Paperfolding and Continued Fractions

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Paperfolding

Start with a piece of paper, and fold it in half. Fold the resulting piece in half again, and so forth. Upon unfolding, we observe a certain pattern of hills $(+1)$ and valleys (-1):

For example, after one fold, and unfolding to 90°, we get the pattern in Figure 1.

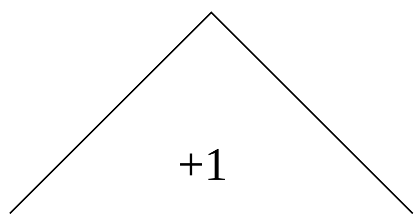

Figure 1: Building the regular paperfolding sequence: one fold

Paperfolding

After two folds, and unfolding to $90°$, we get the pattern in Figure 2.

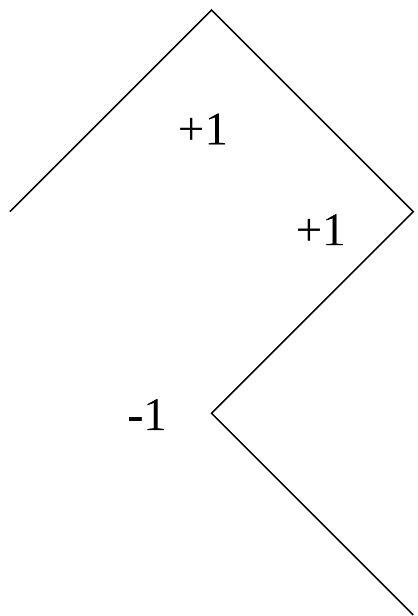

Figure 2: Building the regular paperfolding sequence: two folds

Paperfolding

After twelve folds, we get the interesting "dragon curve" in Figure 3 (where corners have been rounded off for clarity).

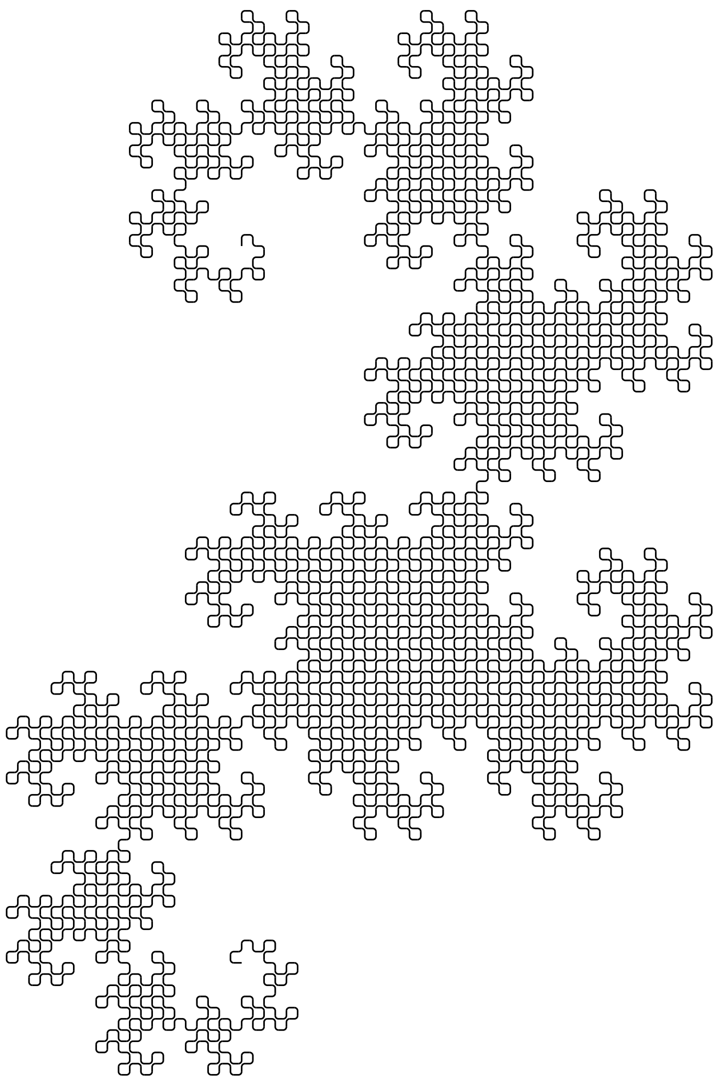

Figure 3: Building the regular paperfolding sequence: twelve folds

Paperfolding

1 fold: $+1$

2 folds: $+1\ +1\ -1$

3 folds: $+1\ +1\ -1\ +1\ +1\ -1\ -1$

Let P_n denote the pattern after n folds. Then

$$\boxed{P_{n+1} = P_n\ ,\ (+1)\ ,\ -P_n^R}$$

Here P_n^R means reverse the list of $+1$'s and -1's.

One can also fold to insert a hill $(+1)$ or a valley (-1).

Paperfolding

Define the folding map F_i to be

$$F_i(L) = L,\ i,\ -L^R.$$

Theorem.
If we fold with instructions

$$a_1, a_2, \ldots, a_n$$

we get the pattern of folds

$$F_{a_1}(F_{a_2}(\cdots(F_{a_n}(\epsilon))\cdots))$$

upon unfolding.

Here ϵ denotes the empty list of folds.

Proof.
Fold the paper once, then $n-1$ times. ∎

Paperfolding

We can now consider folding the paper, always inserting a hill, an infinite number of times. (Any actual piece of paper can be folded in half at most six times.) When we unfold, we get an interesting sequence $\mathbf{R}$, called the *regular paperfolding sequence*.

Here are the first few terms of the sequence $\mathbf{R}$:

$$\begin{array}{rcccccccccccccccc} n & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ R_n & = & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \end{array}$$

It is not hard to prove that

$$\begin{aligned} R_{2n+1} &= (-1)^n; \\ R_{2n} &= R_n. \end{aligned}$$

Paperfolding

We can also fold the paper, alternating whether we insert a hill $(+1)$ or a valley (-1):

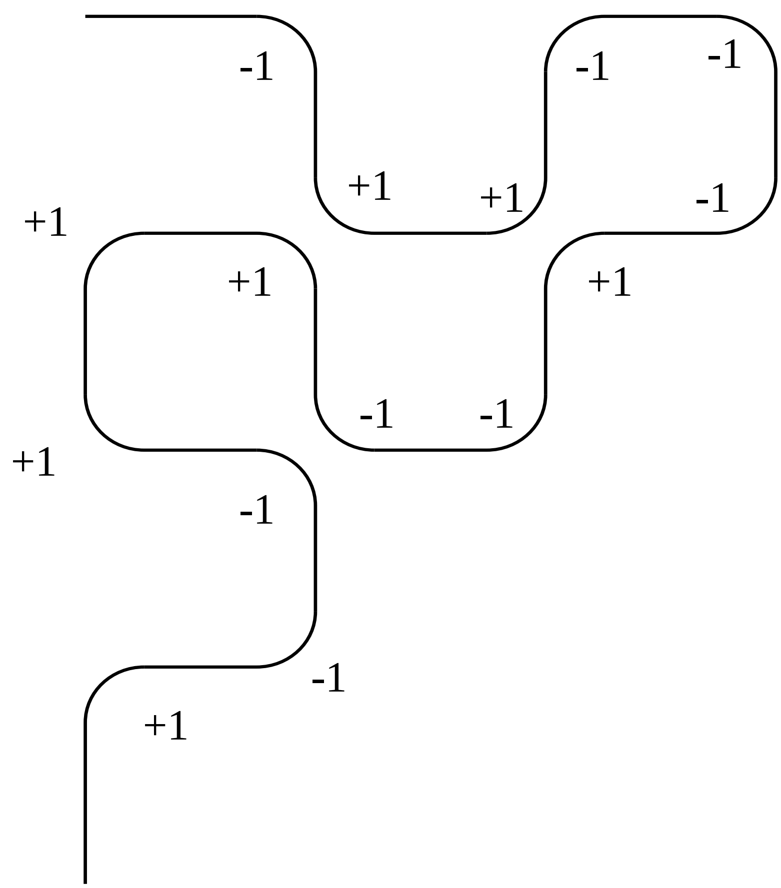

Figure 4: Building the alternate paperfolding sequence: four folds

Alternate Paperfolding

In the limit, we get a curve that fills one-eight of the plane:

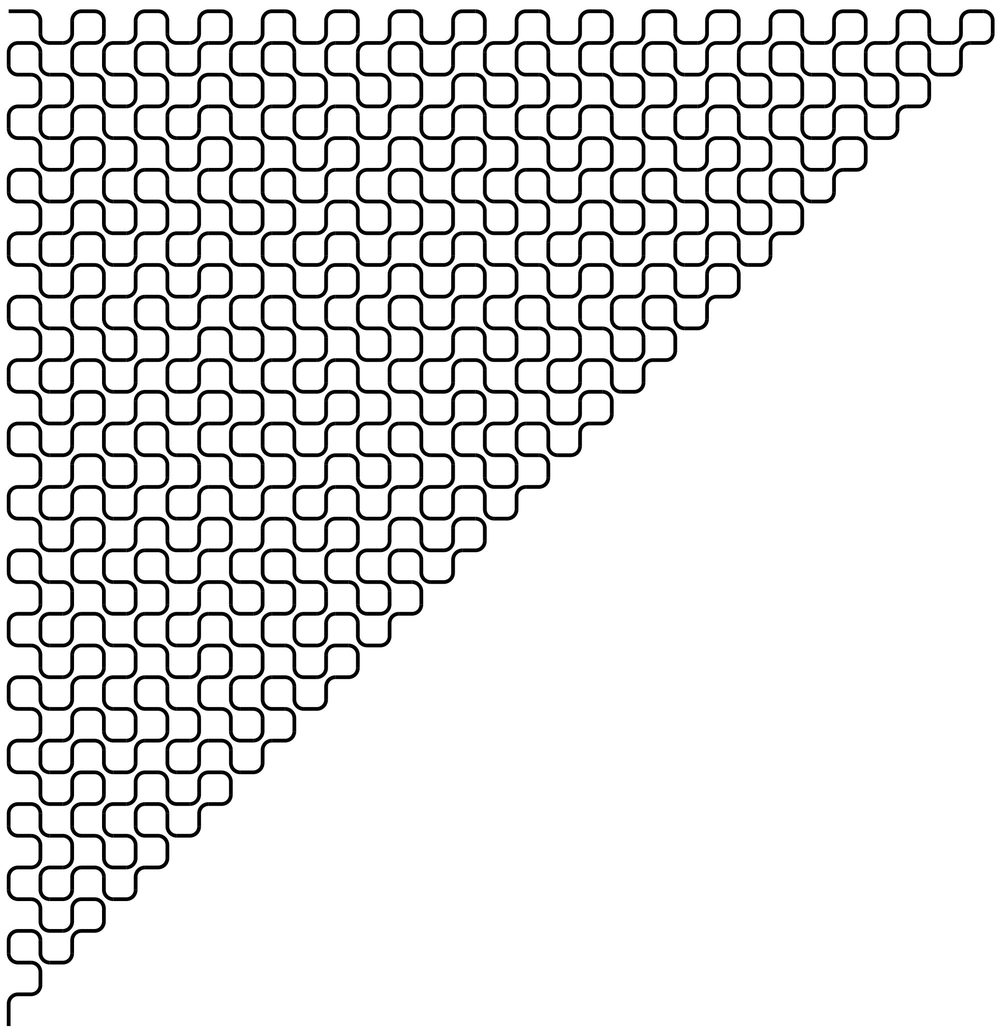

Figure 5: Building the alternate paperfolding sequence: ten folds

General Paperfolding

More generally, write

$$F_{a_1}(F_{a_2}(\cdots(F_{a_n}(\epsilon))\cdots)) = \mathrm{Fold}(a_n, a_{n-1}, \ldots, a_1),$$

and call the sequence $a_n, a_{n-1}, \ldots, a_1$ the *unfolding instructions*.

Now we would like to fold an *infinite* number of times. In this case, we can specify an infinite sequence $b_0, b_1, b_2, \cdots$ of unfolding instructions. The resulting sequence of folds is then given by

$$\mathrm{Fold}(b_0, b_1, b_2, \ldots).$$

For example, the regular paperfolding sequence is

$$\mathbf{R} := \mathrm{Fold}(1, 1, 1, \ldots) = (1, 1, -1, 1, 1, -1, -1, \ldots).$$

The General Paperfolding Theorem

Theorem. Let $(f_i)_{i \geq 1} = \mathrm{Fold}(b_0, b_1, b_2, \ldots)$. Let $n = 2^k(2j+1)$ for some integers $j, k \geq 0$. Then $f_n = (-1)^j b_k$.

Proof. By induction on n. Clearly the result is true for $n = 1$, since then $k = j = 0$, and $f_1 = b_0$.

Now assume the result is true for all $n < 2^m$; we prove it for $2^m \leq n < 2^{m+1}$. If $n = 2^m$, then from the definition of the folding map we have $f_n = b_m$. Otherwise $2^m < n < 2^{m+1}$. In this case, by the folding map we get $f_n = -f_{2^{m+1}-n}$. If $n = 2^k(2j+1)$, then $2^{m+1} - n = 2^k(2j'+1)$, where $j' = 2^{m-k} - j - 1$. Now $2^{m+1} - n < n$, so by induction we have $f_{2^{m+1}-n} = (-1)^{j'} b_k$. But $k < m$, so $j' \equiv j + 1 \pmod 2$. Hence $f_n = (-1)^j b_k$, as desired. ∎

Another Paperfolding Theorem

Theorem. A paperfolding sequence $(f_i)_{i\geq 1}$ is never ultimately periodic.

Proof. Suppose $(f_i)_{i\geq 1}$ is ultimately periodic.

Then there exist integers m, N such that $f_n = f_{m+n}$ for all $n \geq N$.

Let $m = 2^a(2b+1)$. Define the sequence $(n_i)_{i\geq 1}$ by $n_i = 2^{a+1}(2i+1)$. Then $(f_{n_i})_{i\geq 1}$ is purely periodic of period 2 by the previous theorem. But

$$\begin{aligned} m + n_i &= 2^a(2b+1) + 2^{a+1}(2i+1) \\ &= 2^a(2(b+2i+1)+1), \end{aligned}$$

so $(f_{m+n_i})_{i\geq 1}$ is purely periodic of period 1 by the previous theorem, a contradiction. ∎

Yet Another Paperfolding Theorem

Theorem. All the paperfolding curves are non-self-intersecting.

And now for something completely different!

Continued Fractions

"Continued fractions are part of the 'lost mathematics,' the mathematics now considered too advanced for high school and too elementary for college."

— Petr Beckmann, *A History of Pi*

"Continued fractions are hard to like. People who like continued fractions eat pickled okra and drive Citroens."

— R. Gosper

Continued Fractions

$$\pi = 3$$

...but not really 3, more like $3 + \frac{1}{7}$.

...but not really 7, more like $7 + \frac{1}{15}$.

...but not really 15, more like $15 + \frac{1}{1}$.

...but not really 1, more like $1 + \frac{1}{292}$.

That is,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}.$$

Simple Continued Fractions

An expression such as

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}},$$

which we abbreviate as

$$[3, 7, 15, 1, 292, \ldots]$$

is a *simple continued fraction*.

Continued Fractions

There are also more general kinds of continued fractions, such as

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cdots}}}}.$$

Continued fractions were invented (discovered?) by Pietro Antonio Cataldi (1548–1626), who wrote (in modern notation)

$$\sqrt{18} = 4 + \cfrac{2}{8 + \cfrac{2}{8 + \cfrac{2}{8 + \cfrac{2}{8 + \cdots}}}}.$$

Continued Fractions

In 1655 William Brouncker found

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}},$$

which is term-by-term equivalent to Wallis' series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

Figure 6: William Brouncker (1620–1684)

Another Continued Fraction

Also

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cdots}}}},$$

which is term-by-term equivalent to

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots .$$

Simple Continued Fractions

Theorem.

Every real number x *can be written (essentially) uniquely as a simple continued fraction*

$$x = [a_0, a_1, a_2, \ldots,],$$

where

(a) a_i *is an integer for all* $i \geq 0$;

(b) $a_i \geq 1$ *for all* $i \geq 1$.

Other properties include:

(c) the expansion terminates iff x is rational;

(d) the expansion is ultimately periodic iff x is the irrational real root of a quadratic equation with integer coefficients.

The a_i are called the *partial quotients*.

Expanding a Real Number

Given a real number, how do determine its simple continued fraction expansion?

Suppose

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

Then $a_0 \leq x < a_0 + 1$. So $a_0 = \lfloor x \rfloor$, where $\lfloor\ \rfloor$ denotes the *greatest integer function*.

Then

$$x - a_0 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

so...

Expanding a Real Number

$$\frac{1}{x - a_0} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}.$$

Now we can repeat the analysis above to determine a_1, etc.

For example

$$\begin{aligned}
x = x_0 &= \pi \doteq 3.141592653 \\
a_0 &= \lfloor \pi \rfloor = 3 \\
x_1 &= 1/(x_0 - a_0) \doteq 7.062513306 \\
a_1 &= \lfloor x_1 \rfloor = 7 \\
x_2 &= 1/(x_1 - a_1) \doteq 15.996594407 \\
a_2 &= \lfloor x_2 \rfloor = 15 \\
x_3 &= 1/(x_2 - a_2) \doteq 1.003417231 \\
a_3 &= \lfloor x_3 \rfloor = 1 \\
x_4 &= 1/(x_3 - a_3) \doteq 292.634591014
\end{aligned}$$

Examples of Continued Fractions

We can do the same kind of analysis with other real numbers. For example:

$$
\begin{aligned}
x = x_0 &= \sqrt{2} \doteq 1.414 \\
a_0 &= \lfloor \sqrt{2} \rfloor = 1 \\
x_1 &= 1/(x_0 - a_0) = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \doteq 2.414 \\
a_1 &= \lfloor x_1 \rfloor = 2 \\
x_2 &= 1/(x_1 - a_1) = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \doteq 2.414 \\
a_2 &= \lfloor x_2 \rfloor = 2 \\
x_3 &= 1/(x_2 - a_2) = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \doteq 2.414 \\
a_3 &= \lfloor x_3 \rfloor = 2 \\
&\vdots
\end{aligned}
$$

So $\sqrt{2} = [1, 2, 2, 2, \ldots]$.

Examples of Continued Fractions

$$\pi = [3, 7, 15, 1, 292, \ldots]$$

$$22/7 = [3, 7]$$

$$355/113 = [3, 7, 15, 1] = [3, 7, 16]$$

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$$

$$\sqrt{2} = [1, 2, 2, 2, \ldots]$$

$$\frac{1 + \sqrt{5}}{2} = [1, 1, 1, 1, \ldots]$$

Examples of Continued Fractions

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{256}+\cdots=$$

$$[0,1,4,2,4,4,6,4,2,4,6,2,4,6,4,4,2,\ldots]$$

$$2(\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{256}+\cdots)=$$

$$[1,1,1,1,2,1,1,1,1,1,1,1,2,1,1,\ldots]$$

More Examples of Continued Fractions

$$\tan(1) =$$

$$e^2 =$$

$$e^{2/3} =$$

$$2^{-\lfloor \alpha \rfloor} + 2^{-\lfloor 2\alpha \rfloor} + 2^{-\lfloor 3\alpha \rfloor} + \cdots =$$

$$(\alpha = \frac{1+\sqrt{5}}{2})$$

$$2^{1/3} =$$

Big Open Problem

An *algebraic number* is a complex number that is the root of a polynomial equation with integer coefficients. For example, $\sqrt{2}$, $i = \sqrt{-1}$, and $\sqrt[3]{2}$ are all algebraic numbers, but π is not.

The *degree* of an algebraic number α is the lowest degree of any nontrivial polynomial f with $f(\alpha) = 0$.

Are the partial quotients in the continued fraction expansion for a real algebraic number of degree > 2 unbounded? No one knows.

Large Partial Quotients

Figure 7: John Brillhart

John Brillhart found that certain algebraic numbers, such as the real root $\alpha \doteq 3.318628218$ of $x^3 - 8x - 10 = 0$, have large partial quotients at the beginning of their continued fraction expansion. For example,

$$\alpha = [3, 3, 7, 4, 2, 30, 1, 8, 3, 1, 1, 1, 9, 2, 2, 1, 3, 22986, 2, 1, 32, 8, 2, 1, 8, 55, 1, 5, 2, 28, 1, 5, 1, 1501790, \ldots]$$

This was later explained by Harold Stark.

Figure 8: Harold Stark

The explanation is distantly related to the amazing fact that

$$e^{\pi\sqrt{163}} \doteq 262537412640768743.99999999999925007\cdots$$

Distribution of Partial Quotients

Although little is known about the partial quotients for particular numbers such as π, we do know a lot about "almost all" real numbers.

Kuz'min [1928] proved that for almost all real numbers,

41.5% of the partial quotients are 1;

17% of the partial quotients are 2;

9.3% of the partial quotients are 3;

and in general, the fraction of the partial quotients equalling a is $\log_2\left(1 + \frac{1}{a(a+2)}\right)$.

Khintchine proved that the geometric mean

$$(a_1 a_2 \ldots a_n)^{1/n}$$

of the first n partial quotients tends to a limit, about 2.68545, for almost all real numbers.

Convergents

Suppose you have an infinite continued fraction, such as $\pi = [3, 7, 15, 1, 292, \ldots]$.

You can truncate this continued fraction after n terms to get a rational number. This rational number is called a *convergent*, and is a good approximation to the original number. For example,

$$[3] = \frac{3}{1}$$

$$[3, 7] = \frac{22}{7}$$

$$[3, 7, 15] = \frac{333}{106}$$

$$[3, 7, 15, 1] = \frac{355}{113}$$

$$[3, 7, 15, 1, 292] = \frac{103993}{33102}$$

Convergents

Theorem. *Let* $a_0, a_1, \ldots, a_n$ *be real numbers with* $a_i > 0$ *for* $i \geq 1$. *Define* p_n *and* q_n *as follows:*

$$p_{-2} = 0; q_{-2} = 1; \quad p_{-1} = 1; q_{-1} = 0$$

and

$$p_k = a_k p_{k-1} + p_{k-2}; \quad q_k = a_k q_{k-1} + q_{k-2} \tag{1}$$

for $0 \leq k \leq n$. *If* $\alpha > 0$ *is a real number, then*

$$[a_0, a_1, \ldots, a_n, \alpha] = \frac{\alpha p_n + p_{n-1}}{\alpha q_n + q_{n-1}}.$$

Proof. The proof is by induction. The result clearly holds for $n = -1, 0$. Now assume the result holds for n. We find

$$[a_0, a_1, \ldots, a_{n+1}, \alpha] = \left[a_0, a_1, \ldots, a_n, a_{n+1} + \frac{1}{\alpha}\right]$$

$$= \frac{\left(a_{n+1} + \frac{1}{\alpha}\right) p_n + p_{n-1}}{\left(a_{n+1} + \frac{1}{\alpha}\right) q_n + q_{n-1}}$$

(by the induction hypothesis)

$$= \frac{\alpha p_{n+1} + p_n}{\alpha q_{n+1} + q_n} \text{ by (1)}$$

and so the result holds for $n + 1$, and hence for all n. ■

Convergents to π

n	-2	-1	0	1	2	3	4	5
a_n			3	7	15	1	292	1
p_n	0	1	3	22	333	355	103993	104348
q_n	1	0	1	7	106	113	33102	33215

n	6	7	8	9	10
a_n	1	1	2	1	3
p_n	208341	312689	833719	1146408	4272943
q_n	66317	99532	265381	364913	1360120

Convergents

Recall from (1) that

$$p_k = a_k p_{k-1} + p_{k-2}; \quad q_k = a_k q_{k-1} + q_{k-2}.$$

Then

$$\begin{bmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} a_{n+1} & 1 \\ 1 & 0 \end{bmatrix}$$

and hence an easy induction gives

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

By taking the determinants of both sides, we get

Theorem.

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}. \tag{3}$$

Convergents to e

n	-2	-1	0	1	2	3	4	5	6
a_n			2	1	2	1	1	4	1
p_n	0	1	2	3	8	11	19	87	106
q_n	1	0	1	1	3	4	7	32	39

n	7	8	9	10	11
a_n	1	6	1	1	8
p_n	193	1264	1457	2721	23225
q_n	71	465	536	1001	8544

Infinite Simple Continued Fractions

We now define infinite simple continued fractions. Provided the limit exists, we may define the symbol $[a_0, a_1, \ldots, a_n, \ldots]$ as

$$\lim_{n\to\infty}[a_0, a_1, \ldots, a_n].$$

If the continued fraction is simple, that is, if the a_i are integers and $a_i \geq 1$ for $i \geq 1$, then it is easy to see that the limit exists. For by Eq. (3) we see that

$$\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{|p_n q_{n-1} - p_{n-1} q_n|}{q_n q_{n-1}} = \frac{1}{q_n q_{n-1}}.$$

Since $q_n \geq n$ by (1), we see that p_n/q_n is a Cauchy sequence, and hence converges to a limit.

(Recall that a sequence of real numbers (x_n) converges if and only if it is a Cauchy sequence, and that (x_n) is a Cauchy sequence if for all $\epsilon > 0$ there exists an integer N such that for all $m, n \geq N$ we have $|x_n - x_m| < \epsilon$.)

Formal Power Series

You are probably familiar with expansions such as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

These are called power series.

We can also examine *formal power series*, which are, roughly speaking, power series without worrying about convergence.

Formal Laurent Series

We define a formal Laurent series in $1/X$ to be an expression of the form

$$\sum_{i \geq i_0} a_i X^{-i}.$$

Such an expression has only finitely many positive powers of X, but potentially infinitely many negative powers of X. There is a strong analogy with the base-k representation of a real number.

We can expand formal Laurent series in continued fractions, in exactly the same way as for a real number. We only need a definition of $\lfloor \cdot \rfloor$. What could be more natural than to define $\lfloor \cdot \rfloor$ to be the terms with non-negative exponents?

Formal Laurent Series

For example, let

$$
\begin{aligned}
x_0 &= e^{1/X} = 1 + \frac{1}{X} + \frac{1}{2X^2} + \frac{1}{6X^3} + \frac{1}{24X^4} + \cdots \\
a_0 &= \lfloor x_0 \rfloor = 1 \\
x_1 &= 1/(x_0 - a_0) = X - \frac{1}{2} + \frac{1}{12X} - \frac{1}{720X^3} + \cdots \\
a_1 &= \lfloor x_1 \rfloor = X - \frac{1}{2} \\
x_2 &= 1/(x_1 - a_1) = 12X + \frac{1}{5X} - \frac{1}{700X^3} + \cdots \\
a_2 &= \lfloor x_2 \rfloor = 12X \\
&\vdots
\end{aligned}
$$

Continuing in this fashion, we get

$$
e^{1/X} = [1,\ X - \frac{1}{2},\ 12X,\ 5X,\ 28X - 70, \ldots].
$$

An Example

$$f(X) := \frac{1}{\sqrt{X^2 - 1}} = \sum_{k \geq 0} 2^{-2k} \binom{2k}{k} X^{-2k-1}$$

$$= X^{-1} + \frac{1}{2}X^{-3} + \frac{3}{8}X^{-5} + \frac{5}{16}X^{-7} + \frac{35}{128}X^{-9} + \cdots$$

$$= [0,\ X,\ -2X,\ 2X,\ -2X,\ 2X,\ -2X,\ \ldots]$$

Here are the first few convergents to the continued fraction for f:

n	0	1	2	3	4
a_n	0	X	$-2X$	$2X$	$-2X$
p_n	0	1	$-2X$	$-4X^2+1$	$8X^3-4X$
q_n	1	X	$-2X^2+1$	$-4X^3+3X$	$8X^4-8X^2+1$

An Example

By the way, it can be proved that

- $p_n(X) = (-1)^{\lfloor n/2 \rfloor}\, U_{n-1}(X)$;
- $q_n(X) = (-1)^{\lfloor n/2 \rfloor}\, T_n(X)$;

where T and U are the Chebyshev polynomials of the first and second kinds, respectively.

Continued Fractions for Formal Power Series

Theorem. (Artin, 1924) *Let n be an integer. Any formal power series*

$$f(X) = \sum_{-\infty < i \leq n} b_i X^i$$

can be expressed uniquely as a continued fraction

$$f(X) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

$$= [a_0, a_1, a_2, \ldots]$$

where

- a_i *is a polynomial in* X *with rational coefficients for* $i \geq 0$*; and*
- $\deg a_i > 0$ *for* $i > 0$.

Some Real Numbers with Bounded Partial Quotients

Suppose x is a real number with continued fraction $[a_0, a_1, a_2, \ldots]$. Define $K(x) = \sup_{i \geq 1} a_i$.

Theorem. (Kmošek, 1979; Shallit, 1979)
Let $n \geq 2$ be an integer, and define

$$f(X) = \sum_{i \geq 0} X^{-2^i}.$$

Then $K(f(2)) = 6$, $K(f(n)) = n + 2$ for $n \geq 3$, and $K(nf(n)) = n$ for $n \geq 2$.

Examples.

$$f(2) = [0, 1, 4, 2, 4, 4, 6, 4, 2, 4, 6, 2, 4, 6, 4, 4, 2, \ldots]$$
$$f(3) = [0, 2, 5, 3, 3, 1, 3, 5, 3, 1, 5, 3, 1, 3, 3, 5, 3, \ldots]$$
$$f(4) = [0, 3, 6, 4, 4, 2, 4, 6, 4, 2, 6, 4, 2, 4, 4, 6, 4, \ldots]$$
$$f(5) = [0, 4, 7, 5, 5, 3, 5, 7, 5, 3, 7, 5, 3, 5, 5, 7, 5, \ldots]$$

Now what's the relationship between this and paperfolding?

Define

$$h_n(X) = X \sum_{0 \le i \le n} X^{-2^i}.$$

For example

$$\begin{aligned}
h_0(X) &= X(\frac{1}{X}) = 1; \\
h_1(X) &= X(\frac{1}{X} + \frac{1}{X^2}) = 1 + X^{-1}; \\
h_2(X) &= X(\frac{1}{X} + \frac{1}{X^2} + \frac{1}{X^4}) = 1 + X^{-1} + X^{-3}. \\
&\vdots
\end{aligned}$$

We find

$$\begin{aligned}
&h_0(X) = 1 = [1] \\
&h_1(X) = 1 + X^{-1} = [1, X] \\
&h_2(X) = [1, X, -X, -X] \\
&h_3(X) = [1, X, -X, -X, -X, X, X, -X] \\
&h_4(X) = [1, X, -X, -X, -X, X, X, -X, \\
&\qquad\qquad - X, X, -X, -X, X, X, X, -X].
\end{aligned}$$

This suggests that if

$$h_i(X) = [1,\ Y],$$

then

$$h_{i+1}(X) = [1,\ Y,\ -X,\ -Y^R].$$

Thus we conjecture that

$$\lim_{n\to\infty} h_n(X) = Xf(X)$$
$$= [1, \cdots F_{-X}(F_{-X}(\cdots F_{-X}(X)\cdots))\cdots] \quad ,$$
$$= [1, \quad , \quad , \quad , \quad , \quad , \quad , \quad , \cdots]$$

and so we can compute a continued fraction merely by folding a piece of paper.

Now put $X = 2$. We get

$$2 \sum_{i \geq 0} 2^{-2^i} = [1, 2, -2, -2, -2, 2, 2, -2, -2, 2, \cdots].$$

... so we need to "remove" the -2's somehow.

This can be done with the following

Lemma.

$$[a, -b, c] = [a-1, 1, b-2, 1, c-1];$$
$$[a, 0, b] = [a+b].$$

Thus

$$
\begin{aligned}
&2\sum_{i\geq 0} 2^{-2^i} \\
&= [1, 2, -2, -2, -2, 2, 2, -2, -2, 2, -2, -2, \cdots] \\
&= [1, 1, 1, 0, 1, -3, -2, 2, 2, -2, -2, 2, -2, -2, \cdots] \\
&= [1, 1, 2, -3, -2, 2, 2, -2, -2, 2, -2, -2, \cdots] \\
&= [1, 1, 1, 1, 1, 1, -3, 2, 2, -2, -2, 2, -2, -2, \cdots] \\
&= [1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 2, -2, -2, 2, -2, -2, \cdots] \\
&= [1, 1, 1, 1, 2, 1, 1, 1, 2, -2, -2, 2, -2, -2, \cdots] \\
&= [1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 0, 1, -3, 2, -2, -2, \cdots] \\
&= [1, 1, 1, 1, 2, 1, 1, 1, 1, 2, -3, 2, -2, -2, \cdots] \\
&= [1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, -2, -2, \cdots] \\
&\cdots
\end{aligned}
$$

Now how do we prove this all works?

Lemma. *Suppose* $p_n/q_n = [c_0, c_1, c_2, \ldots, c_n]$, *and let* w *represent the list of partial quotients* $c_1, c_2, \ldots, c_n$. *Then*

$$[c_0,\ w,\ t,\ -w^R] = \frac{p_n}{q_n} + \frac{(-1)^n}{tq_n^2}.$$

Proof. By Eq. (2) we have

$$\begin{bmatrix} c_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} c_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix}.$$

Similarly, it is easily proved by induction that

$$\begin{bmatrix} -c_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -c_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} -c_n & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} (-1)^{n+1} p_n & (-1)^n p_{n-1} \\ (-1)^n q_n & (-1)^{n-1} q_{n-1} \end{bmatrix}. \qquad (4)$$

Take the transpose of both sides of Eq. (4); we get

$$\begin{bmatrix} -c_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -c_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} -c_0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} (-1)^{n+1} p_n & (-1)^n q_n \\ (-1)^n p_{n-1} & (-1)^{n-1} q_{n-1} \end{bmatrix}. \tag{5}$$

Multiply Eq. (5) on the right by

$$\begin{bmatrix} -c_0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & c_0 \end{bmatrix}$$

to get

$$\begin{bmatrix} -c_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -c_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} -c_1 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} (-1)^n q_n & * \\ (-1)^{n-1} q_{n-1} & * \end{bmatrix} \tag{6}$$

where the asterisks represent entries that do not concern us.

Hence we find

$$\begin{bmatrix} c_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} c_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -c_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -c_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} -c_1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (-1)^n q_n & * \\ (-1)^{n-1} q_{n-1} & * \end{bmatrix}$$

$$= \begin{bmatrix} (tp_n + p_{n-1})(-1)^n q_n + (-1)^{n-1} p_n q_{n-1} & * \\ (tq_n + q_{n-1})(-1)^n q_n + (-1)^{n-1} q_n q_{n-1} & * \end{bmatrix}.$$

It follows that

$$[c_0, c_1, \ldots, c_n, t, -c_n, -c_{n-1}, \ldots, -c_1]$$

$$= \frac{(-1)^n(tp_nq_n + p_{n-1}q_n - p_nq_{n-1})}{(-1)^n(tq_n^2 + q_{n-1}q_n - q_nq_{n-1})}$$

$$= \frac{tp_nq_n + (-1)^n}{tq_n^2}$$

$$= \frac{p_n}{q_n} + \frac{(-1)^n}{tq_n^2},$$

as desired. ∎

More generally, we have the following

Theorem. (van der Poorten and Shallit, 1990)
Let $a_0 = 1$, $a_i = \pm 1$ *for* $i \geq 1$.
Then the number

$$2 \sum_{i \geq 0} a_i 2^{-2^i}$$

is transcendental and its continued fraction expansion consists solely of 1's *and* 2's.

(A number is *transcendental* if it is not algebraic, i.e., not the root of an algebraic equation with integer coefficients.)

Another Interesting Continued Fraction

Let F_n be the n-th Fibonacci number, and define

$$\begin{aligned}\alpha &= \sum_{n \geq 2} 2^{-F_n} \\ &= 2^{-1} + 2^{-2} + 2^{-3} + 2^{-5} + 2^{-8} + 2^{-13} \cdots \\ &= .111010010000 1 \cdots_{(2)}.\end{aligned}$$

Then we find that α has the following continued fraction expansion:

$$\alpha = [0, 1, 10, 6, 1, 6, 2, 14, 4, 124, 2, 1, 2, 2039, 1, 9, 1, 1, 1, 262111, 2, 8, 1, 1, 1, 3, 1, 536870655, 4, 16, 3, 1, 3, 7, 1, 140737488347135, \ldots].$$

How can we prove that α has **un**bounded partial quotients?

Define

$$s_n = X^{-F_2} + X^{-F_3} + X^{-F_4} + \cdots + X^{-F_n}.$$

We find that

$$s_\infty =$$
$$[0,\ X-1,\ X^2+2X+2,\ X^3-X^2+2X-1,$$
$$-X^3+X-1,\ -X,\ -X^4+X,\ -X^2,$$
$$-X^7+X^2,\ -X-1, X^2-X+1,\ X^{11}-X^3,$$
$$-X^3-X,\ -X,\ X,\ X^{18}-X^5,\ -X,$$
$$X^3+1,\ X,\ -X,\ -X-1,\ -X+1,$$
$$-X^{29}+X^8,\ X-1,\ \cdots]$$

It can be shown (van der Poorten & Shallit, 1991) that if

$$s_n = [0, f_n]$$

then, for $n \geq 11$,

$$s_{n+1} = [0,\ f_n,\ 0,\ -f_{n-4},\ -X^{L_{n-4}},\ f_{n-4}^R,$$
$$0,\ -f_{n-3},\ X^{F_{n-4}},\ f_{n-3}^R].$$

Liouville's Transcendental Number

In 1851, Liouville gave the first explicit example of a real number that is *transcendental* (not the root of an algebraic equation with integer coefficients).

Figure 9: Joseph Liouville (1809–1882)

Liouville's number is

$$L(m) = \sum_{k \geq 1} m^{-k!}.$$

Liouville's Transcendental Number

It can be shown, using the techniques we have seen, that

$$
\begin{aligned}
L(m) &= \sum_{k\geq 1} m^{-k!} \\
&= [0,\ m-1,\ m+1,\ m^2,\ -m-1,\ -m+1, \\
&\quad -m^{12},\ m-1,\ m+1,\ -m^2,\ -m-1, \\
&\quad -m+1,\ -m^{72}, \ldots] \\
&= [0,\ m-1,\ m+1,\ m^2-1,\ 1,\ m,\ m-1, \\
&\quad m^{12}-1,\ 1,\ m-2, \ldots]
\end{aligned}
$$

Books About Continued Fractions

1. C. D. Olds, *Continued Fractions*, Random House, 1963. Good, very elementary introduction.
2. G. Chrystal, *Textbook of Algebra*. Chapter 32 has a good, old-fashioned introduction to the subject.
3. Khinchin, *Continued Fractions*. More advanced treatment, including the metric theory.
4. Hardy and Wright, *An Introduction to the Theory of Numbers*. Good intermediate treatment.

For Further Reading...

- Dekking, Mendès France, and van der Poorten, Folds! *Math. Intelligencer* **4** (1982), 130–138, 173–181, 190–195.

- van der Poorten and Shallit, Folded Continued Fractions, *J. Number Theory*, **40** (1992), 237–250.

- van der Poorten and Shallit, A Specialised Continued Fraction, *Canadian J. Math.* **45** (1993), 1067–1079.

- Shallit, Real Numbers with Bounded Partial Quotients: A Survey, *L'Enseignement Math.*, **38** (1992), 151–187.