(uv)]_k - [\pi_1(u)]_k}{[\pi_2(uv)]_k - [\pi_2(u)]_k}$$

one of the following three cases must hold:

(i) $\mathrm{quo}_k(uw) < \mathrm{quo}_k(uvw) < \mathrm{quo}_k(uv^2w) < \cdots < U$;
(ii) $\mathrm{quo}_k(uw) = \mathrm{quo}_k(uvw) = \mathrm{quo}_k(uv^2w) = \cdots = U$;
(iii) $\mathrm{quo}_k(uw) > \mathrm{quo}_k(uvw) > \mathrm{quo}_k(uv^2w) > \cdots > U$.

Furthermore, $\lim_{i\to\infty} \mathrm{quo}_k(uv^i w) = U$.

sup A is rational or infinite

It follows that if $\sup A$ is finite, and the DFA M has n states, then $\sup A = \max T$, where

$$T = T_1 \cup T_2$$

and

$$\begin{aligned} T_1 &= \{\text{quo}_k(x) \;:\; |x| < n \text{ and } x \in L\}; \\ T_2 &= \{\gamma(u, v) \;:\; |uv| \leq n,\ |v| \geq 1,\ \delta(q_0, u) = \delta(q_0, uv), \\ &\qquad \text{and there exists } w \text{ such that } uvw \in L\}. \end{aligned}$$

$\sup A$ is computable

We know that $\sup A$ lies in the finite computable set T.

For each of $t \in T$, we can check to see if $t \geq \sup A$ by checking if $A \cap (t, \infty)$ is empty.

Then $\sup A$ is the least such t.

Computing the critical exponent

- Previously known to be computable for fixed points of uniform morphisms (Krieger)

Theorem. If $\mathbf{w}$ is a k-automatic sequence, then its critical exponent is rational or infinite. Furthermore, it is computable from the DFAO M generating w.

Proof sketch. Given M, we can transform it into another automaton M' accepting

$$\{(m, n) \,:\, \text{there exists } i \geq 0 \text{ such that } \mathbf{w}[i..i+m-1] \text{ has period } n\}.$$

We then apply our algorithm for computing $\sup(\text{quo}_k(L))$ to $L(M')$.

The Leech word

Leech [1957] showed that the fixed point $\mathbf{l}$ of the morphism

$$\begin{aligned} 0 &\rightarrow 0121021201210 \\ 1 &\rightarrow 1202102012021 \\ 2 &\rightarrow 2010210120102 \end{aligned}$$

is squarefree.

We used our method to compute the critical exponent of this word. It is $15/8$.

Furthermore, if x is a $15/8$-power occurring in $\mathbf{l}$, then $|x| = 15 \cdot 13^i$ for some $i \geq 0$.

Generalization: fixed points of morphisms over $\mathbb{Z}$

- Example: morphism $h : i \to (0, i+1)$
- Then $h^\omega(0) = 0102010301020104\cdots$, the so-called "ruler function"
  - Naturally arises as the lexicographically least squarefree word over $\mathbb{N}$
- The critical exponent of $h^\omega(0)$ is 2

Nørgård's "infinity series"

- The sequence $(0, 1, -1, 2, 1, 0, -2, 3, \ldots)$
- Fixed point of $i \to (-i, i+1)$
- Used by Danish composer Per Nørgård in many of his musical compositions
- Critical exponent is $\frac{4}{3}$

Open Questions

- Extend these ideas to morphic sequences (fixed points of possibly non-uniform morphisms, followed by a coding)
  - Some ideas are extendable to, e.g., the Fibonacci word
  - Carton & Thomas proved that $(\mathbb{N}, <, \text{morphic word})$ is decidable
- Which predicates for automatic sequences (like squarefreeness) are decidable in polynomial time? Leroux has proved it for ultimate periodicity.

More open problems

- Extend these ideas to "infinite state" automata (i.e., fixed points of morphism like $n \to (an+b, cn+d)$) or prove undecidability
- Is $\sup\{x/y : (x,y)_k \in L\}$ computable for context-free languages L?
- Given a regular language $L \subseteq (\Sigma_k \times \Sigma_k)^*$ representing a a set $S \subseteq \mathbb{N} \times \mathbb{N}$ of pairs of natural numbers, is it decidable if S contains a pair (p,q) with $p \mid q$?
  - This is a question of $\exists^1(\mathbb{N}, +, V_k, |)$; of course $\text{Th}(\mathbb{N}, +, |)$ is undecidable and $\exists^1(\mathbb{N}, +, |)$ is decidable (Lipshitz)

More Open Questions

- Prove or disprove: if L is a regular language with $\text{quo}_k(L) = \mathbb{Q}^{\geq 0}$, then L contains infinitely many distinct representations for infinitely many distinct rational numbers.

Which of the following questions is decidable? Given L representing a set of rationals S,

- Is there some rational $p/q \in S$ having infinitely many distinct representations in L?
- Are there infinitely many distinct rationals $p/q \in S$ having infinitely many distinct representations in L?